ON FIBERINGS WITH SINGULARITIES

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1. INTRODUCTION

The original treatment of fiberings goes back to H. Seifert [5] who permitted a certain type of singularity: namely, the case where neighboring fibers wind themselves around a particular fiber. About a dozen years later, in 1945, Montgomery and Samelson [4] studied a different type of singularity which may be roughly described by saying that certain fibers are pinched to points.

In the present paper, we shall first introduce a general definition of fiberings with singularities which includes all of the known fiberings as special cases. Then, by identifying the singular fibers to points, we shall prove that every fibering p with singularities can be decomposed into the composition $p_1 \circ p_2$ of two fiberings p_1 and p_2, where every singular fiber of p_1 is a singleton and every regular fiber of p_2 is a singleton. Furthermore, p_2 is the natural projection of some topological identification. Therefore, after that point, we shall be concerned only with fiberings whose singular fibers are singletons.

As an important and useful fibering with only one singular fiber, we shall introduce and study the extended tangent space $E(X, x_0)$ of a given space X at a given point x_0 and the natural projection $p: E(X, x_0) \to X$. The invariants of the regular fibers of this fibering are closely related with the invariants of the residual space $X \setminus x_0$ and the local invariants of X at x_0.

Our main interest is to investigate the local property of any given fibering $f: X \to Y$ at an isolated singular fiber x_0. For this purpose, we may assume without loss of generality that x_0 is the only singular fiber of f. If $y_0 = f(x_0)$, then the map $f:(X, x_0) \to (Y, y_0)$ induces homomorphisms on the local homology groups and the local homotopy groups defined in [2]. To study these induced homomorphisms, we shall investigate the derived continuous map $\hat{f}: E(X, x_0) \to E(Y, y_0)$. Under a mild condition on f, this map $\hat{f}$ is a fibering with only one singular fiber. We shall say that f is normal at x_0 if and only if $\hat{f}$ is a fibering with only one singular fiber. We shall prove that the *fibering axiom* of the local homotopy groups holds for every fibering with a single normal singular fiber [2, Section 16].

The invariants of the regular fibers $\hat{F}$ of the derived fibering $\hat{f}$ are closely related to the local invariants of the spaces X and Y at the points x_0 and y_0, respectively. If Y is locally euclidean, then we have a local version of Wang's exact sequence [7] which will help determine the homology groups of $\hat{F}$.

Finally, by a cone construction, it will be shown that the usual global theory of fiberings without singularities may be deduced from a special case of the local theory of fiberings at an isolated singularity.

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2. DEFINITIONS

Let E, B be topological spaces, and let A be a subspace of B. A continuous map

$$p: E \to B$$

from E *onto* B will be called a *fibering over* B *with* A *as singularities*, if the restriction of p on $p^{-1}(B \setminus A)$ is a fibering over $B \setminus A$ in the sense of Serre [6, p. 443]. In other words, p is a fibering over B with A as singularities if and only if, for every continuous map $g: X \to E$ from a finitely triangulable space X into E such that the composed map $f = p \circ g$ carries X into $B \setminus A$ and, for every homotopy $f_t: X \to B \setminus A$ $(0 \leq t \leq 1)$ such that $f_0 = f$, there exists a homotopy $g_t: X \to E$ $(0 \leq t \leq 1)$ such that $g_0 = g$ and $p \circ g_t = f_t$ for every t.

This definition of fiberings with singularities includes as special cases those studied by H. Seifert [5] as well as those investigated by Montgomery and Samelson [4]. Furthermore, if p is a fibering over B in the sense of Serre, and A is any subspace of B, then the condition in our definition is obviously satisfied.

Now, let p be a fibering over B with A as singularities. Then the space E is called a *fiber space over the base space* B *with* A *as singularities*. The points of the subspace A are called the *singular points*, and those of $B \setminus A$ are called the *regular points*. For each point b of the base space B, the inverse image $p^{-1}(b)$ is called the *fiber* over b. If b is a singular point, then $p^{-1}(b)$ is said to be a *singular fiber;* otherwise, $p^{-1}(b)$ is said to be a *regular fiber*.

As examples of fiberings with singularities, let us first consider the continuous map

$$p: S^n \to J$$

of the unit n-sphere S^n in the euclidean $(n+1)$-space onto the closed interval $J = [-1, 1]$ of real numbers defined by $p(x_0, x_1, \cdots, x_n) = x_0$ for every point $(x_0, x_1, \cdots x_n)$ of S^n. One can easily see that p is a fibering over J with the end points of J as singularities. The regular fibers are $(n-1)$-spheres; and the two singular fibers are singletons, namely, the south pole $(-1, 0, \cdots, 0)$ and the north pole $(1, 0, \cdots, 0)$.

Next, let us consider the local Hopf map

$$p: R^4 \to R^3$$

of the euclidean 4-space R^4 onto the euclidean 3-space R^3 defined as follows. Consider $R^4 = R^2 \times R^2$ as the space of all pairs (x, y) of complex numbers x and y. Let S^1 denote the unit circle in R^2 consisting of the complex numbers z with $|z| = 1$. Then S^1 is a topological transformation group of R^4 by the operation $z(x, y) = (zx, zy)$ The orbit space may be identified with R^3, and the natural projection $p: R^4 \to R^3$ is the local Hopf map, [2, Section 18]. One can easily see that p is a fibering over R^3 with the origin of R^3 as the only singularity. The regular fibers are circles; and the unique singular fiber is a singleton, namely, the origin of R^4. By using quaternions and Cayley numbers, one can also define the local Hopf maps of R^8 onto R^5, and of R^{16} onto R^9. They are fiberings with the origin as the only singular fiber. Their regular fibers are 3-spheres and 7-spheres respectively.

3. SPECIAL FIBERINGS

Consider a given fibering $p: E \to B$ over the base space B with the subspace A as singularities. The fibering p is said to be *special of the first type* if every singular fiber is a singleton, that is to say, $p^{-1}(b)$ consists of a single point for every singular point $b \in A$. Thus the fiberings studied by Montgomery and Samelson [4] belong to this type. On the other hand, the fibering p is said to be *special of the second type* if every regular fiber is a singleton.

PROPOSITION 3.1. *If the subspace* A *of* B *is closed, then every fibering* $p: E \to B$ *over* B *with* A *as singularities can be factorized into the composition* $p = p_1 \circ p_2$ *of a special fibering* p_1 *of the first type and a special fibering* p_2 *of the second type.*

Proof. Identifying every singular fiber of the fibering p into a single point, we obtain a quotient space D of the space E. Precisely, D is constructed as follows. Introduce an equivalence relation $\sim$ in E by saying that, for any two points x, y of E, $x \sim y$ if and only if either $x = y$ or they belong to the same singular fiber. This equivalence relation divides the points of E into disjoint equivalence classes. Let D denote the set of these classes, and let

$$p_2: E \to D$$

denote the natural projection which sends each point x of E to the class containing x. The topology of D is defined by saying that a set U in D is *open* if and only if the inverse image $p_2^{-1}(U)$ in E is open. Then the natural projection p_2 is a continuous map.

