A GENERALIZED MANIFOLD

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INTRODUCTION

While generalized manifolds share many of the properties of classical manifolds, there has been a general question: How far can a generalized manifold get away from classical manifolds? In particular, it does not seem to have been settled whether or not there exists a generalized manifold that is not locally euclidean at any point. A generalized manifold that fails to be locally euclidean at a single point can easily be obtained by shrinking a bad arc in S^3 to a point [3]. This being the case, one readily suspects that a generalized manifold that is not locally euclidean at any point can be obtained by putting bad arcs densely in S^3 and shrinking each to a point. In the present note, we show that this is indeed the case. In the course of the construction, however, care must be taken so that the decomposition space is still a Hausdorff space and is also finite-dimensional. The first requirement is fulfilled if we put in arcs such that the arcs together with the points not on any of the arcs form an upper-semicontinuous decomposition (see [8] for definition) of S^3.

1. THE CONSTRUCTION OF AN UPPER-SEMICONTINUOUS DECOMPOSITION G

We denote by E^3 and S^3 the euclidean 3-space and 3-sphere, respectively. By a 3-cell we mean a homeomorph of the unit sphere together with its interior in E^3. By the boundary and the interior of a 3-cell we mean the parts that correspond to the unit sphere and its interior, and they will be denoted, sometimes, by Bd and Int, respectively.

It is known [4] that there exists an arc A in E^3 such that E^3 - A is not simply connected. It is easy to see that such an A can be put into any pre-assigned open subset of E^3.

Let $\mathfrak{U}^0 = \{U_1^0, U_2^0, \cdots, U_{k_0}^0\}$ be an open covering of S^3 such that each U_i^0 is the interior of a 3-cell $\overline{U}_i^0$ of diameter less than 1. Let F^0 be the union of Bd U_i^0. In each U_i^0, we can find an arc A_i^0 such that (1) the arcs A_i^0 are pairwise disjoint, (2) each A_i^0 is situated in U_i^0 in the same manner as A is in E^3, and (3) no A_i^0 meets F^0. There exists a positive number $d_1 < 1/2$ such that no $3d_1$-neighborhood of any A_i^0 meets F^0 or any other A_j^0.

Let $\mathfrak{U}^1 = \{U_1^1, U_2^1, \cdots, U_{k_1}^1\}$ be an open overing of S^3 such that each U_i^1 is the interior of a 3-cell $\overline{U}_i^1$ of diameter less than d_1. Let F^1 denote the union of the Bd U_i^1. In each U_i^1, we can find an arc A_i^1 such that (1) the arcs A_p^0 and A_q^1 are pairwise disjoint, (2) each A_i^1 is situated in U_i^1 in the same way as A is in E^3, and (3) no A_i^1 meets F^0 or F^1. There exists a positive number $d_2 < \min(d_1, 1/4)$ such that no $3d_2$-neighborhood of any A_p^0 or A_q^1 meets any other $A_{p'}^0$ or $A_{q'}^1$ and no $3d_2$-neighborhood of A_q^1 meets F^0 or F^1.

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Let $\mathfrak{U}^2 = \{U_1^2, U_2^2, \cdots, U_{k_2}^2\}$ be an open covering of S^3 such that each U_i^2 is the interior of a 3-cell $\overline{U}_i^2$ of diameter less than d_2. In each U_i^2 we can find an arc A_i^2 such that (1) the arcs A_p^0, A_q^1 and A_r^2 are pairwise disjoint, (2) each A_i^2 is situated in U_i^2 in the same manner as A is in E^3, and (3) no A_i^2 meets F^0, F^1 or F^2, where F^2 denotes the union of the Bd U_j^2. There exists a positive number $d_3 < \min(d_2, 1/8)$ such that no $3d_3$-neighborhood of any A_p^0, A_q^1 or A_r^2 meets any other $A_{p'}^0$, $A_{q'}^1$ or $A_{r'}^2$, and no $3d_3$-neighborhood of A_r^2 meets F^2.

Continuing in this manner, we have a sequence of coverings

$$\mathfrak{U}^p = \{U_1^p, U_2^p, \cdots, U_{k_p}^p\} \quad (p = 0, 1, 2, \cdots)$$

and arcs

$$A_q^p \quad (q = 1, 2, \cdots, k_p;\ p = 0, 1, 2, \cdots),$$

and positive numbers $d_1 > d_2 > d_3 > \cdots$ $(d_i < 2^{-i})$ such that (1) the arcs A_q^p are pairwise disjoint, (2) each A_q^p is situated in U_q^p in the same manner as A is in E^3, (3) no A_q^p meets $F^{p'}$, the union of the Bd $U_i^{p'}$, for $p' \leq p$, and (4) no $3d_{p+1}$-neighborhood of $A_{i_0}^0$, $A_{i_1}^1$, $\cdots$, or $A_{i_p}^p$ meets any other $A_{i_0'}^0$, $A_{i_1'}^1$, $\cdots$, or $A_{i_p'}^p$, and no $3d_{p+1}$-neighborhood of A_i^p meets F^0, F^1, $\cdots$, or F^p. Observe that our choice of the d_i, implies that (5) each element of $\mathfrak{U}^p$ contains an element of $\mathfrak{U}^{p+1}$ whose boundary meets no arcs.

