A GENERALIZATION OF BAGEMIHL'S THEOREM ON AMBIGUOUS POINTS

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Let f[z] be a function mapping the open unit disk $|z| < 1$ into the Riemann sphere. The point p on $|z| = 1$ is an *ambiguous point of* f (see [4]) if there exist two arcs A_1 and A_2, each with one end point at p, lying in the open disk except for p, and such that the limits of f at p along A_1 and along A_2 exist and are unequal. Bagemihl [1] proved a remarkable theorem: *even if* f *is not assumed to be continuous, it can have at most countably many points of ambiguity.* (Bagemihl's result was actually stronger; it is the case $n = 2$ of the Theorem below.) Bagemihl and Seidel [4] showed that every countable set on the unit circle is contained in the set of ambiguous points of some meromorphic function of bounded characteristic; and Lohwater and Piranian [7] have strengthened this by proving that every countable set on $|z| = 1$ is exactly the set of ambiguous points for some such function. It follows immediately from these results (or indeed from the existence of even one ambiguous point for a function in the unit disk) that if ambiguous points of a function in the (n - 1)-sphere of n-space are defined in terms of asymptotic behavior on arcs, in the obvious fashion, then there exist functions in the (n - 1)-sphere which have uncountably many ambiguous points.

But several other possible generalizations to higher dimensions suggest themselves. One might expect, for example, that the ambiguous points can not fill a cell on the (n - 1)-sphere. This possibility has been pretty thoroughly demolished by Bagemihl [2], [3], Piranian [9], and Church [5], who give examples of functions on the interior of the 2-sphere in E^3 for which the set of ambiguous points is a 2-cell; Church's example is a differentiable homeomorphism. In this note, I give a generalization in the spirit of Bagemihl's theorem.

I now define a "cell of disjoint cluster sets." Let D be a domain in Euclidean n-space E^n, and let $f: D \to S$ be a function from D into a topological space. A closed r-cell I in the boundary of D is an *r-cell of disjoint cluster sets for* f provided there exist two closed (r + 1)-cells J_1 and J_2, lying in D except for I (which is in the combinatorial boundary of each), such that the cluster set on I from J_1 of f does not meet the cluster set on I from J_2 of f; in other words, such that if $\{p_k\}$ is a sequence of points in J_1 - I, converging to a point p of I, and $\{q_k\}$ is a sequence of points in J_2 - I, converging to a point q of I, and $\lim f(p_k)$, $\lim f(q_k)$ exist in S, then they are not equal. (Clearly, this definition can be freed from the concept of sequences; however, I intend to apply it only where S is a compact metric space).

The examples mentioned in the second paragraph can easily be modified to show that there exist functions in the interior of the unit (n - 1)-sphere S^{n-1} in E^n such that there are uncountably many disjoint (n - 3)-cells of disjoint cluster sets that fill an (n - 1)-cell in S^{n-1} $(n > 2)$. The principal theorem of this paper is this, that there cannot be uncountably many disjoint (n - 2)-cells of disjoint cluster sets.

For completeness, we prove a lemma of a familiar type about the image space to be used.

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LEMMA 1. *Let* $\{H_\alpha, K_\alpha\}$ *be an uncountable collection of pairs of disjoint closed sets in the compact metric space* M. *Then there exist disjoint open sets* U *and* V *in* M *such that, for uncountably many values of* α, H_α *is in* U *and* K_α *is in* V.

Proof. Let d(x, y) be the metric for M. Setting

$$d(H_\alpha, K_\alpha) = \inf [d(h_\alpha, K_\alpha); h_\alpha \in H_\alpha, k_\alpha \in K_\alpha],$$

we can find an integer n such that $d(H_\alpha, K_\alpha) > 3/n$ for uncountably many α; there is no loss in assuming that the inequality holds for all α. The hyperspace 2^M of all closed subsets of M has a countable basis [6, p. 120] and so, considering the sets H_α as points of 2^M, we see that one of them, H_β, is a point of condensation of the rest; hence the spherical neighborhood of H_β of radius $1/n$ in the hyperspace contains, as points, uncountably many sets H_α. The hyperspace metric defined in [6] is such that this implies that the set U of all points x for which $d(x, H_\beta) < 1/n$ contains all these H_α. For these values of α, we find that one of the sets K_α, say K_γ, has the property that the set V of all points of M at distance less than $1/n$ from K_γ contains uncountably many of the K_α. Then U and V are the desired sets.

