ON SINGULAR FIBERINGS BY SPHERES

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The purpose of this note is to discuss the properties of certain types of singular fibrations as introduced by Montgomery and Samelson [9]. We repeat their definition: A fiber space with singularities is a quadruple $[(X, A), (Y, B), \pi, F]$ such that

1. $\pi: (X, A) \to (Y, B)$ is a proper open onto mapping,

2. $\pi^{-1}(B) = A$ and $\pi|A$ is a homeomorphism, and

3. $\pi|X - A: X - A \to Y - B$ is a fiber mapping with fiber F.

We recall that a mapping is proper if the inverse image of every compact set is compact. This is not the only possible definition of singular fibration (see for example [10]), but it is a first step in studies related to the problems of compact transformation groups. Generally speaking, if we are given a singular fibration $[(X, A), (Y, B), \pi, F]$, the question is: what may one conclude about spaces A and Y from knowledge of X and F? As our title indicates, we are mainly concerned with the case where F is a sphere (cohomology sphere), and we obtain results closely resembling those of Smith for the stationary point set of an involution [11]. Our principal tool is of course the Gysin sequence; however, the use we shall make of it appears to be new.

This note will consider only arcwise connected, arcwise semi-locally 1-connected spaces. Although our theorems are actually valid for a fiber which is only a cohomology sphere, we shall only consider topological spheres, for convenience. We shall deal only with the Čech cohomology groups with coefficients in Z_2 (the integers mod 2), and we shall not denote the coefficient group explicitly. We shall denote the cohomology with compact supports by a subscript c [2]. In particular, we shall use $H^i_c(X - A)$ for the relative cohomology groups of the compact pair (X, A).

1. ACYCLICITY AND LOCAL CONNECTIVITY

THEOREM 1.1. *If* $[(X, A), (Y, B), \pi, S^r]$ *is a singular fibering by an* r*-sphere* $(r > 0)$ *and* X *is a compact space such that for some integer* m, $H^i(X) = 0$ *for* $i \geq m$, *then* $H^i(A) = 0$ *for* $i \geq m$, *and* $H^i(Y) \cong H^i(B)$ *for* $i \geq m - r$.

We first consider the natural diagram

$$\begin{array}{ccccccc} \to H^i(A) & \xrightarrow{\delta^*} & H_c^{i+1}(X - A) & \to & H^{i+1}(X) & \to \\ \uparrow \pi_1^* & & \uparrow \pi^* & & \uparrow & \\ \to H^i(B) & \to & H_c^{i+1}(Y - B) & \to & H^{i+1}(Y) & \to . \end{array}$$

Since $H^i(X) = 0$ $(i \geq m)$, the homomorphism δ^* is onto for $i \geq m - 1$, but π_1^* is an isomorphism; thus, π^* is onto for $i \geq m - 1$. Now we shall employ the Gysin

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sequence associated with $\pi|X - A: X - A \to Y - B$. Since we only consider coefficients in Z_2, the Gysin sequence has the form

$$\to H_c^i(Y - B) \xrightarrow{\phi} H_c^{i+r+1}(Y - B) \xrightarrow{\pi^*} H_c^{i+r+1}(X - A) \xrightarrow{\delta^*} H_c^{i+1}(Y - B) \to ,$$

where the homomorphism ϕ may be interpreted as multiplication by a cohomology class $\phi \in H^{r+1}(Y - B)$ [2]. Since $\pi^*: H_c^i(Y - B) \to H_c^i(X - A)$ is onto for $i \geq m$, it follows that $\phi: H_c^i(Y - B) \to H_c^{i+r+1}(Y - B)$ is an isomorphism into for $i \geq m - r$. It is known [3] that if X is locally compact, $h \in H_c^i(X)$ and $e \in H^j(X)$ $(j \geq 1)$, then for some integer n, $h \cup (e)^n = 0$. In our present situation, if $H_c^i(Y - B) \neq 0$ for some $i \geq m - r$, then there is an $h \in H_c^i(Y - B)$ such that $h \cup (\phi)^n \neq 0$ for any $n \geq 0$. Thus, $H_c^i(Y - B) = 0$ for $i \geq m - r$. This implies that $H_c^i(X - A) = 0$ for $i \geq m$. We may use the exact sequence of the pair (X, A) to conclude that $H^i(A) \cong H^i(X) = 0$ for $i \geq m$. Furthermore, from the exact sequence of (Y, B), $H^i(Y) \cong H^i(B)$ for $i \geq m - r$.

