THE POSITIVITY SETS OF THE SOLUTIONS OF A TRANSPORT EQUATION

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1. Let

$$Df(x, t) = 0, \; f(x, 0) = g(x) \geq 0 \qquad (x, t \geq 0) \tag{1}$$

be an initial-value problem whose operator D and initial function $g(x)$ are such that the following properties hold:

(i) there exists a unique continuous solution $f(x, t)$ valid for $x, t \geq 0$,

(ii) $f(x, t)$ is analytic in t for each x,

(iii) $f(x, t) \geq 0 \; (> 0)$ if $g(x) \geq 0 \; (> 0)$.

Let $P = \{x \mid g(x) > 0\}$ and $Z = \{x \mid g(x) = 0\}$, and define

$$Z_n = \left\{ x \,\middle|\, \frac{\partial^k f(x, 0)}{\partial t^k} = 0 \; (k = 0, 1, \cdots, n-1), \; \frac{\partial^n f(x, 0)}{\partial t^n} > 0 \right\} \qquad (n = 1, 2, \cdots),$$

$$Z_\omega = Z - \bigcup_1^\infty Z_n.$$

Z_n is called the n-th positivity set, and Z_ω is called the residual set; the totality of these gives some information about the behaviour of $f(x, t)$, especially for small t. For example, $f(x, t) > 0$ for $t > 0$ if and only if $x \in Z \cup P - Z_\omega$; $f(x, t) = 0$ for all t if and only if $x \in Z_\omega$; and over Z_n, $f(x, t) = O(t^n)$ for small t.

In this note there will be considered an example of a nonlinear integrodifferential operator D for which the sets Z_n can be completely described in terms of Z and P alone.

2. The equation

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{2}\int_0^x f(y, t) f(x-y, t)\phi(y, x-y)\,dy - f(x, t)\int_0^\infty f(y, t)\phi(x, y)\,dy \tag{2}$$

has been considered, as a special case, in [1]. It satisfies the above conditions (i), (ii), and (iii) under the following hypotheses:

(H_1) $f(x, 0)$ is a continuous, nonnegative, integrable and uniformly bounded function for $x \geq 0$, and

(H_2) $\phi(x, y) = \phi(y, x)$ is a continuous, nonnegative and uniformly bounded function for $x, y \geq 0$.

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Let $f(x, 0) \leq B$, $\phi(x, y) \leq A$, $\int_0^\infty f(x, 0)\,dx = N$ and $m = 3AN/2$. Then, for $t < 1/m$, the following estimates are valid, by Section 3 of [1]:

$$\left|\frac{\partial^n f(x, t)}{\partial t^n}\right| \leq \frac{B\, n!\, m^n}{(1 - mt)^{n+1}}, \quad \int_0^\infty \left|\frac{\partial^n f(x, t)}{\partial t^n}\right| dx \leq \frac{N\, n!\, m^n}{(1 - mt)^{n+1}}. \tag{3}$$

These will be needed later on.

Let X and Y be arbitrary sets of nonnegative real numbers. Then the vector sum $X + Y$ is defined by

$$X + Y = \{z \mid z = x + y \ (x \in X,\ y \in Y)\},$$

the operation of vector addition of sets is therefore commutative and associative. Define now $X^1 = X$, $X^{n+1} = X^n + X$, and let Z and P be as before. The theorem of this note can now be stated as follows.

THEOREM 1. *Let* $\phi(x, y) > 0$. *Then the positivity sets of the solutions* $f(x, t)$ *of equation* (2) *are given by*

$$Z_n = Z \cap \left(P^{n+1} - \bigcup_1^n P^i\right).$$

3. The proof of Theorem 1 is based on a lemma which describes the form of the n-th derivative $\partial^n f(x, t)/\partial t^n$. Before the statement of the lemma, some special notation will be introduced. Let

$$f * f = \frac{1}{2}\int_0^x f(y, t) f(x - y, t)\phi(x - y, y)\,dy, \qquad \bar{f} = \int_0^\infty f(y, t)\phi(x, y)\,dy.$$

Then (2) may be written as

$$\frac{\partial f}{\partial t} = f * f - f\bar{f}.$$

The operation * has the usual properties of a convolution, and expressions like $(f*)^n$ and $\Pi_1^n*(f\, g_i)$ can be formed in the usual manner, the g_i being continuous and integrable functions of x and t. If in addition $g_i \neq 0$, then the hypothesis $\phi(x, y) > 0$ of Theorem 1 implies the equivalence of the following three statements:

$$(f*)^n = 0, \qquad x \notin P^n, \qquad \prod_1^n {}^*(f\, g_i) = 0. \tag{4}$$

The first two of these are always equivalent.