THEOREM. *Suppose that* $f: D \to M$ *is a function defined from the interior* D *of the unit* (n - 1)*-sphere* S^{n-1} *in* E^n *into the compact metric space* M. *Then* S^{n-1} *does not contain uncountably many disjoint* (n - 2)*-cells of disjoint cluster sets of* f.

Proof. Let $\{I_\alpha\}$ be a collection of disjoint (n - 2)-cells of disjoint cluster sets for such a function f. For each α, let $J_{1\alpha}$, $J_{2\alpha}$ be the (n - 1)-cells containing the cell I_α of the definition. Let $C_{1\alpha}$, $C_{2\alpha}$ denote the cluster sets at I_α from $J_{1\alpha}$ and $J_{2\alpha}$, respectively. Suppose $\{I_\alpha\}$ is uncountable. By means of nine lemmas, we shall arrive at a contradiction.

LEMMA 2. *There is no loss in assuming that, for each* α, $(J_{1\alpha} - I) \cap (J_{2\alpha} - I)$ *is empty.*

Proof. For each α, the cluster sets $C_{1\alpha}$, $C_{2\alpha}$ form a pair of disjoint closed sets in M. By Lemma 1, there exist disjoint open sets U_1 and U_2 in M such that, for uncountably many values of α, $C_{1\alpha}$ is in U_1 and $C_{2\alpha}$ is in U_2. For each such α, there exists an open set W_α in E^n containing I_α such that $f[(J_{i\alpha} - I_\alpha) \cap W_\alpha]$ is in U_i (i = 1, 2). Replace $J_{1\alpha}$, $J_{2\alpha}$ by (n - 1)-cells containing I_α and lying in W_α. With these new cells as $J_{1\alpha}$, $J_{2\alpha}$, and for this uncountable subcollection of the indices α, the conclusion of the lemma holds.

The remainder of the proof of our theorem consists in showing that we can also assume that $J_{1\alpha} \cap J_{1\beta}$ and $J_{2\alpha} \cap J_{2\beta}$ are empty $(\alpha \neq \beta)$, and in deriving a contradiction from this. In the course of the argument, we shall both drop some of the indices α, and replace the remaining cells I_α by subcells. Presumably, it would be possible to keep the original cells I_α; but this would complicate the argument.

LEMMA 3. *Let* V *be an open* n*-cell in* E^n *(or in* S^n*). Let* J *be an* (n - 1)*-cell which intersects* V *and whose boundary lies in* $E^n - V$. *Let* K *be a component of* $J \cap V$. *Then* K *separates* V *into exactly two components, each having* K *as boundary in* V.

Proof. Considered as a space, V is an orientable acyclic n-manifold, and K is an orientable (n - 1)-manifold; therefore the lemma is an immediate consequence of a result of Wilder [10, Chap. X, Theorem 3.1].

LEMMA 4. *Let* $\{J_\alpha\}$ *be an uncountable collection of closed* (n - 1)*-cells in* E^n *(or* S^n*). Then there exist an open* n*-cell* V *and an uncountable subcollection* $\{J_{\alpha'}\}$

of $\{J_\alpha\}$ *such that* $\overline{V}$ *contains no point of the (combinatorial) boundary of any cell* $J_{\alpha'}$, *and such that* V *is separated by each* $J_{\alpha'}$.

