# FH-SPACES AND INTERSECTIONS OF FK-SPACES

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## 1. INTRODUCTION

We present a new concept, the FH-space (a specialization of the (F)-space of Bourbaki or $B_0$-space of Mazur), which is general (see Examples in Section 2), but in which we are able to develop the theory of FK-spaces, and a little of the commutative Banach algebra theory.

In Section 4 we fill in the remaining gap in the theory of intersections in summability. Here we introduce a new property which is stronger than perfectness but weaker than the property which has on various occasions been called boundedness, the PMI-, AK-, and mean-value property. We show that a familiar Nörlund matrix has this property.

## 2. FH-SPACES

We begin with a fixed Hausdorff space H, not necessarily a linear space. An FH-space is an F-space (linear, metric, complete, and locally convex) which is a subset of H, and whose topology is stronger than that of H. (Throughout this article, "stronger than" means "stronger than or equal to.")

*Convention.* If $L_1$ and $L_2$ are FH-spaces and the symbols $L_1 \subset L_2$, $L_1 \cap L_2$, $L_1 \cup L_2$ occur, we assume only set-theoretical inclusion, intersection, union, and that the linear operations have the same formal meaning in $L_1$ and in $L_2$.

THEOREM 1. *Let* $L_1$ *and* $L_2$ *be* FH-*spaces with* $L_1 \subset L_2$. *Then the topology of* $L_1$ *is stronger than that of* $L_2$.

*Proof.* Let i: $L_1 \to L_2$ be the inclusion map ix = x. Then i is closed, since if $x^n \to x$ in $L_1$ and $x^n \to y$ in $L_2$, we have $x^n \to x$ and $x^n \to y$ in H, so that x = y. The closed-graph theorem now yields the continuity of i and concludes the proof.

COROLLARY. *The topology of an* FH-*space is uniquely determined; that is, a linear space cannot be given two different* FH-*topologies.*

*Example 1.* Let H be the set s of all real sequences x with the usual coordinatewise topology (the F-topology determined by the seminorms $p_n(x) = |x_n|$ for $x = \{x_n\}$). In this case, the FH-spaces are the well-known FK-spaces, that is, F-spaces of sequences with continuous coordinates. Here Theorem 1 plays an important role in connection with summability (see [5]).

*Example 2.* Let B be a commutative semisimple complex Banach algebra. Let H be B, but with the weak topology generated by the multiplicative linear functionals (scalar homomorphisms); H is a Hausdorff space, since the set of homomorphisms is separating; and it is weaker than B, since the homomorphisms are continuous on

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B. If now we attempt to put another norm on B to make it a Banach algebra, it is equivalent to the original norm, by Theorem 1.

*Example 3.* In connection with Example 2, we make the following observation: If a linear space allows two complete norms with a common separating family of continuous linear functionals, then it follows from Theorem 1 that the norms are equivalent; indeed instead of the family of functionals, it is sufficient to have a Hausdorff topology (not necessarily linear) which is weaker than both.

*Example 4.* Let a set X be given, and let H be the set of all complex functions on X. We give H the product, or pointwise topology; that is, $H = C^X$ is the product of a number of copies of C (the space of complex numbers) equal to the cardinality of X; the topology being the weakest such that for each $x \in X$, $f(x)$ is a continuous function of $f \in H$. Then *any Banach algebra* B *of complex functions on* X *is an* FH-*space,* since for each $x \in X$, $f(x)$ is a multiplicative, linear functional of $f \in B$ and hence must be continuous. Thus the topology of B is stronger than that of H. (We are supposing, as usual, that the operations in B are the pointwise ones.) Example 1 is a special case in which X is the set of positive integers, and *any Banach algebra of sequences is an* FK-space.

Certain facts, known for FK-spaces, can be generalized immediately. For example, *if an* FH-*space is a proper subset of another, it is of first category in it,* since it is the range of the inclusion map. From this it follows that *the union of a strictly expanding sequence of* FH-*spaces cannot be an* FH-*space, since it would be of first category in itself.*

THEOREM 2. *For each* $n = 1, 2, \cdots$, *let* $E^n$ *be an* FH-*space. Let* $E = \bigcap E^n$ *be given all the seminorms of all the* $E_n$. *With this topology,* E *is an* FH-space.