LEMMA 1. *With the notation above, the* k*-th derivative is of the form*

$$\frac{\partial^k f}{\partial t^k} = a_k (f*)^{k+1} + \sum_{j=2}^k S_{kj} + f\, G_k, \tag{5}$$

where

$$a_k = k!, \qquad S_{kj} = h_{kj} \prod_{i=1}^{j}{}^{*} (f\, g_{kji}) + \text{a sum of a finite number of similar terms,}$$

and g_{kji}, h_{kj} *and* G_k *are functions of* x *and* t.

For $k = 1$, (5) is the equation (2) itself. Call $(f^*)^{k+1}$, S_{kj} and fG_k terms of type A_k, B_k and C_k, respectively. Assume that (5) holds for $k = 1, 2, \cdots, n$. Differentiating (5) with respect to t, for $k = n$, one obtains formally

$$\frac{\partial^{n+1} f}{\partial t^{n+1}} = a_n \frac{\partial (f^*)^{n+1}}{\partial t} + \sum_{j=2}^{n} \frac{\partial S_{nj}}{\partial t} + \frac{\partial f}{\partial t} G_n + f \frac{\partial G_n}{\partial t}. \tag{6}$$

By an induction on k and by the estimates (3), this result may be justified: all terms are defined, and the necessary conditions of differentiability and integrability hold. The convolutions are differentiated by the ordinary Leibniz product rule. Substituting $f^* f - f \bar{f}$ for $\partial f/\partial t$, where necessary, one finds that the last two terms on the right are of type C_{n+1} and B_{n+1}. The second term is a sum of a finite number of expressions of type

$$\frac{\partial h_{nj}}{\partial t} \prod_{k=1}^{j}{}^{*} (f\, g_{njk}) \qquad \text{or} \qquad h_{nj} \frac{\partial (f\, g_{nj1})}{\partial t} \prod_{k=2}^{j}{}^{*} (f\, g_{njk}),$$

both of which are of type B_{n+1}. Finally, the first term may be written as

$$a_n \frac{\partial (f^*)^{n+1}}{\partial t} = (n+1)\, a_n (f^*)^{n+2} + \text{type } B_{n+1} \text{ term.}$$

This shows that $a_n = n!$ and that (5) is valid for all k.

4. The proof of Theorem 1 now follows at once. Let $x \in Z_n$; then $x \in Z$, so that $\partial f(x, 0)/\partial t = f * f$. Since $x \notin P^2$, it follows from (4) that $\partial f(x, 0)/\partial t = 0$. Similarly it follows from (4) and (5) that

$$\frac{\partial^k f(x, 0)}{\partial t^k} = 0 \qquad (k = 1, 2, \cdots, n-1).$$

But $x \in P^{n+1}$, and therefore

$$\frac{\partial^n f(x, 0)}{\partial t^n} = n!\, (f^*)^{n+1} > 0.$$

In exactly the same way one shows that if $\partial^k f(x, 0)/\partial t^k = 0$ for $k = 0, 1, \cdots, n-1$ but $\partial^n f(x, 0)/\partial t^n > 0$, then $x \in Z_n$.

The following corollary of Theorem 1 and the conditions (i), (ii) and (iii) is obvious: $f(x, t) > 0$ for $t > 0$ and for all x if and only if $Z \subset \bigcup_2^\infty P^n$.

It appears to be an interesting problem to find other equations obeying the conditions (i), (ii) and (iii) and to determine their positivity sets.

REFERENCE

1. Z. A. Melzak, *A scalar transport equation,* Trans. Amer. Math. Soc. 85 (1957), 547-560.

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