COROLLARY 1.1. *If* $[(X, A), (Y, B), \pi, S^r]$ *is a singular fibering of a compact space which is acyclic* mod 2, *then both* A *and* Y *are acyclic* mod 2.

We made no assumptions about the finite dimensionality of X for Theorem 1.1. We shall now give a local form of the preceeding argument. Dually to homology local connectedness, a space X is n-clc at $x \in X$ if for each closed neighborhood U of x there is a closed neighborhood V of x $(V \subset U)$ such that the homomorphism $i^*: \tilde{H}^n(U) \to \tilde{H}^n(V)$ induced by inclusion is trivial, where $\tilde{H}$ denotes reduced cohomology. The space X is clc^n at $x \in X$ if it is m-clc at x for all $m \leq n$. Finally, a space is n-clc or clc^n if it has that property at each of its points; and it is clc if it is clc^n for all n.

THEOREM 1.2. *Let* $[(X, A), (Y, B), \pi, S^r]$ $(r > 0)$ *be a singular fibering of a compact space* X. *Suppose that the cohomology dimension* mod 2 *of* X *is finite and that* X *is in* clc. *Then both* A *and* Y *are* clc.

The cohomology dimension mod 2 of X is max $\{i | H_c^i(U) \neq 0$ for some open set U in $X\}$. It is known [1] that under this hypothesis both A and Y have finite cohomology dimension mod 2.

Let $x \in A$, and choose closed neighborhoods $U \supset V \supset W$ of x such that for every integer i, the induced homomorphisms $\tilde{H}^i(U) \to \tilde{H}^i(V)$ and $\tilde{H}^i(V) \to \tilde{H}^i(W)$ are trivial. Notice here that the finite dimensionality mod 2 of X is used. Let m be the least integer for which sets $U \supset V \supset W$ can be chosen such that $\tilde{H}^i(U) \to \tilde{H}^i(V)$ and $\tilde{H}^i(V) \to \tilde{H}^i(W)$ are trivial for $i \geq m$. Such an m exists, by finite dimensionality; but then $m = 0$. Let $\hat{U} = \pi(U)$, $A_U = A \cap U$, $\hat{A}_U = \pi(A_U) = B \cap \hat{U}$, and so forth. Then we have the diagram

$$\begin{array}{ccccc} \tilde{H}^i(A_V) & \xrightarrow{\delta^*} & H^{i+1}(V - A_V) & \to & \tilde{H}^{i+1}(V) \\ & & \uparrow i_0^* & & \uparrow i^* \\ & & H_c^{i+1}(U - A\ ) & \to & \tilde{H}^{i+1}(U). \end{array}$$

Since i^* is trivial, im $i_0^* \subset$ im δ^*. In the diagram

$$\begin{array}{ccc} \tilde{H}^i(A_V) & \xrightarrow{\delta^*} & H_c^{i+1}(V - A_V) \\ \uparrow \pi^* & & \uparrow \pi_0^* \\ \tilde{H}^i(\hat{A}_V) & \rightarrow & H_c^{i+1}(\hat{V} - \hat{A}_V) \end{array}$$

π^* is an isomorphism; hence, im $\delta^* \subset$ im π_0^*. Thus, im $i_0^* \subset$ im π_0^*, a fact we shall use repeatedly. A similar statement holds for the pair (V, W).