Proof. For each α, let p_α be a point of J_α which does not lie in the boundary BJ_α of J_α. There exists an integer k such that, for uncountably many α, $d(p_\alpha, BJ_\alpha) > 2/k$. The set of all points p_α with such indices α has a point of condensation, p. Let V be the spherical neighborhood of radius 1/k about p. Then for uncountably many α, V contains p_α while $\overline{V} \cap BJ_\alpha$ is empty. For any of these values of α, each component of $V \cap J_\alpha$ separates V, by Lemma 3, so that, *a fortiori*, $V \cap J_\alpha$ separates V.

We now return to the main proof. Since we are concerned only with the part of the cell $J_{i\alpha}$ near S^{n-1}, there is no loss in assuming that the origin does not lie in any set $J_{i\alpha}$. Hence we can perform an inversion h: $(E^n - O) \rightarrow (E^n - O)$ about S^{n-1}, with the result that each set $h(J_{i\alpha}) = \tilde{J}_{i\alpha}$ is a closed (n - 1)-cell, and such that $J_{i\alpha} \cup \tilde{J}_{i\alpha}$ is also a closed (n - 1)-cell. (This "reflection" is convenient, but not really essential.) A double application of Lemma 4 shows that in E^n there exists an open n-cell V intersecting S^{n-1} in an open (n - 1)-cell V*, and having the properties that for an uncountable set of indices α, V is separated by $J_{i\alpha} \cup \tilde{J}_{i\alpha}$ but does not meet $B(J_{i\alpha} \cup \tilde{J}_{i\alpha})$, and that V* is separated by each I_α. Again suppose that this is true for all α. For each α, let $H_{1\alpha}$ be a component of $J_{1\alpha} \cup \tilde{J}_{1\alpha}$ that meets V*; let C_α be a component of $I_\alpha \cap V^*$ in $H_{1\alpha}$; and let $H_{2\alpha}$ be a component of $J_{2\alpha} \cup \tilde{J}_{2\alpha}$ containing C_α. (It may easily happen that $H_{1\alpha}$ and $H_{2\alpha}$ can intersect V* in other components.)

LEMMA 5. *Each set* $H_{i\alpha}$ (i = 1, 2) *is the union of a component of* $J_{i\alpha}$ *and its reflection under* h. *Furthermore,* $H_{i\alpha} \cap (J_{i\alpha} - I_\alpha)$ *is connected.*

Proof. First, let $K = H_{i\alpha} \cap J_{i\alpha}$, and let $\tilde{K} = H_{i\alpha} \cap \tilde{J}_{i\alpha}$. The symmetry of the construction assures us that $h(K) = \tilde{K}$ and $h(\tilde{K}) = K$. For otherwise $h(K \cup \tilde{K})$ would be connected, would contain $J_{i\alpha} \cap I_\alpha$, and would be larger than $H_{i\alpha}$, contrary to the fact that $H_{i\alpha}$ is a component. Now, by local connectivity of I_α and $J_{i\alpha}$, C_α is open and closed in $I_\alpha \cap V^*$, and the union of the closures of components of $K - I_\alpha$ that have limit points in C_α is therefore open and closed in $J_{i\alpha} \cap V$. This shows that each component of $K - I_\alpha \cap K$ has limit points in C_α. For otherwise K would not be connected, and symmetry would imply the same for $H_{i\alpha}$. Finally, it follows that $K - I_\alpha$ is connected; for K is an open connected subset of an (n - 1)-cell, and it cannot be disconnected by a subset of the boundary of the cell.

By Lemma 3, $H_{i\alpha}$ separates V into two connected open sets $A_{i\alpha}$ and $B_{i\alpha}$ (i = 1, 2).

LEMMA 6. *The set* $H_{2\alpha} - I_\alpha$ *lies entirely in* $A_{1\alpha}$ *or entirely in* $B_{1\alpha}$.

Proof. By Lemma 5, the set $H_{2\alpha} \cap J_{2\alpha} - I_\alpha$ lies entirely in $A_{1\alpha}$ or entirely in $B_{1\alpha}$, say in $A_{1\alpha}$. By construction, $h(A_{1\alpha}) = A_{1\alpha}$, and it follows that $H_{2\alpha} \cap \tilde{J}_{2\alpha} - I_\alpha$ also lies in $A_{1\alpha}$.