Let C denote the subspace of D which consists of all singular fibers of the fibering p. Since $p_2^{-1}(C) = p^{-1}(A)$ and A is assumed to be closed, it follows that C is closed in D. Then one can easily prove that p_2 maps $p_2^{-1}(D \setminus C)$ homeomorphically onto $D \setminus C$. This proves that p_2 is a fibering over D with C as singularities. Since every regular fiber of p_2 is a singleton, p_2 is a special fibering of the second type.

On the other hand, since the given fibering p sends each equivalence class of E into a single point of B, there is a unique function

$$p_1: D \to B$$

such that $p_1 \circ p_2 = p$. For an arbitrary open set V in B, the inverse image $U = p_1^{-1}(V)$ is open in D because $p_2^{-1}(U) = p^{-1}(V)$ is open in E. Hence p_1 is a continuous map. On the open subspace $D \setminus C = p_1^{-1}(B \setminus A)$, p_1 is equal to the composition $p \circ p_2^{-1}$. This implies that p_1 is a fibering over B with A as singularities. Since every singular fiber of p_1 is a singleton, p_1 is a special fibering of the first type.

Since $p = p_1 \circ p_2$, the proof of (3.1) is complete.

The fibering p_2 constructed above is merely the natural projection of a topological identification, and we may therefore focus our effort on the special fiberings of the first type. Throughout the remainder of the paper, we are concerned only with the special fiberings of the first type. For economy of language, hereafter when we say fiberings, we shall always mean special fiberings of the first type.

4. THE EXTENDED TANGENT SPACE

Let X be a given topological space, and x_0 any given point in X. In defining the local algebraic invariants of the space X at the point x_0, the notion of the *tangent space* $T(X, x_0)$ of X at x_0 was introduced in [2, Section 2]. We recall the definition as follows.

By a *path* in X, we mean a continuous map $\sigma: I \to X$ of the closed unit interval $I = [0, 1]$ into X. The set $W(X)$ of all paths in X forms a topological space with the usual compact-open topology. Then, $T(X, x_0)$ is the subspace of $W(X)$ defined by the formula

$$T(X, x_0) = \{\sigma \in W(X): \sigma(t) = x_0 \text{ if and only if } t = 0\}.$$

Let $e_0 \in W(X)$ denote the degenerate path $e_0(I) = x_0$. Then, by definition, e_0 is not contained in the tangent space $T(X, x_0)$. If we adjoin this degenerate path e_0 to $T(X, x_0)$, we obtain a subspace

$$E = E(X, x_0) = e_0 \cup T(X, x_0)$$

of $W(X)$ which will be called the *extended tangent* space of X at the point x_0.

PROPOSITION 4.1. *The space* E *is contractible to the point* e_0.

Proof. Define a homotopy $h_t: E \to E$ $(0 \leq t \leq 1)$ by the following formula:

$$[h_t(\sigma)](s) = \sigma(st) \qquad (\sigma \in E,\ s \in I,\ t \in I).$$

Then $h_0(E) = e_0$, h_1 is the identity map on E, and $h_t(e_0) = e_0$ for every t. This proves that E is contractible to the point e_0.

Now let us consider the continuous map

$$p: E \to X$$

defined by $p(\sigma) = \sigma(1)$ for every $\sigma \in E$. This map σ will be called the *natural projection*.

THEOREM 4.2. *If* X *is a pathwise connected* T_1*-space, then the natural projection* $p: E \to X$ *is a fibering over* X *with the point* e_0 *as the only singular fiber.*

Proof. First, let us prove that p carries E onto X. For this purpose, let x_1 be any point of X different from x_0. Since X is pathwise connected, there exists a path $\xi: I \to X$ such that $\xi(0) = x_0$ and $\xi(1) = x_1$. Since X is a T_1-space and ξ is continuous, the inverse image $\xi^{-1}(x_0)$ is a closed set of the unit interval I. Let k denote the least upper bound of $\xi^{-1}(x_0)$. Then $0 \leq k < 1$, since $\xi(1) = x_1$. Also, $\xi(k) = x_0$ and $\xi(t) \neq x_0$ whenever $k < t \leq 1$. Define a path $\eta: I \to X$ by setting

$$\eta(t) = \xi(k + t - kt) \qquad (t \in I).$$

Then $\eta(0) = x_0$, $\eta(1) = x_1$, and $\eta(t) \neq x_0$ whenever $0 < t \leq 1$. Hence $\eta \in T(X, x_0) \subset E$ and $p(\eta) = x_1$. This proves that p carries E onto X.

By the definition of the space E, it is clear that the inverse image $p^{-1}(x_0)$ consists of a single point e_0. Hence the restriction of p on the tangent space $T = T(X, x_0)$ is a continuous map

$$\pi: T \to B$$

of the tangent space T onto the residual subspace $B = X \setminus x_0$ of X. It remains to prove that π is a fibering without singularity, in other words, that π is a fibering in the sense of Serre.

In fact, we shall prove that π has a stronger property, namely, the *path-lifting property* (abbreviated PLP hereafter) defined as follows. Let Z denote the subspace of the topological product $T \times B^I$ of the space T and the space B^I of all paths in B defined by

$$Z = \{(\sigma, \tau) \in T \times B^I : \pi(\sigma) = \tau(0)\}.$$

Define a continuous map $f: T^I \to Z$ of the space T^I of all paths in T into the space Z by setting $f(\zeta) = (\zeta(0), \pi \circ \zeta)$ for each $\zeta: I \to T$ in the space T^I. Then the map $\pi: T \to B$ is said to have the PLP if and only if there exists a continuous map $g: Z \to T^I$ such that the composed map $f \circ g$ is the identity map on Z. According to a result of Hurewicz [3], the PLP is equivalent to the absolute covering homotopy property, and hence it follows that π is a fibering without singularity.

To establish the PLP of π, we have to construct a continuous map $g: Z \to T^I$ such that $f \circ g$ is the identity map on Z. For this purpose, let us consider the unit square $Q = I \times I$ in the euclidean 2-space and its closed subspace

$$P = (I \times 0) \cup (1 \times I) \subset Q.$$

Let $\rho: Q \to P$ denote the retraction of Q onto P constructed as follows: For an arbitrary point q of the unit square Q, consider the straight line determined by q and the point (0, 2) in the euclidean 2-space R^2. Then $\rho(q)$ is defined to be the unique point of intersection of this line and the set P. It follows that the inverse image $\rho^{-1}(0, 0)$ is the set $0 \times I$.

Next, consider the space X^P of all continuous maps from P into X with the usual compact-open topology. Let M denote the subspace of X^P defined by the formula:

$$M = \{\mu \in X^P : \mu(y) = x_0 \text{ if and only if } y = (0, 0)\}.$$

For each $\mu: P \to X$ in M, define two paths $\sigma: I \to X$ and $\tau: I \to B$ by taking

$$\sigma(t) = \mu(t, 0), \qquad \tau(t) = \mu(1, t)$$

for every $t \in I$. Then $\sigma \in T$, $\tau \in B^I$, and $\pi(\sigma) = \tau(0)$. Hence the assignment $\mu \to h(\mu) = (\sigma, \tau)$ determines a function $h: M \to Z$. One can verify that h is a homeomorphism of M onto Z.

Using the inverse of the homeomorphism h, we can construct a continuous map $g: Z \to T^I$ by taking

$$\{[g(z)](t)\}(s) = [h^{-1}(z)]\, \rho(s, t)$$

for every $z \in Z$, $s \in I$ and $t \in I$. Since ρ is a retraction, it follows immediately that $f[g(z)] = z$ for every $z \in Z$. This completes the proof of (4.2).