*Proof.* Only completeness is in doubt. If $\{x^n\}$ is a Cauchy sequence in E, it is also a Cauchy sequence in each of $E^1, E^2, \cdots$, converging to $y^1, y^2, \cdots$, respectively. Since $x^n \to y^1$, $x^n \to y^2, \cdots$, in H, it follows that $y^1 = y^2 = \cdots = y$, say. Thus $y \in E$ and $x^n \to y$ in E.

THEOREM 3. *If the sequence* $E^n$ *in Theorem 2 is decreasing, and* f *is a continuous linear functional on* E, *then there exists an integer* N *such that* f *is continuous on* E *with the topology of* $E^N$.

*Proof.* In an F-space, every continuous linear functional f is bounded; that is, there exists a finite number N of the seminorms $p_n$ defining the topology such that $f \leq M(p_1 + p_2 + \cdots + p_N)$. Since we have assumed that $E^1 \supset E^2 \supset \cdots \supset E^N$, the result follows.

REMARK. *In Theorem* 3, *we may replace "functional" by "map into a Banach space."* Instead of f, we use $\|f\|$, a continuous seminorm, and essentially the same proof applies.

THEOREM 4. *If, in Theorem* 2, E *contains a sequence* S *which is a basis for each* $E^n$ *and which has a single biorthogonal set of functionals good for each* $E^n$, *then* S *is a basis for* E.

*Proof.* For $x \in E$, we have $x = \Sigma a_k s^k$ $(s^k \in S)$, with the infinite series being taken in the topology of each and every $E^n$. Hence the series converges to x in E, by definition of the topology of E.

*Example 5.* The result holds for FK-spaces, which have $\{\delta^k\}$ as basis, where $\delta^k = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ (1 in the $k^{th}$ place); the coordinates are the required biorthogonal set.

## 3. THE DOMAIN THEOREM

By $(E, p_i)$ we shall denote an FH-space E with seminorms $p_0, p_1, \cdots$.

LEMMA. *Let* A, B *be subsets of a Hausdorff space* H *which are given topologies stronger than that of* H. *If* $f{:}A \to B$ *is continuous in the* H *topology, then it is closed as a function from* A *to* B.

*Proof.* Let $x_n \to x$ in A, $f(x_n) \to y$ in B. Then, in H, $x_n \to x$, $f(x_n) \to y$, $f(x_n) \to f(x)$. Hence $y = f(x)$.

THEOREM 5 (the domain theorem). *Let* H *be a Hausdorff space, and let* $(E, p_i)$ *and* $(F, q_i)$ *be* FH-*spaces. Let* $f\colon E \to H$ *be continuous, and its restriction to* $f^{-1}(F)$ *linear. Then*

(i) $f^{-1}(F)$ *is an* FH-*space with seminorms* $p_i$ *and* $q_i f$ $(i = 0, 1, \cdots)$;

(ii) *if* f *is one-to-one and onto* F, *use only* $q_i f$ *in* (i).

This generalizes 4.10 of [5]. Compare the space E in [1, pp. 47-48], where, incidentally, Banach omits the necessary assumption that coordinates are continuous (see [7]).

*Proof.* The topology of $f^{-1}(F)$ as given in (i) is stronger than that induced by E (since it has more seminorms), hence stronger than that of H. Thus only completeness remains to be proved. Let $\{x^n\}$ be a Cauchy sequence in $f^{-1}(F)$. Then it is also a Cauchy sequence in E, hence it converges in E to x, say. Moreover, $\{f(x^n)\}$ is a Cauchy sequence in F, hence it converges in F to y, say.

Then $y = f(x)$ by the Lemma. Hence $x \in f^{-1}(F)$. Finally, $x^n \to x$ in $f^{-1}(F)$, since $p_i(x^n - x) \to 0$ and $q_i f(x^n - x) \to 0$.