We have a similar lemma for $H_{1\alpha} - I_\alpha$.

Since there are only two cases in Lemma 6 and uncountably many α, we can suppose that one of the cases occurs for uncountably many α, and therefore we may also suppose that it occurs for all α. We now assume that for each α the set $H_{2\alpha} - I_\alpha$ lies entirely in $A_{1\alpha}$. The sets $A_{2\alpha}$ and $B_{2\alpha}$ are connected, and therefore one of them lies in $A_{1\alpha}$. Suppose, for all α, that $A_{2\alpha} \subset A_{1\alpha}$. Then $A_{2\alpha}$ is in $A_{1\alpha}$, and $B_{1\alpha}$ in $B_{2\alpha}$, and therefore both $H_{1\alpha}$ and $H_{2\alpha}$ separate $A_{2\alpha}$ from $B_{1\alpha}$.

LEMMA 7. *There exist two points of* V^* *which, for uncountably many* α, *are separated in* V *both by* $H_{1\alpha}$ *and by* $H_{2\alpha}$.

Proof. By symmetry, $A_{2\alpha}$ and $B_{1\alpha}$ both meet $D \cap V$ and also $h(D \cap V)$, and therefore both meet V^*. If X is a countable dense subset of V^*, there is a point of X in $A_{2\alpha}$, and one in $B_{1\alpha}$. Some two of these points, p and q, are separated by $H_{1\alpha}$ and by $H_{2\alpha}$, for uncountably many α. Let p be the one that is in $A_{2\alpha}$. Then p is separated from $H_{1\alpha} - I_\alpha \cap H_{1\alpha}$ by $H_{2\alpha}$ for every such α.

Again, suppose that the situation described in Lemma 7 holds for all α. Let α and β be different indices. By Lemma 2, the sets $H_{2\alpha}$ and $H_{1\beta}$ are disjoint, so that one separates the other from p. Suppose first that $H_{2\alpha}$ separates $H_{1\beta}$ from p. We need a more detailed analysis of the separation of V by the several sets involved.

LEMMA 8. *The sets* $B_{2\alpha} \cap A_{1\beta}$ *and* $B_{1\alpha} \cap A_{2\beta}$ *are connected; the first has as its boundary in* V *the set* $H_{2\alpha} \cup H_{1\beta}$; *and the second has as its boundary in* V *the set* $H_{1\alpha} \cup H_{2\beta}$.

Proof. Suppose that the set $B_{2\alpha} \cap A_{1\beta}$ is not connected. Since the set is open, each of its components is open, and it is therefore the union of a countable collection $\{U_k\}$ of connected open sets. For each k the set U_k has boundary points in $H_{2\alpha}$ and in $H_{1\beta}$, since otherwise U_k is either a component of $V - H_{2\alpha}$ or of $V - H_{1\beta}$, and U_k cannot contain p or q. This contradicts Lemma 3. Let W be an open set in V containing no point of $\overline{B}_{1\beta}$ and containing $H_{2\alpha}$. Using again Lemma 3, we see that $W - H_{2\alpha}$ is the union of two disjoint connected open sets W_1 and W_2. One of these lies in $A_{2\alpha}$, and the other, say W_2, in $B_{2\alpha}$. But W_2 intersects both U_i and U_j, so that $U_i \cup W_2 \cup U_j$ is connected; since this set lies in $B_{2\alpha} \cap A_{1\beta}$, this contradicts the fact that U_i and U_j are components. Hence $B_{2\alpha} \cap A_{1\beta}$ is connected. An application of Lemma 3 proves the assertion about its boundaries. We note that we have not used anything about $B_{2\alpha} \cap A_{1\beta}$ that is not equally true of $B_{1\alpha} \cap A_{2\beta}$.