COROLLARY 4.3. *If* X *is a pathwise connected* T_1*-space, then the natural projection*

$$\pi: T(X, x_0) \to X \setminus x_0$$

is a fibering over $X \setminus x_0$ *without singularity.*

If X is a T_1-space but fails to be pathwise connected, then the image of the natural projection $p: E(X, x_0) \to X$ is the path-component X_0 of X which contains x_0. It follows from the proof of (4.2) that $T(X, x_0)$ is a fiber space over $X_0 \setminus x_0$ without singularity, and hence $E(X, x_0)$ is a fiber space over X_0 with e_0 as its only singular fiber.

5. LOCAL INVARIANTS OF $E(X, x_0)$ AT e_0

In a previous paper of the author [2], local homology groups and local homotopy groups of X at x_0 are defined by means of the tangent space $T(X, x_0)$. Since

$$E = E(X, x_0) = e_0 \cup T(X, x_0),$$

it is natural to inquire about the local invariants of the space E at the point e_0. It turns out that the local invariants of E at e_0 are the same as those of X at x_0. In fact, we shall prove that the tangent spaces $T(X, x_0)$ and $T(E, e_0)$ are of the same homotopy type.

For this purpose, let us begin with the natural projection

$$p: (E, e_0) \to (X, x_0)$$

defined in the preceding section. Since $p^{-1}(x_0) = e_0$, p is an *admissible map* in the sense of [2, Section 4]. Therefore, p induces a continuous map

$$\hat{p}: T(E, e_0) \to T(X, x_0)$$

of the corresponding tangent spaces defined by $\hat{p}(\sigma) = p \circ \sigma$ for each $\sigma: I \to E$ in the $T(E, e_0)$.

PROPOSITION 5.1. *The map* $\hat{p}$ *is a homotopy equivalence.*

Proof. Consider the unit square $Q = I \times I$ and its closed subspace

$$L = (I \times 0) \cup (0 \times I).$$

Every path $\sigma: I \to E$ gives rise to a continuous map $\phi_\sigma: Q \to X$ defined by

$$\phi_\sigma(s, t) = [\sigma(t)](s) \qquad (s \in I,\ t \in I).$$

One can easily verify that $\sigma \in T(E, e_0)$ if and only if $\phi_\sigma^{-1}(x_0) = L$, and that the assignment $\sigma \to \phi_\sigma$ determines a homeomorphism of $T(E, e_0)$ onto a subspace of the space X^Q of all continuous maps from Q into X. Thus, we may identify σ with ϕ_σ and consider $T(E, e_0)$ as the subspace of X^Q defined by

$$T(E, e_0) = \{\phi \in X^Q: \phi^{-1}(x_0) = L\}.$$

After this identification, the induced map $\hat{p}$ reduces to the map given by

$$[\hat{p}(\phi)](t) = \phi(1, t) \qquad (\phi \in T(E, e_0),\ t \in I).$$

Next, let us define a homotopy $h_t\colon Q \to Q$ $(0 \leq t \leq 1)$ by setting

$$h_t(x, y) = \begin{cases} (x, 2tx - 2ty + y) & \text{if } x \leq y \text{ and } 0 \leq t \leq 1/2, \\ (2x - 2tx + 2t - 1, x) & \text{if } x \leq y \text{ and } 1/2 \leq t \leq 1, \\ (x, y) & \text{if } x \geq y \text{ and } 0 \leq t \leq 1/2, \\ (2x - 2tx + 2t - 1, y) & \text{if } x \geq y \text{ and } 1/2 \leq t \leq 1. \end{cases}$$

Then one can verify that h_0 is the identity map on Q, h_1 is a retraction of Q onto its subspace $K = 1 \times I$, $h_t(L) \subset L$, and $h_t | K$ is the inclusion map for every $t \in I$.

Using the retraction $h_1\colon Q \to K$, we define a continuous map

$$\hat{q}\colon T(X, x_0) \to T(E, e_0)$$

as follows. For each $\sigma\colon I \to X$ in $T(X, x_0)$, let $\sigma^*\colon K \to X$ denote the map given by $\sigma^*(1, t) = \sigma(t)$ for every $t \in I$. Then $\hat{q}(\sigma)$ is defined to be the continuous map of Q into X given by

$$[\hat{q}(\sigma)](x, y) = \sigma^*[h_1(x, y)] \qquad ((x, y) \in Q).$$

It remains to prove that $\hat{q}$ is a two-sided homotopy inverse of $\hat{p}$.

First, the composition $\hat{p} \circ \hat{q}$ is the identity map on $T(X, x_0)$. In fact, for every $\sigma\colon I \to X$ in $T(X, x_0)$, we have

$$[\hat{p} \circ \hat{q}(\sigma)](t) = [\hat{q}(\sigma)](1, t) = \sigma^*[h_1(1, t)] = \sigma^*(1, t) = \sigma(t)$$

for every $t \in E$. Hence $\hat{p} \circ \hat{q}(\sigma) = \sigma$. This proves the assertion that $\hat{p} \circ \hat{q}$ is the identity map.

Second, the composition $\hat{q} \circ \hat{p}$ is homotopic to the identity map on $T(E, e_0)$. To prove this assertion, let us consider the homotopy $k_t\colon T(E, e_0) \to T(E, e_0)$ $(0 \leq t \leq 1)$ defined as follows: For each $\phi\colon Q \to X$ in $T(E, e_0)$, $k_t(\phi)$ is taken to be the composed map $\phi \circ h_t$ for each $t \in I$. Then k_0 is the identity map and $k_1 = \hat{q} \circ \hat{p}$. This proves the assertion and also completes the proof of (5.1).

For the use in the sequel, we are going to establish the following

PROPOSITION 5.2. *The map* $\hat{p}$ *is a fibering over* $T(X, x_0)$ *without singularity.*

Proof. First, let us prove that $\hat{p}$ sends $T(E, e_0)$ onto $T(X, x_0)$. For this purpose, let $\sigma\colon I \to X$ be any point in the tangent space $T(X, x_0)$. Let $\tau\colon I \to E$ denote the path defined by

$$[\tau(t)](s) = \sigma(st) \qquad (s \in I,\ t \in I).$$

Then one can easily verify that $\tau \in T(E, e_0)$ and $\hat{p}(\tau) = p \circ \tau = \sigma$. This proves that $\hat{p}$ is onto.

Next, let us prove that $\hat{p}$ has the PLP. For this purpose, let

$$\hat{X} = T(X, x_0), \qquad \hat{E} = T(E, e_0).$$

$$Z = \{(\sigma, \tau) \in \hat{E} \times \hat{X}^I\colon \hat{p}(\sigma) = \tau(0)\}.$$

Let $f: \hat{E}^I \to Z$ denote the continuous map defined by $f(\xi) = (\xi(0), \hat{p} \circ \xi)$ for each $\xi: I \to \hat{E}$ in the space $\hat{E}^I$. Then it suffices to prove the existence of a continuous map $g: Z \to \hat{E}^I$ such that $f \circ g$ is the identity map on Z.