Let y denote a point of B, and let $x = \pi^{-1}(y)$. Since the cohomology dimension mod 2 of Y is finite, there is an integer k such that for each closed neighborhood N of y, there is a closed neighborhood N' of y such that $H_c^i(N - N_B) \rightarrow H_c^i(N' - N'_B)$ is trivial for all $i \geq k$. Let $k_0 + 1$ denote the least such integer; we shall show that $k_0 = -1$.

Let N be a closed neighborhood of y. Let U be a closed neighborhood of x such that $\pi(U) \subset N$ and $U = \pi^{-1}\pi(U)$. There is a closed neighborhood V_1 of x such that $\tilde{H}^i(U) \rightarrow \tilde{H}^i(V_1)$ is trivial for all i, and there is a closed neighborhood N_1 of x such that $H_c^i(\hat{U} - \hat{U}_A) \rightarrow H_c^i(\hat{N}_1 - \hat{N}_{1_A})$ is trivial for all $i \geq k_0 + 1$. Let V denote a closed neighborhood of x lying in $V_1 \cap N_1$ such that $V = \pi^{-1}\pi V$, and let $W \subset V$ be defined in a similar way.

Consider the natural diagram of Gysin sequences

$$\begin{array}{ccccccc} & & & & H_c^i(N - N_B) & & \\ & & & & \downarrow & & \\ H_c^{i+r}(\hat{U} - \hat{U}_A) & \rightarrow & H_c^{i+r}(U - U_A) & \rightarrow & H_c^i(\hat{U} - \hat{U}_A) & \xrightarrow{\phi_1^*} & H_c^{i+r+1}(\hat{U} - \hat{U}_A) \\ \downarrow & & \downarrow i_0^* & & \downarrow i_2^* & & \downarrow i_4^* \\ H_c^{i+r}(\hat{V} - \hat{V}_A) & \xrightarrow{\pi_0^*} & H_c^{i+r}(V - V_A) & \xrightarrow{\delta_1^*} & H_c^i(\hat{V} - \hat{V}_A) & \xrightarrow{\phi_2^*} & H_c^{i+r+1}(\hat{V} - \hat{V}_A) \\ \downarrow & & \downarrow i_1^* & & \downarrow i_3^* & & \downarrow \\ H_c^{i+r}(\hat{W} - \hat{W}_A) & \xrightarrow{\pi_1^*} & H_c^{i+r}(W - W_A) & \xrightarrow{\delta_2^*} & H_c^i(\hat{W} - \hat{W}_A) & \rightarrow & H_c^{i+r+1}(\hat{W} - \hat{W}_A) \end{array}$$

in which i_4^* is trivial, im $\pi_0^* \supset$ im i_0^*, and im $\pi_1^* \supset$ im i_1^*. We shall show that $\text{im}(i_3^* i_2^*) = 0$. Suppose $i_3^* i_2^*(h) \neq 0$. Since i_4^* is trivial, $\phi_2^*(i_2^*(h)) = 0$; therefore, there is an element $k \in H_c^{i+r}(V - V_A)$ such that $\delta_1^*(k) = i_2^*(h)$. Since im $\pi_1^* \supset$ im i_1^*, there is an element $\ell \in H_c^{i+r}(\hat{W} - \hat{W}_A)$ such that $\pi_1^*(\ell) = i_1^*(k)$. Thus,

$$0 = \delta_2^* \pi_1^*(\ell) = \delta_2^* i_1^*(k) = i_3^* \delta_1^*(k) = i_3^* i_2^*(h).$$

Hence, $k_0 = -1$.

Consequently, for a given closed neighborhood U of x, we can choose closed neighborhoods V and W of x $(W \subset V \subset U)$ such that in the above diagram both i_2^* and i_3^* are trivial for all i. We next show that this implies the triviality of $i_1^* i_0^*$. This follows immediately from the triviality of i_3^* (for $i + r$) and the fact that im $i_0^* \subset$ im π_0^*.