To prove (ii), let $f^{-1}(F)$ be given the seminorms $q_i f$ $(i = 0, 1, 2, \cdots)$. Then f is a linear isometry between $f^{-1}(F)$ and F. Hence $f^{-1}(F)$ is complete.

This completes the proof. We shall not go on to announce the general form of the continuous linear functional on $f^{-1}(F)$; see [5, 4.11].

By the convergence domain $c_A$ of a matrix A we mean the set of sequences x such that $Ax \in c$ (where c is the space of convergent sequences). By $c_A^0$ we mean the same with c replaced by $c^0$, the space of null sequences.

The domain theorem immediately yields the fact that *the convergence domain of a row-finite matrix* A *is an* FK-*space with seminorms*, $\sup_n |\Sigma_k a_{nk} x_k|$, $|x_0|, |x_1|, \cdots$. *If the mapping* $x \to Ax$ *is one-to-one, we may omit all but the first seminorm.* For in the domain theorem we choose $E = H = s$, $F = c$. If A is one-to-one, apply (ii) with $F = c \cap As$, using the fact that As is closed in s [3, p. 419].

For matrices that are not necessarily row-finite, we shall omit the development of FH-spaces which yields the following result [5, 5.1]: *The convergence domain of a matrix* A *is an* FK-*space with seminorms*

$$\sup_m \left| \sum_{k=0}^{m} a_{nk} x_k \right| \quad (n = 0, 1, 2, \cdots), \qquad |x_n| \quad (n = 0, 1, 2, \cdots), \qquad \textit{and} \ \sup_n \left| \sum_k a_{nk} x_k \right|.$$

By a triangle, we mean a matrix A with $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for each n.

REMARK. *Suppose that* A *can be made into a triangle by striking out certain of its rows. Then in the preceding result we can omit the second set of seminorms.*

*Proof.* Let numbers $m_n$ be chosen so that the matrix $B = (b_{nk}) = (a_{m_n k})$ is a triangle. Then $c_B$ is an FK-space with norm $\sup_n |\Sigma_k b_{nk} x_k|$. Since each $|x_n|$ is thus continuous in this topology, it is *a fortiori* continuous in the topology generated by the larger norm $\sup_n |\Sigma_k a_{nk} x_k|$.

## 4. FK-SPACES

In the remainder of this article we deal exclusively with FK-spaces. An FK-space which has $\{\delta^k\}$ as basis is said to have the AK-property, while if $\{\delta^k\}$ is fundamental, the space is said to have the AD-property. For a matrix A, we say that A has the AK- or AD-property if $c_A^0$ has the property. These concepts were introduced in [6]. A regular matrix A is *perfect*, that is, c is dense in $c_A$, if and only if it has the AD-property. We say that $\{a_n\}$ is a sequence of *convergence factors* for a set E of sequences if $\Sigma a_n x_n$ is convergent for all $x \in E$.

THEOREM 6. *Let* $\{E^n\}$ *be a decreasing sequence of* FK-*spaces, each of which has the* AK-*property; let* $E = \bigcap E^n$; *and let* $\{a_n\}$ *be a sequence of convergence factors for* E. *Then* $\{a_n\}$ *is a set of convergence factors for* $E^N$, *for some* N.

*Proof.* The function f given by $f(x) = \Sigma a_k x_k$ for $x \in E$ is continuous, by the usual convergence principle of functional analysis. Let N be as in Theorem 3, and let F be the unique extension of f to $E^N$. Note that E is dense in $E^N$. For $x \in E^N$, $x = \Sigma x_k \delta^k$, thus

$$F(x) = \sum x_k F(\delta^k) = \sum x_k f(\delta^k) = \sum x_k a_k .$$

The following result is known; parts of it are proved here for completeness. By a *decreasing sequence of matrices* we mean a sequence of matrices whose convergence domains form a decreasing sequence of sets.

THEOREM 7. *Let* $\{A^n\}$ *be a decreasing sequence of regular matrices.*

(i) *If each* $A^n$ *has the* AK-property, *there exists no matrix* A *with* $c_A = \bigcap c_{A^n}$.