Consider the unit cube $Q = I \times I \times I$ in the euclidean 3-space and its closed subspace

$$P = (I \times I \times 0) \cup (1 \times I \times I) \subset Q.$$

Let $\rho: Q \to P$ denote the retraction of Q onto P constructed as follows: For an arbitrary point (x, y, z) of Q, consider the straight line joining the points $(0, y, 2)$ and (x, y, z) in the euclidean 3-space. Then $\rho(x, y, z)$ is defined to be the unique point of intersection of this line and the set P.

Now, let us consider the space X^P of all continuous maps from P into X with the usual compact-open topology. Let M denote the subspace of X^P defined by the formula:

$$M = \{\mu \in X^P: \mu^{-1}(x_0) = (0 \times I \times 0) \cup (I \times 0 \times 0) \cup (1 \times 0 \times I)\}.$$

For each $\mu: P \to X$ in M, define two paths $\sigma: I \to E$ and $\tau: I \to \hat{X}$ by taking

$$[\sigma(t)](s) = \mu(s, t, 0),$$

$$[\tau(t)](s) = \mu(1, s, t)$$

for every $s \in I$ and $t \in I$. Then $\sigma \in \hat{E}$, $\tau \in \hat{X}^I$, and $\hat{p}(\sigma) = \tau(0)$. Hence, the assignment $\mu \to h(\mu) = (\sigma, \tau)$ determines a function $h: M \to Z$. One can verify that h is a homeomorphism of M onto Z.

Using the inverse of the homeomorphism h, we can construct a continuous map $g: Z \to \hat{E}^I$ by taking

$$\{[(gz)(t)](s)\}(r) = [h^{-1}(z)]\rho(r, s, t)$$

for every $z \in Z$ and r, s, t in I. Since ρ is a retraction, it follows immediately that $f[g(z)] = z$ for every $z \in Z$. Hence $f \circ g$ is the identity map on Z. This implies that $\hat{p}$ has PLP, and it completes the proof of (5.2).

6. REGULAR FIBERS IN $E(X, x_0)$

Let x_1 be any point of X different from x_0, and write

$$F = p^{-1}(x_1) \subset E(X, x_0),$$

where $p: E(X, x_0) \to X$ stands for the natural projection. Then F is a regular fiber in $E(X, x_0)$ of the fibering p. The homotopy properties of F depend on both the global properties of X and the local properties at x_0; however, they are independent of the choice of x_1 if $X \setminus x_0$ is pathwise connected. As a subspace of the space $W(X)$ of all paths in X, F is defined by the following formula:

$$F = \{\sigma \in W(X): \sigma(1) = x_1 \text{ and } \sigma^{-1}(x_0) = 0\}.$$

Since the tangent space $T = T(X, x_0)$ is a fiber space over the residual space $B = X \setminus x_0$ without singularity, and since F is the fiber over the point $x_1 \in B$, the invariants of F are related to those of T and B. For example, let $e_1 \in F$. Then the homotopy sequence of a fibering without singularity gives an exact sequence:

$$\cdots \to \pi_n(F, e_1) \xrightarrow{\phi} \pi_n(T, e_1) \xrightarrow{\psi} \pi_n(B, x_1) \xrightarrow{\tau} \pi_{n-1}(F, e_1) \to \cdots,$$

where ϕ is induced by the inclusion $F \subset T$, ψ is induced by the projection, and τ is the transgression. For the case where X is pathwise connected around the point x_0 in the sense of [2, Section 12], $\pi_n(T, e_1)$ was defined to be the n-th local homotopy group of X at x_0 [2, Section 13]; in symbols,

$$\lambda_n(X, x_0) = \pi_n(T, e_1).$$

In some cases, the exact sequence helps to determine the invariants of F. For example, let X be the m-sphere S^m with $m > 1$. Then the residual space B is contractible to the point x_1, and hence ϕ is an isomorphism of $\pi_n(F, e_1)$ onto $\pi_n(T, e_1)$ for every $n \geq 1$. According to [2, Section 7], T has the same homotopy type as the $(m - 1)$-sphere. Hence, the homotopy groups and the homology groups of the space F are isomorphic to the corresponding groups of the $(m - 1)$-sphere S^{m-1}. This example shows that the invariants of F are in general different from those of the space

$$G = \{\sigma \in W(X) : \sigma(0) = x_0 \text{ and } \sigma(1) = x_1\}.$$

For in the case of $X = S^m$, it is a well-known result of Morse that $H_n(G)$ is infinite cyclic if n is a multiple of $m - 1$, and that $H_n(G) = 0$ otherwise.

Next, let X be the euclidean m-space R^m with $m > 1$, and let x_0 be the origin of R^m. In this case, we may define a continuous map $q: X \to E(X, x_0)$ by setting $[q(x)](t) = tx$ for every $x \in X$ and $t \in E$. Then the composition $p \circ q$ is the identity map on X. By the method used in [2, Section 7], one can construct a homotopy

$$h_t: E(X, x_0) \to E(X, x_0) \qquad (0 \leq t \leq 1)$$

such that h_0 is the identity map, $h_1 = q \circ p$, and $h_t^{-1}(e_0) = e_0$ for every $t \in E$. This implies that ψ is an isomorphism of $\pi_n(T, e_1)$ onto $\pi_n(B, x_1)$. It follows from the exactness of the sequence that $\pi_n(F, e_1) = 0$ for every $n \geq 1$, and hence $H_n(F) = 0$ for every $n \geq 1$.

Finally, let X be the real projective space P^m of dimension $m > 1$. If $m > 2$, then T and B are homotopically equivalent to S^{m-1} and P^{m-1} respectively. Furthermore, it can also be seen that the natural projection of T onto B is homotopically equivalent to the universal covering map of S^{m-1} onto P^{m-1}. Hence ψ is an isomorphism of $\pi_n(T, e_1)$ onto $\pi_n(B, x_1)$, for every $n > 1$. Since $\pi_1(T, e_1) = 0$, it follows from the exactness of the sequence that $\pi_n(F, e_1) = 0$ for every $n > 0$. If $m = 2$, then both T and B are of the same homotopy type as the circle S^1, and the natural projection of T onto B is of degree 2. This implies also that $\pi_n(F, e_1) = 0$ for every $n > 0$. Hence, for every $m > 1$, we have $\pi_n(F, e_1) = 0$ and $H_n(F) = 0$ for every $n > 0$. By means of the exact sequence, we can also deduce that F has two path-components.

7. THE DERIVED FIBERING

Let $f: X \to Y$ be a given fibering over the base space Y and with a set $Y_0 \subset Y$ as singularities. Consider an isolated point y_0 of Y_0. Then there is an open neighborhood N of y_0 in Y which contains no singular point other than y_0. According to our assumption made at the end of Section 3, the singular fiber $f^{-1}(y_0)$ consists of a single point x_0 of X. Let $M = f^{-1}(N)$. Then M is an open neighborhood of x_0 in X, and the restriction of f on M is a fibering over N with x_0 as the only singular fiber. In the remainder of this paper, we are concerned with the local properties of the fibering f at an isolated singular fiber x_0. Hence we may assume that x_0 is the only singular fiber of the given fibering $f: X \to Y$.