LEMMA 9. *The set* $B_{1\alpha} \cap A_{2\beta}$ *is a subset of* $B_{2\alpha} \cap A_{1\beta}$.

Proof. If not, the set intersects $A_{2\alpha}$ or $B_{1\beta}$. If it intersects $A_{2\alpha}$, then the set $B_{1\beta} \cup (H_{1\beta} \cap I_\beta) \cup (B_{1\alpha} \cap A_{2\beta}) \cup A_{2\alpha}$ is connected (since $H_{1\beta} \cap I_\beta$ intersects $H_{2\beta} \cap I_\alpha$); it contains p and q, but does not meet $H_{2\alpha}$. A similar argument disposes of the other possibility.

LEMMA 10. *The sets* $H_{1\alpha} \cap H_{1\beta}$ *and* $H_{2\alpha} \cap H_{2\beta}$ *are both empty.*

Proof. Suppose y is a point in $H_{1\alpha} \cap H_{1\beta}$. Then y is not in I_α or I_β. The set $B_{1\beta} \cup y \cup (B_{1\alpha} \cap A_{2\beta}) \cup H_{2\alpha} \cup A_{2\alpha}$ is a connected set, since $H_{2\alpha} \cap I_\alpha$ contains limit points of $B_{1\alpha} \cap A_{2\beta}$; and it is a connected set from q to p that does not intersect $H_{2\beta}$. The other case is handled similarly.

This proves that all the sets $H_{i\alpha}$ and $H_{j\beta}$ $(i, j = 1, 2; \alpha \neq \beta)$ are disjoint. The same, then, is true for the sets $K_{i\alpha}$ and $K_{j\beta}$ $(i, j = 1, 2; \alpha \neq \beta)$.

To complete the proof of the theorem, we need the following generalization [11] of Moore's theorem on triods in the plane; E^n *does not contain uncountably many disjoint sets each of which is the union of an* $(n - 1)$*-cell* C *and an arc* A *having in common with* C *a point of the (combinatorial) interior of* C. Such a set is a T_{n-1}-*set*. Each of the sets $K_{1\alpha} \cup K_{2\alpha}$ contains a closed $(n - 1)$-cell C_α having an $(n - 1)$-cell of I_α in its (combinatorial) interior. For each α, there exists a rectilinear interval F_α which is perpendicular to S^{n-1}, has one end point in the interior of C_α, and is exterior to S^{n-1} except for that end point. Then $C_\alpha \cup F_\alpha$ is a T_{n-1}-set, and the collection of all sets $C_\alpha \cup F_\alpha$ is an uncountable collection of disjoint T_{n-1}-sets in E^n. This contradiction shows that there are only a countable number of disjoint ambiguous $(n - 1)$-cells in S^{n-1}.

Two comments on the role of the hypotheses may be worthwhile. First, R. L. Moore [8] has given an example (which deserves to be better known) of a 2-manifold S which is a Moore space, and therefore regular, but which is not metric, which has a countable dense subset, and which satisfies the Jordan curve theorem, so that it is very close to being the plane. The upper half-plane P is dense in this space, and if we consider the identity map $i: P \to S$, $i(p) = p$, then every point on the x-axis is a point of ambiguity. In fact, on every straight line in the upper half-plane intersecting the x-axis at a point p, there is a unique limit as p is approached, and if two lines from p have slopes with different absolute values, then the functional values have different limits.

Second, the reader may notice that Lemma 1 is true if the closed subsets are required to be compact and the space M is required merely to be separable; for a separable metric space can be imbedded in a compact metric space. However, our theorem would not be true if we replaced the requirement that the space M be compact by the requirement that the cluster sets along the arcs be compact, or even by the requirement that the union of all the cluster sets be compact. The reason for this is that even though the cluster set is compact, it may still happen that along an arc a sequence of functional values may contain no convergent subsequence. When the space is imbedded in a compact space, these sequences have limits, and the cluster sets are no longer disjoint.

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