(ii) *If each* $A^n$ *is perfect, there exists no row-finite matrix* A *with* $c_A = \bigcap c_{A^n}$. *However*

(iii) *There exists a decreasing sequence* $\{A^n\}$ *of regular matrices with* $\bigcap c_{A^n} = c = c_I$, I *being the identity matrix.*

For (iii), see [8, p. 5].

To prove (i), assume on the contrary the existence of such a matrix A as a map of $c_A$ into c. By the Remark on Theorem 3, the map is continuous with the topology of $A^N$, and it is defined on a dense subset of $c_{A^N}$, hence can be extended to all of $c_{A^N}$. Apply Theorem 6 to each row, to see that the extension is still given by the matrix A. This contradicts the choice of A.

To prove (ii): as in (i), extend A to, say F, defined on $c_{A^N}$. Then $F = \{F_n\}$, where $F_n$, a functional, is the $n^{th}$ coordinate of F. Let $x \in c_{A^N}$. Then $x = \lim y^n$, where each $y^n$ is in $c_A$ and the limit is in the $A^N$-topology. Then

$$F_n(x) = \lim_k F_n(y^k) = \lim_k A_n(y^k) = A_n(y),$$

since A is row-finite. Hence F is given by A, again a contradiction.

A crucial point in the proof of (ii) is the fact that a function of the form $\Sigma_{n=0}^m a_n x_n$ has the same form when extended. As part of Theorem 8 we show that this is false if $m = \infty$, the extension being the Cesàro or $(C, 1)$-$\lim \Sigma a_n x_n$.

We now complete these results by showing that "row-finite" cannot be omitted in (ii).

THEOREM 8. *There exists a decreasing sequence* $\{A^n\}$ *of regular perfect matrices and a regular matrix* H *with* $c_H = \bigcap c_{A^n}$.

Let

$$Z = \begin{pmatrix} 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & \cdots \\ 0 & 0 & 1/2 & 1/2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is well known that Z is perfect. Indeed, it is of type M, that is,

$$\sum_n |t_n| < \infty \quad \text{and} \quad \sum_n t_n a_{nk} = 0 \quad (k = 0, 1, 2, \cdots)$$

together imply that $t_n = 0$ for all n; see [2, Theorem 3.2.1 (d)]. But Z does not have AK. (Easiest proof: Suppose it did. Then for every $x \in c_Z^0$,

$$0 = \lim_n z_{nn} x_n = \lim 1/2\, x_n$$

by [4, page 263, Section 5]. But Z sums $\{(-1)^n\}$ .)

However Z has a property which lies between perfectness and the AK-property.

LEMMA 1. $\{\delta^k\}$ *is a* $(C, 1)$ *basis for* $c_Z^0$; *in other words, for each* $x \in c_Z^0$, *let* $y^n = \Sigma_{k=0}^n x_k \delta^k$; *then* $\{y^n\}$ *is* $(C, 1)$ *summable to* x, *that is, for each* $x \in c_Z^0$ *we have*

$$x = \lim_n \frac{1}{n+1} \sum_{k=0}^n y^k.$$

We compute: let $u^m = \frac{1}{m+1} \Sigma_{k=0}^m y^k$. Then

$$x - u^m = \left\{0, \frac{1}{m+1} x_1, \frac{2}{m+1} x_2, \cdots, \frac{m}{m+1} x_m, x_{m+1}, x_{m+2}, \cdots\right\},$$

$$\|x - u^m\| = \sup_n \left| \sum_{k=0}^{n} z_{nk}(x_k - u_k^m) \right| = \frac{1}{2} \sup_n |(x_{n-1} - u_{n-1}^m) + (x_n - u_n^m)|.$$

(Here and elsewhere, $x_{-1} = 0$.)