Throughout the remainder of this section, let $f: X \to Y$ be a given fibering over the base space Y and with a point $x_0 \in X$ as the only singular fiber. Let $y_0 = f(x_0) \in Y$. Then $f^{-1}(y_0) = x_0$, and therefore the continuous map

$$f: (X, x_0) \to (Y, y_0)$$

is admissible in the sense of [2, Section 4]. Consider the extended tangent spaces $E(X, x_0)$ and $E(Y, y_0)$. Then the admissible map f induces a continuous map

$$\hat{f}: E(X, x_0) \to E(Y, y_0)$$

defined by $\hat{f}(\xi) = f \circ \xi$ for every $\xi: I \to X$ in $E(X, x_0)$. Let us denote by ξ_0 and η_0 the degenerate paths $\xi_0(I) = x_0$ and $\eta_0(I) = y_0$. Then we have

$$E(X, x_0) = \xi_0 \cup T(X, x_0),$$

$$E(Y, y_0) = \eta_0 \cup T(Y, y_0),$$

$$f^{-1}(\eta_0) = \xi_0.$$

Hence, $\hat{f}$ is an admissible map of $(E(X, x_0), \xi_0)$ into $(E(Y, y_0), \eta_0)$.

It is natural to inquire whether $\hat{f}$ is a fibering. For this purpose, let us introduce the following notion. The given fibering $f: X \to Y$ is said to be *convergent* at the singular fiber x_0 if, for every open neighborhood U of x_0 in X, there exists an open neighborhood V of y_0 in Y such that $f^{-1}(V) \subset U$. Intuitively, f is convergent at the singular fiber x_0 if and only if the fibers over points near to y_0 are to be near x_0.

PROPOSITION 7.1. *If the fibering* $f: X \to Y$ *is convergent at its only singular fiber* x_0, *then the induced map* $f: E(X, x_0) \to E(Y, y_0)$ *sends* $E(X, x_0)$ *onto* $E(Y, y_0)$.

Proof. Let η be an arbitrary point in $E(Y, y_0)$. We shall construct a point ξ in $E(X, x_0)$ such that $\hat{f}(\xi) = \eta$. Since $\hat{f}(\xi_0) = \eta_0$, we may assume that η is in $T(Y, y_0)$. Hence η is a path $\eta: I \to Y$ such that $\eta^{-1}(y_0) = 0$. Let $y_1 = \eta(1)$, and pick a point x_1 in X with $f(x_1) = y_1$. For each positive integer n, let I_n denote the subspace of the unit interval I defined by

$$I_n = \{t \in I: 1/n \leq t \leq 1\}.$$

We shall construct inductively a sequence of continuous maps $\xi_n: I_n \to X$ $(n = 1, 2, \cdots)$ satisfying the following three conditions for every positive integer n:

(1) $$\xi_1(I_1) = x_1;$$

(2) $$\xi_{n+1}|I_n = \xi_n;$$

(3) $$f \circ \xi_n = \eta | I_n.$$

In fact, ξ_1 is determined by the condition (1). Assume that $n \geq 1$ and that ξ_n has already been constructed. Let J_n denote the subspace of I defined by

$$J_n = \{ t \in I: 1/(n+1) \leq t \leq 1/n \}.$$

Since the restriction of f on $X \setminus x_0$ is a fibering over $Y \setminus y_0$ without singularity, there exists a continuous map $\phi_n: J_n \to X$ such that

(4) $$\phi_n(1/n) = \xi_n(1/n),$$

(5) $$f \circ \phi_n = \eta | J_n.$$

The condition (4) enables us to define a continuous map $\xi_{n+1}: I_{n+1} \to X$ by taking

$$\xi_{n+1}(t) = \begin{cases} \xi_n(t) & \text{if } t \in I_n, \\ \phi_n(t) & \text{if } t \in J_n. \end{cases}$$

Then we have (2) and $f \circ \xi_{n+1} = \eta | I_{n+1}$. This completes the inductive construction of the sequence $\{\xi_n\}$.

Because of the condition (2), we can define a function $\xi: I \to X$ by taking

$$\xi(t) = \begin{cases} x_0 & \text{if } t = 0, \\ \xi_n(t) & \text{if } t \in I_n. \end{cases}$$

By the convergence of f at x_0, it follows that ξ is continuous. By the condition (3), we have $f \circ \xi = \eta$. Hence $\xi \in T(X, x_0)$ and $\hat{f}(\xi) = \eta$. This completes the proof of (7.1).

If f is not convergent at x_0, then $\hat{f}$ might fail to be onto. For example, let X be the subspace of the euclidean plane R^2 consisting of a point $x_0 = (s_0, 0)$, where $s_0 \leq 0$, and the totality of points (s, t) satisfying $0 < s \leq 1$ and $0 \leq t \leq 1$. Take the unit interval $I = [0, 1]$ to be the base space Y with $y_0 = 0$. Consider the map $f: X \to Y$ defined by

$$f(s, t) = \begin{cases} 0 & \text{if } s = s_0 \text{ and } t = 0, \\ s & \text{if } 0 < s \leq t \text{ and } 0 \leq t \leq 1. \end{cases}$$

Then f is a fibering over Y with x_0 as the only singular fiber. In the case $s_0 < 0$, x_0 is an isolated point of X, and hence $E(X, x_0) = \xi_0$. Since $T(Y, y_0)$ is not empty, this shows that $\hat{f}$ is not onto if $s_0 < 0$. For the remaining case $s_0 = 0$, one can easily see that $\hat{f}$ is onto. Since f is obviously not convergent at x_0, this example shows also that the condition in (7.1) is by no means necessary.

THEOREM 7.2. *If the fibering* $f: X \to Y$ *is convergent at its only singular fiber* x_0, *then the induced map* $\hat{f}: E(X, x_0) \to E(Y, y_0)$ *is a fibering over* $E(Y, y_0)$ *with* ξ_0 *as its only singular fiber.*

Proof. By (7.1), $\hat{f}$ maps $E(X, x_0)$ onto $E(Y, y_0)$. Hence, it remains to prove that the restriction

$$\bar{f}: T(X, x_0) \to T(Y, y_0)$$

of $\hat{f}$ has the covering homotopy property for every given finitely triangulable space K. For this purpose, let us consider any given continuous map $\phi: K \to T(X, x_0)$. Let $\psi = \bar{f} \circ \phi$, and let $\psi_t: K \to T(Y, y_0)$ $(0 \leq t \leq 1)$ be a given homotopy with $\psi_0 = \psi$. We have to construct a homotopy $\phi_t: K \to T(X, x_0)$ $(0 \leq t \leq 1)$ such that $\phi_0 = \phi$ and $\bar{f} \circ \phi_t = \psi_t$ for every $t \in I$.