Thus, if U is a closed neighborhood of $x \in A$, there are closed neighborhoods V and W of x ($V \subset W \subset U$) such that all of the homomorphisms $\tilde{H}^i(U) \to \tilde{H}^i(V)$, $\tilde{H}^i(V) \to \tilde{H}^i(W)$, $H_c^i(U - U_A) \to H_c^i(V - V_A)$, and $H_c^i(V - V_A) \to H_c^i(W - W_A)$ are trivial. We have the natural diagram

$$\begin{array}{ccccc} & & \tilde{H}^i(U_A) & \to & H_c^{i+1}(U - U_A) \\ & & \downarrow & & \downarrow \\ \tilde{H}^i(V) & \to & \tilde{H}^i(V_A) & \to & H_c^{i+1}(V - V_A) \\ \downarrow & & \downarrow & & \\ \tilde{H}^i(W) & \to & \tilde{H}^i(W_A) & & \end{array}$$

in which the vertical homomorphisms to the left and right are trivial and the middle row is exact. Hence, the composition of the homomorphisms in the middle column is trivial. This shows that A is clc mod 2. Since B is also clc mod 2, an argument just like the immediately preceeding one shows that Y is clc mod 2.

Theorem 1.2 is a localization of Corollary 1.1. It shows that the singular set in a singular fibering by a sphere is not perfectly arbitrary, but must inherit properties from the total space X. We shall find more of these inherited properties.

We point out here that Z_2 is used as the coefficient group, so that the coefficients in the Gysin sequence will always be ordinary. There is another interesting case to which our methods apply. Let (S^1, M^n) denote the action of a circle group as a group of transformations on a compact n-manifold M^n. Let $A \subset M^n$ be the set of stationary points in M^n. Floyd showed [7] that in this case there are only a finite number of distinct isotropy groups. Let $G \subset S^1$ denote the cyclic subgroup generated by the isotropy groups. If M^n/G denotes the quotient space of G acting on M^n, then $S^1/G = S^1$ acts on M^n/G and gives a singular fibering in the sense of Montgomery and Samelson. Furthermore, this is a principal singular fibering; therefore by [8], the Gysin sequences may be used with integral coefficients. Since M^n/G is an ANR [6], we may use Theorem 1.2 to show

THEOREM (Floyd). *If* (S^1, M^n) *denotes the action of a circle group on a compact manifold, then both the stationary point set and the orbit space are* clc *over the integers.*

This compresses into one step some results proved in [4] and [7].

2. LOCAL AND GLOBAL BETTI GROUPS

THEOREM 2.1. *If* $[(X, A), (Y, B), \pi, S^r]$ *is a singular fibering and* X *is a* mod 2 *cohomology* n-*sphere, then there is an integer* k *such that* A *is a* mod 2 *cohomology* $(n - k\cdot(r + 1))$-*sphere.*

We again consider the natural diagram

$$\begin{array}{ccccccccc} \to & \tilde{H}^{i-1}(X) & \to & \tilde{H}^{i-1}(A) & \to & H_c^i(X - A) & \to & \tilde{H}^i(X) & \to \\ & \uparrow & & \uparrow \cong & & \uparrow \pi^* & & \uparrow & \\ \to & \tilde{H}^{i-1}(Y) & \to & \tilde{H}^{i-1}(B) & \to & H_c^i(Y - B) & \to & \tilde{H}^i(Y) & \to \end{array}$$

If $i \neq n$, $\pi^*: H_c^i(Y - B) \to H_c^i(X - A)$ is onto. The Gysin sequence

$$\to H_c^i(Y - B) \xrightarrow{\phi^*} H_c^{i+r+1}(Y - B) \xrightarrow{\pi^*} H_c^{i+r+1}(X - A) \to \cdots$$

shows that if $i + r \neq n$, then $\phi^*: H_c^i(Y - B) \to H_c^{i+r+1}(Y - B)$ is an isomorphism into.