For $n = 0$, the expression following $\sup_n$ is 0. For $1 \leq n \leq m + 1$, it is

$$\left| \frac{n-1}{m+1} x_{n-1} + \frac{n}{m+1} x_n \right| = \left| \frac{n}{m+1}(x_{n-1} + x_n) - \frac{x_n}{m+1} \right|$$

$$\leq \begin{cases} \dfrac{\sqrt{m}}{m+1} \sup\limits_n |x_{n-1} + x_n| + \dfrac{1}{m+1} \max\limits_{n \leq \sqrt{m}} |x_n| & \text{if } n \leq \sqrt{m}, \\[2ex] \sup\limits_{n > \sqrt{m}} |x_{n-1} + x_n| + \dfrac{1}{m+1} \max\limits_{\sqrt{m} < n \leq m+1} |x_n| & \text{if } n > \sqrt{m}. \end{cases}$$

Call the last two expressions $a_m$ and $b_m$, respectively.

Meanwhile, for $n > m + 1$, the expression following $\sup_n$ is

$$|x_{n-1} + x_n| \leq \sup_{n > m+1} |x_{n-1} + x_n| \leq b_m.$$

Hence $\|x - u^m\| \leq \max(a_m, b_m)$, and we complete the proof by showing that $a_m \to 0$ and $b_m \to 0$.

Clearly $x_n = o(n)$, since $\{x_n/(n+1)\}$ is the $(C, 1)$ transform of

$$\{(-1)^n (x_{n-1} + x_n)\},$$

and our hypothesis is that the latter is a null sequence. Thus $a_m \to 0$ and $b_m \to 0$.

LEMMA 2. *Given* $\Sigma |b_n| < \infty$, *define the continuous linear functional* $f$ *on* $c_Z$ *by*

$$f(x) = \sum_{n=0}^{\infty} b_n(x_{n-1} + x_n),$$

*and let* $a_n = b_n + b_{n+1}$. *Then, for all* $x \in c_Z$, $f(x) = (C, 1)\text{-}\lim \Sigma a_k x_k$ *(the Cesàro limit).*

From the identity

$$\sum_{n=0}^{m} b_n(x_{n-1} + x_n) = \sum_{n=0}^{m-1} a_n x_n + b_m x_m$$

it follows that $f(x) = \Sigma a_n x_n$, at least for convergent $x$. At this stage, in the proof of Theorem 7, we were able to say that $f$ had the same form when extended. For $x \in c_Z^0$, $x = \lim (n+1)^{-1} \Sigma_{k=0}^{n} y^k$ (see Lemma 1). Hence

$$f(x) = \lim \frac{1}{n+1} \sum_{k=0}^{n} f(y^k) = \lim \frac{1}{n+1} \sum_{k=0}^{n} \sum_{r=0}^{k} a_r x_r$$

(since $y^k \in c$). For $x \in c_Z$ we apply this to $\{x_n - t\}$, where $t$ is the Z-limit of $x$.

LEMMA 3. *Let a regular matrix* A *be formed by placing a finite number of rows on top of* Z. *Then* A *is perfect.*

Since A is regular, it is sufficient to prove that $c^0$ is dense in $c_A^0$.

We shall prove this with just one row adjoined. The extension to any finite number will then be obvious. Let the adjoined row be $a_0, a_1, a_2, \cdots$. By the Remark in Section 3, $c_A$ is an FK-space with seminorms

$$p(x) = \sup_m \left|\sum_{k=0}^{m} a_k x_k\right|, \qquad q_n(x) = \max\left(\left|\frac{1}{2}x_{n-1}\right|, \left|\frac{1}{2}x_{n-1} + \frac{1}{2}x_n\right|\right) \quad (n = 0, 1, 2, \cdots),$$

$$r(x) = \max\left(\left|\sum_{k=0}^{\infty} a_k x_k\right|, \sup_n\left|\frac{1}{2}x_{n-1} + \frac{1}{2}x_n\right|\right).$$