Define a continuous map $\Psi: K \times I \times I \to Y$ by setting

$$\Psi(k, s, t) = [\psi_t(k)](s) \qquad (k \in K,\ s \in I,\ t \in I).$$

Then it suffices to construct a continuous map $\Phi: K \times I \times I \to X$ such that $f \circ \Phi = \Psi$ and

$$\Phi(k, s, 0) = [\phi(k)](s) \qquad (k \in K,\ s \in I).$$

To construct such a map Φ, we shall make use of the subspaces I_n and J_n of the unit interval I defined in the proof of (7.1). We shall construct inductively a sequence of continuous maps $\Phi_n: K \times I_n \times I \to X$ $(n = 1, 2, \cdots)$ satisfying the following three conditions for every positive integer n:

(1) $$\Phi_{n+1} | K \times I_n \times I = \Phi_n;$$

(2) $$f \circ \Phi_n = \Psi | K \times I_n \times I;$$

(3) $$\Phi_n(k, s, 0) = [\phi(k)](s) \qquad (k \in K,\ s \in I_n).$$

Since the restriction of f on $X \setminus x_0$ is a fibering over $Y \setminus y_0$ without singularity, the existence of a map Φ_1 satisfying (2) and (3) follows from the covering homotopy property of f. Assume that $n \geq 1$ and that Φ_n has already been constructed. Since the closed subspace

$$(K \times J_n \times 0) \cup (K \times 1/n \times I)$$

of the space $K \times J_n \times I$ is a strong deformation retract of the latter, it is a well-known result of the covering homotopy property of f over $Y \setminus y_0$ that there exists a continuous map

$$F_n: K \times J_n \times I \to X$$

satisfying the following three conditions:

(4) $$F_n(k, s, 0) = [\phi(k)](s) \qquad (k \in K,\ s \in J_n),$$

(5) $$F_n(k, 1/n, t) = \Phi_n(k, 1/n, t) \qquad (k \in K,\ t \in I),$$

(6) $$f \circ F_n = \Psi | K \times J_n \times I.$$

The condition (5) enables us to define a continuous map $\Phi_{n+1}: K \times I_{n+1} \times I \to X$ by taking

$$\Phi_{n+1}(k, s, t) = \begin{cases} \Phi_n(k, s, t) & (k \in K, \ s \in I_n, \ t \in I), \\ F_n(k, s, t) & (k \in K, \ s \in J_n, \ t \in I). \end{cases}$$

This completes the inductive construction of the sequence $\{\Phi_n\}$.

Because of the condition (1), we can define a function $\Phi: K \times I \times I \to X$ by taking

$$\Phi(k, s, t) = \begin{cases} x_0 & \text{if } s = 0, \\ \Phi_n(k, s, t) & \text{if } x \in I_n. \end{cases}$$

By the convergence of f at x_0, it follows that Φ is continuous. By the condition (2), we have $f \circ \Phi = \Psi$. By the condition (3), $\Phi(k, s, 0) = [\phi(k)](s)$ for every $k \in K$ and $s \in I$. This completes the proof of (7.2).

The converse of (7.2) is false. For example, if

$$X = E(Y, y_0), \qquad x_0(I) = y_0,$$

and if the given fibering $f: X \to Y$ is the natural projection defined in Section 4, then the induced map $\hat{f}: E(X, x_0) \to E(Y, y_0)$ is a fibering over $E(Y, y_0)$ with x_0 as its only singularity according to Proposition 5.2. On the other hand, if Y is a pathwise connected T_1-space and has more than one point, then it is easy to see that the natural projection $f: X \to Y$ as a fibering with only one singular fiber x_0 is by no means convergent at x_0.

The example given above shows that the condition that a fibering $f: X \to Y$ be convergent at x_0 is rather unsatisfactory. Therefore, we suggest the following artificial notion of a *normal singularity*. A fibering $f: X \to Y$ with only one singular fiber x_0 is said to be *normal* at x_0 if and only if the induced map

$$\hat{f}: E(X, x_0) \to E(Y, y_0)$$

is a fibering over $E(Y, y_0)$ with ξ_0 as the only singular fiber. If this is the case, then y_0 is called a *normal singularity*, x_0 is called a *normal singular fiber*, and $\hat{f}$ is called the *derived fibering*.

8. THE INDUCED HOMOMORPHISMS

Consider two triplets (X, A, x_0), (Y, B, y_0) in the sense of [1, p. 491], and a continuous map

$$f: (X, A, x_0) \to (Y, B, y_0)$$

which is admissible in the sense that $f^{-1}(y_0) = x_0$ or, equivalently, $f(X \setminus x_0) \subset Y \setminus y_0$. Let

$$\hat{X} = T(X, x_0), \qquad \hat{A} = T(A, x_0), \qquad \hat{Y} = T(Y, y_0), \qquad \hat{B} = T(B, y_0).$$

Then the admissible map f induces a continuous map

$$\hat{f}: (\hat{X}, \hat{A}) \to (\hat{Y}, \hat{B})$$

defined by $\hat{f}(\sigma) = f \circ \sigma$ for every $\sigma: I \to X$ in $\hat{X}$. According to [2, Section 4], the map $\hat{f}$ gives rise to the induced homomorphisms

$$f_*: L_n(X, A, x_0; G) \to L_n(Y, B, y_0; G),$$

$$f_*: L^n(Y, B, y_0; G) \to L^n(X, A, x_0; G)$$

of the local homology and cohomology groups for any abelian coefficient group G. Let $\sigma: I \to A$ be any point in $\hat{A}$, and let $\tau = f \circ \sigma \in \hat{B}$. Then, by [2, Section 13], the map $\hat{f}$ also gives rise to the induced homomorphisms

$$f_\#: \lambda_n(X, A, x_0; \sigma) \to \lambda_n(Y, B, y_0; \tau)$$

of the local homotopy groups.

Now let us assume that f is a fibering over Y with x_0 as the only singular fiber, and that $A = f^{-1}(B)$. If f is normal at x_0 in sense of the preceding section, then the induced map $\hat{f}$ is a fibering over $\hat{Y}$ without singularity. Furthermore, the condition $A = f^{-1}(B)$ implies that $\hat{A} = \hat{f}^{-1}(\hat{B})$. Hence, the fibering theorem for the global homotopy groups, [1, p. 495], gives the following *fibering theorems* for the local homotopy groups:

THEOREM 8.1. *If* $f: X \to Y$ *is a fibering over* Y *normal at its only singular fiber* x_0, *if* B *is a subspace of* Y *containing the point* $y_0 = f(x_0)$, *and if* $A = f^{-1}(B)$, *then*

$$f_\#: \lambda_n(X, A, x_0; \sigma) \approx \lambda_n(Y, B, y_0; f \circ \sigma)$$

for every $\sigma: I \to A$ *in* $T(A, x_0)$ *and every* $n > 0$.

On the other hand, for every $\sigma: I \to X$ in $\hat{X} = T(X, x_0)$ and $\tau = f \circ \sigma \in \hat{Y} = T(Y, y_0)$, the induced map $\hat{f}$ also gives the induced homomorphisms

$$f_\#: \lambda_n(X, x_0; \sigma) \to \lambda_n(Y, y_0; \tau)$$

for every $n > 0$. Write

$$\hat{F} = \hat{f}^{-1}(\tau) \subset T(X, x_0).$$

If f is a fibering over Y normal at its only singular fiber, then it follows that $\hat{f}$ is a fibering over $\hat{Y}$ without singularity and with $\hat{F}$ as the fiber over the point $\tau \in \hat{Y}$. Hence, the homotopy sequence of the fibering $\hat{f}$ gives the exact sequence

$$\cdots \to \lambda_{n+1}(Y, y_0; \tau) \to \pi_n(\hat{F}, \sigma) \to \lambda_n(X, x_0; \sigma) \to \lambda_n(Y, y_0; \tau) \to \cdots \to \pi_0(\hat{F}, \sigma),$$

which will be called the *local homotopy* sequence of the fibering f at the normal singular fiber x_0.