Suppose i is of the form $n - r - k(r + 1) + j$, where $1 \leq j \leq r$. If

$$(n - r - k(r + 1) + j + k'(r + 1)) + r = n,$$

for some k', then $j = (k - k')(r + 1)$, which is not possible. Thus, if $0 \neq h \in H_c^i(Y - B)$, then $h \cup (\phi)^{k'} \neq 0$ for all k', but this is not possible as we have seen before. Hence, $H_c^i(Y - B) = 0$ if $i = n - r - k(r + 1) + j$ and $1 \leq j \leq r$. Since π^* is onto for $i \neq n$, $H_c^i(X - A) = 0$ for the same i except possibly for $k = 0$ and $j = r$; that is, except possibly for $H_c^n(X - A)$. The relative sequence of the pair (X, A) then implies that $\tilde{H}^i(A) = 0$ for $i = n - k(r + 1) + j$ $(1 \leq j \leq r)$, except possibly for $H^{n-1}(A)$. Since $H_c^n(Y - B) = 0$, $\delta^*: \tilde{H}^{n-1}(A) \to H_c^n(X - A)$ is trivial, by the above diagram. Then $\tilde{H}^{n-1}(X) = 0$ implies that $\tilde{H}^{n-1}(A) = 0$. Thus we have the exact sequence

$$0 \to H_c^n(X - A) \to \tilde{H}^n(X) \to \tilde{H}^n(A) \to 0.$$

Since $H_c^n(Y - B) \cong H_c^{n+1}(Y - B) = 0$, it follows from the Gysin sequence that

$$H_c^n(X - A) \cong H_c^{n-r}(Y - B).$$

Since π^* is onto for $i \neq n$ and ϕ^* is an isomorphism into for $i + r \neq n$, we have the exact sequences

$$0 \to H_c^{n-r-(r+1)}(Y - B) \xrightarrow{\phi^*} H_c^{n-r}(Y - B) \xrightarrow{\pi^*} H_c^{n-r}(X - A) \to 0,$$

$$0 \to H_c^{n-r-2\cdot(r+1)}(Y - B) \xrightarrow{\phi^*} H_c^{n-r-(r+1)}(Y - B) \xrightarrow{\pi^*} H_c^{n-r-(r+1)}(X - A) \to 0,$$

and so forth. Since the vector space $H^n(X)$ is finite-dimensional, we see inductively that each of the vector spaces $\tilde{H}^n(A)$, $H_c^n(X - A)$, $H_c^{n-r}(Y - B)$, $H_c^{n-r-k(r+1)}(X - A)$, and $H_c^{n-r-k(r+1)}(Y - B)$ is finite-dimensional. Hence, we have the following equations:

$$\begin{aligned}\sum_{k=0}^{\infty} \dim H_c^{n-r-k(r+1)}(X - A) &= \dim H_c^{n-r}(Y - B)\\ &= \dim H_c^n(X - A)\\ &= \dim \tilde{H}^n(X) - \dim \tilde{H}^n(A).\end{aligned}$$

But $\tilde{H}^{n-r-k(r+1)-1}(A) \cong H_c^{n-r-k(r+1)}(X - A)$; therefore

$$\sum_{k=0}^{\infty} \dim \tilde{H}^{n-k(r+1)}(A) = \dim \tilde{H}^n(X) = 1.$$

This completes the proof.