Clearly $q_n(x) \leq |x_{n-1}|/2 + r(x)$, and therefore the seminorms $q_n$ can be omitted, the coordinate seminorms $|x_{n-1}|$ being already disposed of in the Remark. Next, it is clear that $r(x)$ can be replaced by $s(x) = \sup_n |x_{n-1} + x_n|/2$, without alteration of the topology of $c_A$, because $s \leq r \leq p + s$. Hence we consider $c_A$ with $p$ and $s$ as seminorms. Now, insofar as $s$ is concerned, we have already seen that given $x \in c_A^0$, $s(x - u^m) \to 0$, $u^m$ being the (C, 1) sums of the segments of $x$ (see Lemma 1). It remains to show that the same is true for the seminorm $p$. But in this case we have even more, namely $p(x - y^m) \to 0$ (see Lemma 1 for $y^m$); for

$$p(x - y^m) = \sup_r \left|\sum_{k=m+1}^{r} a_k x_k\right| \to 0 \qquad (m \to \infty),$$

since the series $\Sigma a_k x_k$ is convergent.

We are now ready to construct the sequence of Theorem 8. We first construct three sequences $\{a^n\}, \{b^n\}, \{c^n\}$ of sequences of nonnegative numbers satisfying, for $r = 1, 2, \cdots$, the following five conditions:

(1) $$\sum_n b_n^{\,r} < 1/r,$$

(2) $$a_n^{\,r} = b_n^{\,r} + b_{n+1}^{\,r},$$

(3) $$\sum_n a_n^{\,k} c_n^{\,r} < \infty \ \text{ for } k = 1, 2, 3, \cdots, r-1,$$

(4) $$\limsup a_n^{\,r} c_n^{\,r} \geq 1,$$

(5) $$c_n^{\,r} \uparrow \infty \ \text{ as } n \to \infty, \text{ and } c_n^{\,r} - c_{n-1}^{\,r} \to 0 \text{ as } n \to \infty.$$

(For example, for each $r = 1, 2, \cdots$, we may choose an increasing sequence $N(r)$ of integers greater than 2 such that

$$\sum_{n\in N(r)} 1/\log n < \infty \qquad \text{and} \qquad \sum_{n\in N(r)} n^{-1/(r+1)} < 1/r,$$

and we set $c_n^r = n^{1/(r+1)}$; $b_n^r = n^{-1/(r+1)}$ if $n \in N(r)$, otherwise $b_n^r = 0$.)

Let $A^r$ be the matrix Z with r new rows placed on top, these rows being, reading from the top down, $a^r, a^{r-1}, \cdots, a^2, a^1$. The sequence $\{A^n\}$ is strictly decreasing, for we have $c_{A^r} \subset c_{A^{r-1}}$, since $A^{r-1}$ is a submatrix of $A^r$. But also $c_{A^{r-1}} \neq c_{A^r}$ since, by (3) and (5), $\{(-1)^n c_n^r\}$ is in $c_{A^{r-1}}$, but, by (4), not in $c_{A^r}$.

By Lemma 3, each $A^n$ is perfect. To complete the proof of Theorem 8, it remains to construct the matrix H mentioned in its statement.

We first note that $x \in \bigcap c_{A^n}$ if and only if

(6) $$x \in c_Z$$

and

(7) $$\sum_k a_k^n x_k \quad \text{converges for each } n.$$

Let

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ a_1^1 & a_2^1 & a_3^1 & a_4^1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & \cdots \\ 0 & 0 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

that is, let D consist of rows of the form $a_j^i$ alternating with zero rows; let

$$E = \begin{pmatrix} 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(Z, with each row repeated); and let $H = D + E$. Then H is regular, since E is regular, and because of (1) and (2).

If $x \in c_H$, it clearly satisfies (6) and (7). Conversely, suppose x satisfies (6) and (7). To show that $x \in c_H$, it will be sufficient to prove that $\sum_k a_k^n x_k \to 0$ as $n \to \infty$. Now, by Lemma 2 (here we do not need the (C, 1)-limit, since the series involved converges),

$$\left|\sum_k a_k^n x_k\right| = \left|\sum b_k^n (x_k + x_{k-1})\right| \leq \sup \left|x_k + x_{k-1}\right| \cdot \sum_k \left|b_k^n\right| = O(1/n),$$

by (1) and (6).

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