For the given fibering $f: X \to Y$, the induced transformations

$$f_*: L_n(X, x_0; G) \to L_n(Y, y_0; G),$$

$$f_*: L^n(Y, y_0; G) \to L^n(X, x_0; G),$$

$$f_\#: \lambda_n(X, x_0; \sigma) \to \lambda_n(Y, y_0; f \circ \sigma)$$

are the main local invariants of f at the isolated singular fiber x_0. In the study of these local invariants, it turns out that the global homology and homotopy structure of the subspace $\hat{F}$ of the tangent space $T(X, x_0)$ will play an important role.

9. THE SPECTRAL SEQUENCE

Consider a given fibering

$$f: X \to Y$$

over Y with x_0 as its only singular fiber which is normal in the sense of Section 7. Let

$$y_0 = f(x_0), \qquad \hat{X} = T(X, x_0), \qquad \hat{Y} = T(Y, y_0).$$

Then the induced map $\hat{f}: \hat{X} \to \hat{Y}$ is a fibering over $\hat{Y}$ without singularity. Pick a point τ in $\hat{Y}$ and denote by $\hat{F}$ the fiber $f^{-1}(\tau)$ over τ.

For the sake of simplicity, we assume that the space Y is simply connected around the point y_0 in the sense of [2, Section 15]; in other words, we assume that the tangent space $\hat{Y}$ is simply connected in the usual global sense.

Applying Leray's theory of spectral sequence to the singular homology groups by means of the fibering $\hat{f}$, we obtain a spectral sequence

$$\{E^n: n = 1, 2, 3, \cdots\}$$

of bigraded groups as in [6], where $E^2_{p,q}$ is isomorphic with the local homology group

$$L_p(Y, y_0; H_q(\hat{F}))$$

of the base space Y at the point y_0 with the (global) homology group $H_q(\hat{F})$ as coefficient group. Let E^∞ denote the bigraded limit group of this spectral sequence. Then the total local homology group $L(X, x_0)$ of the space X at the point x_0 is filtered with E^∞ as its associated graded group; precisely, we have a natural sequence of subgroups of $L_m(X, x_0)$ satisfying the relations

$$L_m(X, x_0) = G_{m,0} \supset G_{m-1,1} \supset \cdots \supset G_{0,m} \supset G_{-1,m+1} = 0,$$

$$G_{p,q}/G_{p-1,q+1} \approx E^\infty_{p,q}.$$

Just as in the global theory, one can deduce various consequences and applications from this spectral sequence. For example, if we assume that Y is locally euclidean of dimension $r + 1$ $(r \geq 2)$ at the point y_0, then $\hat{Y}$ is a homology r-sphere, and hence we obtain an exact sequence

$$\cdots \to H_{n-r+1}(\hat{F}) \to H_m(\hat{F}) \to L_m(X, x_0) \to H_{m-r}(\hat{F}) \to \cdots$$

analogous to Wang's sequence [7] in the global theory. In many cases, this exact sequence can be used to determine the homology groups of $\hat{F}$. A few examples of these cases are given as follows.

First, consider the case where x_0 is a boundary point of X in the sense of [2, Section 19], that is to say, where the tangent space $T(X, x_0)$ is contractible. In this case, $L_0(X, x_0) \approx Z$ and $L_m(X, x_0) = 0$ for every $m \neq 0$. Then the exactness of the sequence implies that

$$H_0(\hat{F}) \approx L_0(X, x_0) \approx Z,$$

$$H_m(\hat{F}) \approx H_{m-r+1}(\hat{F}) \qquad (m \geq 1).$$

Hence, for any $m \geq 0$, $H_m(\hat{F})$ is infinite cyclic if m is a multiple of $r - 1$, and $H_m(\hat{F}) = 0$ otherwise.

Next, consider the case where $r > 2$ and X is locally euclidean of dimension $q + 1$ with

$$q = k(r - 1) \qquad (k > 1).$$

In this case, $L_m(X, x_0) = 0$ if $m(m - q) \neq 0$, and $L_0(X, x_0) \approx Z \approx L_q(X, x_0)$. Then the exactness of the sequence implies that

$$H_0(\hat{F}) \approx L_0(X, x_0) \approx Z,$$

$$H_m(\hat{F}) \approx H_{m-r+1}(\hat{F}) \qquad (0 < m < q - 1 \text{ or } q < m).$$

Hence, if $0 \leq m < q - 1$, $H_m(\hat{F})$ is infinite cyclic in case m is a multiple of $r - 1$, and $H_m(\hat{F}) = 0$ otherwise. To determine $H_{q-1}(\hat{F})$, consider the following part of the exact sequence:

$$H_{q-r}(\hat{F}) \to H_{q-1}(\hat{F}) \to L_{q-1}(X, x_0).$$

Since $q - r = (k - 1)(r - 1) - 1$ and $r > 2$, $q - r$ is not a multiple of $r - 1$, and hence $H_{q-r}(\hat{F}) = 0$. On the other hand, $L_{q-1}(X, x_0) = 0$. Hence we obtain $H_{q-1}(\hat{F}) = 0$. To determine the group $H_q(\hat{F})$, consider the following part of the exact sequence:

$$L_{q+1}(X, x_0) \to H_{q-r+1}(\hat{F}) \to H_q(\hat{F}) \to L_q(X, x_0) \to H_{q-r}(\hat{F}).$$

Since $L_{q+1}(X, x_0) = 0 = H_{q-r}(\hat{F})$ and $H_{q-r+1}(\hat{F}) \approx Z \approx L_q(X, x_0)$, it follows that $H_q(\hat{F}) \approx Z + Z$. Thus, for each $m \geq 0$, the homology group $H_m(\hat{F})$ is as follows:

$$H_m(\hat{F}) \approx \begin{cases} 0 & \text{if } m \not\equiv 0 \pmod{r - 1}, \\ Z & \text{if } m \equiv 0 \pmod{r - 1},\ m < q, \\ Z + Z & \text{if } m \equiv 0 \pmod{r - 1},\ m \geq q. \end{cases}$$

Next, consider the case where $r > 3$ and X is locally euclidean of dimension $q + 1$ with

$$q = k(r - 1) + j \qquad (k > 1,\ 1 < j < r - 1).$$

By methods similar to those used in the previous case, the homology groups $H_m(\hat{F})$ can be computed. The result is as follows: $H_m(\hat{F})$ is infinite cyclic if $m = p(r - 1)$ or $m = q + p(r - 1)$ $(p \geq 0)$, and $H_m(\hat{F}) = 0$ otherwise.

Finally, consider the case where $r > 2$ and X is locally euclidean of dimension $q + 1$ with

$$q = k(r - 1) + 1 \qquad (k \geq 1).$$

For this case, the following part of the exact sequence is crucial:

$$0 \to H_q(\hat{F}) \to L_q(X, x_0) \xrightarrow{\phi} H_{q-r}(\hat{F}) \to H_{q-1}(\hat{F}) \to 0,$$

where we have $L_q(X, x_0) \approx Z$ and $H_{q-r}(\hat{F}) \approx Z$. The homomorphism ϕ determines a nonnegative integer h such that, if $\alpha \in L_q(X, x_0)$ and $\beta \in H_{q-r}(\hat{F})$ are generators, then $\phi(\alpha) = \pm h\beta$. This integer h will be called the *local Hopf invariant* of the map f at x_0. The homology groups $H_m(\hat{F})$ depend on this integer h as follows:

(1) If $h = 0$, then $H_{q-1}(\hat{F}) \approx Z$ and $H_q(\hat{F}) \approx Z$. Hence, $H_m(\hat{F})$ is infinite cyclic if $m = p(r - 1)$ or $m = q + p(r - 1)$ $(p \geq 0)$, and $H_m(\hat{F}) = 0$ otherwise.