The localization of this theorem is tied to Wilder's notion of generalized manifolds [12]. We shall give the necessary definitions here. If (U, V) is a pair of open sets in X, the pair is called *stable* if $V \subset U$ and if for each open set $W \subset V$,

$$\mathrm{im}\,(H^*(X, X - W) \to H^*(X, X - U)) = \mathrm{im}\,(H^*(X, X - V) \to H^*(X, X - U)).$$

We say that X has *property* $P_{n(x)}$ *at a point* $x \in X$ provided for each open neighborhood W of x there is a stable pair (U, V) $(x \in V \subset U \subset W)$ such that

$$\mathrm{im}\,(H^i(X, X - V) \to H^i(X, X - U)) = 0$$

for $i \neq n(x)$ and $= Z_2$ for $i = n(x)$. If X is connected and has property $P_{n(x)}$ at each of its points, then $n(x) = n(y)$, and we say simply that X has *property* P_n. If the locally compact, paracompact, connected Hausdorff space X has finite mod 2 cohomology dimension, then we say that X is a *locally orientable generalized* n-*manifold* provided

1. X has property P_n,

2. X is clc^n.

It is known that the mod 2 cohomology dimension of X is n [7]; hence, X is clc. Furthermore, Poincaré duality holds for mod 2 locally orientable generalized n-manifolds [12]; therefore $H^n_c(X) \cong Z_2$. P. A. Smith has defined a property Q quite closely related to the property P_n above: A locally compact Hausdorff space X has *property* Q if for each point $x \in X$ and each open neighborhood U_x of x there is an open neighborhood V_x of x $(V_x \subset U_x)$ such that, if W is an open subset of V_x and $y \in W$, then there is an open neighborhood V_y of y $(V_y \subset W)$ such that

$$H^i_c(V_x - V_y) \to H^i_c(U_x - W) \text{ is trivial for all } i.$$

C. T. Yang has shown that (for a fixed field F of coefficients) property P_n implies property Q; and P. E. Conner and E. E. Floyd have shown [5] that is a locally compact, paracompact, connected Hausdorff space of finite cohomology dimension mod F has property Q, then it is a locally orientable generalized n-manifold.

We say that a space X is *strongly paracompact* provided each open subset of X is paracompact. We shall use a particular construction which contains a proof of a special case of Yang's theorem.

LEMMA 2.2. *Let* X *denote a locally compact, locally connected, strongly paracompact Hausdorff space having property* P_n. *Let* U_x *and* V_x *denote connected open neighborhoods of a point* $x \in X$ $(V_x \subset U_x)$ *having compact closures and such that* (U_x, V_x) *is a stable pair,* $H^i(X, X - V_x) \to H^i(X, X - U_x)$ *is trivial for* $i \neq n$ *and is nontrivial for* $i = n$. *Let* W *denote an open subset of* V_x, y *a point of* W, *and* V_y *a connected open neighborhood of* y *lying in* W *such that* $H^i(X, X - V_y) \to H^i(X, X - W)$ *is trivial for* $i \neq n$. *Then* $H^i_c(V_x - V_y) \to H^i(U_x - W)$ *is trivial for all* i.

By the hypothesis, each of the sets U_x and V_y is a connected locally orientable generalized n-manifold; therefore $H^n_c(V_y) \cong H^n_c(U_y) \cong Z_2$. Hence, since (U_x, V_x) is stable, $H^n_c(V_y) \to H^n_c(U_x)$ is an isomorphism. Let $A_1 = U_x - W$, $A_2 = V_x - W$ and $A_3 = V_x - V_y$. Consider the diagram

$$\begin{array}{l} H^n_c(U_x, A_1) \to H^n_c(U_x) \to H^n_c(A_1) \to 0 \\ \quad\uparrow \cong \\ H^n_c(W) \\ \quad\uparrow \\ H^n_c(V_y) \end{array}$$