Our G will consist of the arcs A_q^p and the points not on any of these arcs.

2. THE UPPER-SEMICONTINUITY OF G

A subset T of S^3 will be called *saturated* if every element of G that meets T is in T. Thus a saturated set is one that does not contain an element of G partially. We shall establish the upper-semicontinuity of G by showing that each element of G has arbitrarily close saturated neighborhoods.

Let $g \in G$ be an arc A_q^p. For each $q' > p$, the union U of the elements of $\mathfrak{U}^{q'}$ that meet A_q^p is a $d_{q'}$-neighborhood of g. The set U is saturated, and Bd U does not meet any arc, by (4) of the preceding section. If g is a point and d is any positive number, there exists an integer i such that $d_i < d$. Let e be a positive number such that no e-neighborhood of g meets any arc A_j^r $(r < i)$, and let t be an integer such that $d_t < \min(d_i, e)$. Let U be an element of $\mathfrak{U}^t$ that contains g. Then Bd U does not meet any arc A_j^r $(r < i$ or $r > t)$. Let V be the union of the elements of $\mathfrak{U}^t$ that meet some arc that meets $\overline{U}$. Then V is a saturated $3d$-neighborhood of g such that Bd V does not meet any arc. Thus each element of G has an arbitrarily close saturated neighborhood whose boundary consists entirely of degenerate elements of G. This shows that G is upper-semicontinuous, and that moreover the decomposition space is at most 3-dimensional, since a saturated open set is mapped onto an open subset.

3. THE DECOMPOSITION SPACE

We shall denote by X and f the decomposition space and the quotient map of G, respectively. From the preceding section and the mapping theorem of Wilder [10], it follows that X is a sphere-like closed generalized 3-manifold. We want to show that X is not locally euclidean at any point. Suppose this were not the case. Then there would exist an open set U of X which is homeomorphic to E^3. Since the elements of all the $\mathfrak{U}^p$ form a basis for open sets of S^3, it follows that $V = f^{-1}(U)$ contains an element U_i^{q-1} of $\mathfrak{U}^{q-1}$ for some q. By our construction, there exists an element U_j^q of $\mathfrak{U}^q$ which is contained in U_i^{q-1} and whose boundary Bd U_j^q meets no arc. Since U_j^q is saturated, $U' = f(U_j^q)$ is an open subset of X contained in U. Hence, U' must be locally euclidean. Moreover, by a theorem due to Smale [6], it follows that U' is simply connected. Let $x = f(A_j^q) \in U'$, and let W be a 3-cell in U' such that $x \in$ Int W. The space U' = Int W becomes the space U' when W is attached to U' - Int W along Bd W by means of the identity map. By the proof of Theorem 2.10.1 of [7], U' - Int W has its fundamental group isomorphic to that of U' and is therefore simply connected. But U' - Int W is a deformation retract of U' - x, and consequently U' - x is simply connected. Again by the above-mentioned theorem of Smale, this would imply that $U_j^q - A_j^q$ is simply connected, contrary to our construction.

4. SOME REMARKS

In a joint paper with F. Raymond [5], we have demonstrated similarities between generalized cells in generalized manifolds and euclidean cells in classical manifolds (see [9] for the definition of the generalized cells). In this connection, it would be interesting to know whether or not there exists an orientable spherelike generalized manifold that cannot be covered by the interiors of generalized closed cells. (It is easy to find one that cannot be covered by "arbitrarily small" generalized cell neighborhoods.) Our most non-euclidean space X is covered by the interiors of generalized 3-cells $f(U_1^0), f(U_2^0), \cdots, f(U_{k_0}^0)$.

Although we have here used a countable number of wild arcs, it is possible to use uncountably many tame arcs. If in our construction we put into each U_q^p, instead of A_q^p, Bing's Cantor set of tame arcs [1] and proceed similarly, we get the decomposition space Y. Then Y has no locally simply connected neighborhood at any point. To see this, we use the above-mentioned theorem of Smale and a result due to M. L. Curtis [2].

It would be interesting to know whether $X \times R$ (or $Y \times R$), where R is the real line, is topologically equivalent to $S^3 \times R$.

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