(2) If $h = 1$, then $H_{q-1}(\hat{F}) = 0$ and $H_q(\hat{F}) = 0$. Hence, $H_m(\hat{F})$ is infinite cyclic if $m = p(r - 1)$ with $p = 0, 1, \cdots, k - 1$, and $H_m(\hat{F}) = 0$ otherwise.

(3) If $h \neq 0$ and $h \neq 1$, then $H_{q-1}(\hat{F}) \approx Z_h$ and $H_q(\hat{F}) = 0$. Hence $H_m(\hat{F})$ is infinite cyclic if $m = p(r - 1)$ with $p = 0, 1, \cdots, k - 1$; $H_m(\hat{F})$ is cyclic of finite order h if $m = p(r - 1)$ with $p \geq k$; and $H_m(\hat{F}) = 0$ otherwise.

10. THE CONE CONSTRUCTION

Consider a giving fibering

$$p: E \to B$$

over B without singularity. Define a continuous map

$$\phi: E \times I \to B \times I$$

by setting $\phi(e, t) = (p(e), t)$ for every $e \in E$ and $t \in I$. Then ϕ is a fibering over $B \times I$ without singularity. If we identify the subset $E \times 1$ of the space $E \times I$ to a single point x_0, we obtain a quotient space X of $E \times I$ known as the *cone* over E with x_0 as its *vertex*. Similarly, we identify the subset $B \times 1$ of the space $B \times I$ to a single point y_0 and obtain the cone Y over B with y_0 as vertex. Let

$$\xi: E \times I \to X, \qquad \eta: B \times I \to Y$$

denote the natural projections. Then the map ϕ induces a unique map $f: X \to Y$ such that the rectangle

$$\begin{array}{ccc} E \times I & \xrightarrow{\phi} & B \times I \\ \downarrow \xi & & \downarrow \eta \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative. It is obvious that f is a fibering over Y with $x_0 = f^{-1}(y_0)$ as its only singular fiber. Furthermore, one can easily see that f is convergent at x_0, and hence that x_0 is a normal singular fiber.

Let $\hat{X} = T(X, x_0)$ and $\hat{Y} = T(Y, y_0)$. Then the normality of x_0 implies that the induced map

$$\hat{f}: \hat{X} \to \hat{Y}$$

is a fibering over $\hat{Y}$ without singularity. We shall see that this fibering $\hat{f}$ is essentially the given fibering $p: E \to B$.

For this purpose, define two continuous maps $\iota: E \to \hat{X}$ and $\kappa: B \to \hat{Y}$ as follows. For each $e \in E$ and $b \in B$, let $\iota(e)$ and $\kappa(b)$ denote the paths in X and Y respectively given by

$$[\iota(e)](t) = \xi(e, 1 - t), \qquad [\kappa(b)](t) = \eta(b, 1 - t) \qquad (t \in I).$$

It is clear that $\iota(e) \in \hat{X}$ and $\kappa(b) \in \hat{Y}$. One can also verify that ι and κ are homeomorphisms of E and B into the tangent spaces $\hat{X}$ and $\hat{Y}$. Furthermore, the rectangle

$$\begin{array}{ccc} E & \xrightarrow{\quad p \quad} & B \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \kappa} \\ \hat{X} & \xrightarrow{\quad \hat{f} \quad} & \hat{Y} \end{array}$$

is commutative. Hence, we may consider E, B as subspaces of $\hat{X}$, $\hat{Y}$, respectively, and then p becomes the restriction of $\hat{f}$.

Now, let us introduce a notion as follows. The map $p: E \to B$ is said to be a *strong deformation retract* of the map $\hat{f}: \hat{X} \to \hat{Y}$ if $E \subset \hat{X}$, $B \subset \hat{Y}$, $p = \hat{f}|E$, and if there exist two homotopies

$$h_t: \hat{X} \to \hat{X}, \; k_t: \hat{Y} \to \hat{Y} \qquad (0 \le t \le 1)$$

satisfying the following conditions:

(SDR 1) h_0 and k_0 are identity maps;

(SDR 2) h_1 and k_1 are retractions of $\hat{X}$ and $\hat{Y}$ onto E and B respectively;

(SDR 3) $h_t|E$ and $k_t|B$ are the inclusion maps for each $t \in I$;

(SDR 4) commutativity holds in the rectangle

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\quad \hat{f} \quad} & \hat{Y} \\ \downarrow{\scriptstyle h_t} & & \downarrow{\scriptstyle k_t} \\ X & \xrightarrow{\quad \hat{f} \quad} & Y \end{array}$$

THEOREM 10.1. *The given fibering* $p: E \to B$ *is a strong deformation retract of* $\hat{f}: \hat{X} \to \hat{Y}$.

Proof. We have to construct the homotopies h_t and k_t satisfying (SDR 1-4).

We first define a continuous real function $\chi: X \to I$ by setting

$$\chi(x) = q\,\xi^{-1}(x) \qquad (x \in X),$$

where $\xi: E \times I \to X$ and $q: E \times I \to I$ denote the natural projections. Then we define a homotopy $h_t: \hat{X} \to \hat{X}$ $(0 \leq t \leq 1)$ as follows. Let $\sigma: I \to X$ be any point in $\hat{X} = T(X, x_0)$. If $0 \leq t \leq 1/2$, $h_t(\sigma) \in \hat{X}$ is obtained by replacing the part of the path σ up to the parametric value $2t$ by the line-segment joining x_0 to $\sigma(2t)$; in particular, $h_{1/2}(\sigma)$ is the line-segment joining x_0 to $\sigma(1)$. If $1/2 \leq t \leq 1$, then $h_t(\sigma) \in \hat{X}$ is obtained by extending the line-segment $h_{1/2}(\sigma)$ to a position where the value of the function χ is $2a - 2at$, where $a = \chi[\sigma(1)]$. Similarly, one can define the homotopy $k_t: \hat{Y} \to \hat{Y}$ $(0 \leq t \leq 1)$. Then the conditions (SDR 1) through (SDR 4) can be verified without difficulty. Hence (10.1) has been proved.

Now, let $b \in B$ be a given point. Write

$$F = p^{-1}(b), \qquad \tau = \kappa(b) \in \hat{Y}, \qquad \hat{F} = \hat{f}^{-1}(\tau).$$

Since $k_t(\tau) = \tau$ by (SDR 3), it follows from (SDR 4) that h_t maps $\hat{F}$ into itself. Hence we get the following

COROLLARY 10.2. *The spaces* E, B, F *are strong deformation retracts of the spaces* $\hat{X}$, $\hat{Y}$, $\hat{F}$ *respectively.*

The significance of (10.1) and (10.2) is that the theory of fiberings with a single normal singularity includes the classical theory of fiberings without singularity as a special case.

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