Since $H^n_c(V_y) \to H^n_c(U_x)$ is onto, $H^n_c(A_1) = 0$. Thus, the conclusion of the theorem holds for $i = n$. Now consider the diagram

$$\begin{array}{ccccccc} H^i_c(U_x) & \to & H^i_c(A_1) & \to & H^{i+1}_c(W) & \to & H^{i+1}_c(U_x) \\ \uparrow k_1^* & & \uparrow j_1^* & & \uparrow & & \\ H^i_c(V_x) & \to & H^i_c(A_2) & \xrightarrow{\delta_2^*} & H^{i+1}_c(W) & & \\ \uparrow & & \uparrow j_2^* & & \uparrow k_2^* & & \\ H^i_c(V_x) & \xrightarrow{k_3^*} & H^i_c(A_3) & \to & H^{i+1}_c(V_y) & & \end{array}$$

For i and $i + 1$ different from n, since k_2^* is trivial,

$$\operatorname{im}(H^i_c(A_3) \to H^i_c(A_2)) \subset \operatorname{im}(H^i_c(V_x) \to H^i_c(A_2)) = \ker \delta_2^*.$$

But k_1^* is trivial; therefore, $j_1^* j_2^*$ is trivial. For $i + 1 = n$, $H^n_c(V_y)$ maps isomorphically onto $H^n_c(U_x)$. Thus, k_3^* is onto, and again by the triviality of k_1^*, $j_1^* j_2^*$ is trivial.

THEOREM 2.3. *If* $[(X, A), (Y, B), \pi, S^r]$ *is a singular fibering* $(r > 0)$, *and* X *is a compact, strongly paracompact, locally orientable generalized* n-*manifold, then for each component of* A *there is an integer* k *such that the component is a compact, strongly paracompact, locally orientable generalized* $(n - k \cdot (r + 1))$-*manifold.*

We have previously shown that A is clc. We shall show that A has property Q. The conclusion of the theorem will then follow from a certain observation and the theorem of Conner and Floyd referred to above.

Let Φ denote the class of those open subsets U of X such that $U = \pi^{-1}\pi U$. It is clear that if V is a subset of A which is open relative to A, there is an element U of Φ such that $V = A_U$; also, if V is connected, it is possible to find a connected U in Φ such that $V = A_U$. It is also clear that the set of elements of Φ containing a point $x \in A$ is cofinal in the set of all open neighborhoods of x. Hence, there is no difficulty in assuming throughout that all open subsets of X mentioned belong to Φ; this will be done without further mention.

Let $e = 1 + [n + 1/r + 1]$, and let $d = 1 + 2e$. Let x denote a point of A, and U_x an open neighborhood of x (in X). Let $U_1, \cdots, U_d$ denote a sequence of open neighborhoods of x such that

(i) $U_1 \subset U_x$, $U_{i+1} \subset U_i$, and each U_i is connected,

(ii) each of the pairs (U_i, U_{i+1}) is stable, and

(iii) $\operatorname{im}(H^j(X, X - U_{i+1}) \to H^j(X, X - U_i)) = \begin{cases} 0 & \text{if } j \neq n \\ Z_2 & \text{if } j = n. \end{cases}$

Let W denote an open subset of U_d intersecting A in a point y, and let $V_1, \cdots, V_d$ denote a sequence of open neighborhoods of y having the properties (i), (ii), and (iii) above except that in (i) it is assumed that $V_1 \subset W$. Let $C = U - W$ and $C_i = U_i - V_i$. It follows from Lemma 2.2 that for each i

$$H_c^j(C_{i+1}) \to H_c^j(C_i) \quad \text{is trivial for all } j.$$

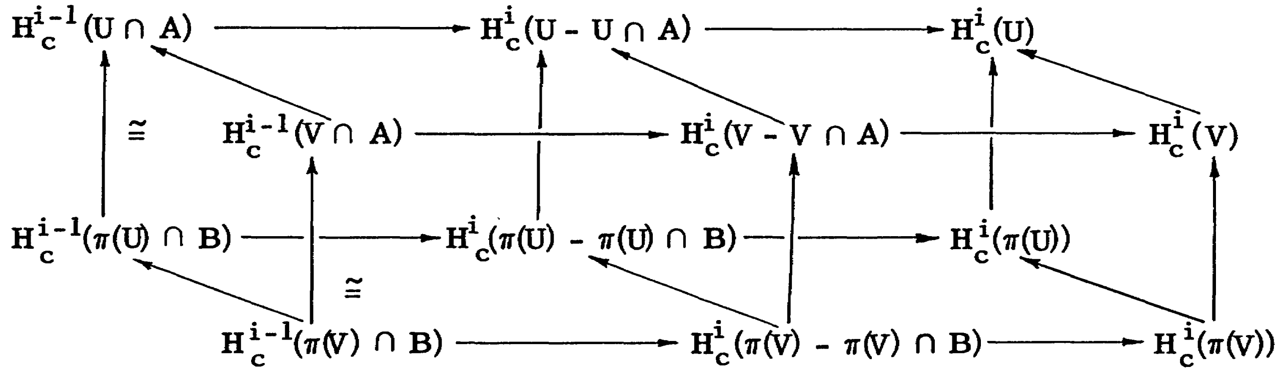
We shall see that

$$(1) \qquad H_c^j(A_{C_d}) \to H_c^j(A_{C_1}) \quad \text{is trivial for all } j.$$

This implies that $H_c^j(A_{C_d}) \to H_c^j(A_C)$ is trivial for all j; and hence, A has property Q. The demonstration of triviality asserted in (1) is exactly that given in the proof of Theorem 1.2 with the C_i's substituted for the U's and V's appearing there.

Thus, we have shown that each component of A is a locally orientable generalized manifold of some dimension. It remains only to establish that this dimension is of the form asserted. This will be done by showing that the local co-Betti groups of A at each of its points vanish, except in one dimension which is of the form $n - k \cdot (r + 1)$. As might be expected, this will be done by an argument much like that appearing in the proof of Theorem 2.1.

For $x \in A$ and U and V open subsets of X, $(x \in V \subset U)$, we have the commutative diagram



Since inverse limits commute with exact sequences of vector spaces and $\{\pi(U) \mid U$ an open neighborhood of $x\}$ is cofinal in $\{V \mid V$ an open neighborhood of $y = \pi(x)\}$, the following diagram is commutative, the horizontal sequences being exact,

$$\begin{array}{ccccccccc}
\longrightarrow & \varprojlim_{x \in U} H_c^{i-1}(U \cap A) & \longrightarrow & \varprojlim_{x \in U} H_c^i(U - U \cap A) & \longrightarrow & \varprojlim_{x \in U} H_c^i(U) & \longrightarrow \\
& \uparrow \cong & & \uparrow \pi^* & & \uparrow & \\
\longrightarrow & \varprojlim_{y \in V} H_c^{i-1}(V \cap B) & \longrightarrow & \varprojlim_{y \in V} H_c^i(V - V \cap B) & \longrightarrow & \varprojlim_{y \in V} H_c^i(V) & \longrightarrow
\end{array}$$

Since $\varprojlim_{x \in u} H_c^i(U) = 0$ for $i \neq n$, π^* is onto for $i \neq n$. Also, by taking inverse limits of Gysin sequences, we see that the sequence

$$\cdots \varprojlim_{y\in V} H_c^i(V - V \cap B) \to \varprojlim_{y\in V} H_c^{i+r+1}(V - V \cap B) \xrightarrow{\pi^*} \varprojlim_{x\in U} H_c^{i+r+1}(U - U \cap A) \to$$

$$\varprojlim_{y\in V} H_c^{i+1}(V - V \cap B) \to \cdots$$

is exact. Formally the situation is identical to that in the proof of Theorem 2.1. Hence, for $x \in A$, there is an integer $m(x) = n - k(x) \cdot (r + 1)$ such that

$$\varprojlim_{y\in V} H_c^i(U \cap A) = \begin{cases} 0 & \text{for } i \neq m(x), \\ Z_2 & \text{for } i = m(x); \end{cases}$$

this completes the proof.

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