DIFFERENTIATION ON MANIFOLDS WITHOUT A CONNECTION

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This note treats higher differentiations on a manifold in an invariant manner, without using any sort of connection. This is accomplished by considering a straightforward generalization of the ordinary Jacobian, which gives rise to new types of tensors. Some decomposition theorems are given, and speculations are made regarding other possible uses of the new tensors. For the sake of simplicity, only cases of lower order are treated.

Other work somewhat along these lines has been done by C. Ehresmann [1], [2] and A. Weil [3]; but their definitions and results are not required in what follows.

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The following conventions and notations will be observed. The manifolds considered allow at least three differentiations (real manifolds of class C^3). Sets of local coordinate variables are distinguished by different types of indices; for example, x^a, x^α, x^A stand for the coordinate functions of different coordinate systems. The symbols ∂_a, ∂_{ab}, ∂_α, $\partial_{\alpha\beta}$, $\cdots$ stand for the partial derivative operators $\partial/\partial x^a$, $\partial^2/\partial x^a \partial x^b$, $\partial/\partial x^\alpha$, $\partial^2/\partial x^\alpha \partial x^\beta$, $\cdots$. However, differentiation of the coordinate variables themselves will be denoted by capital D's; for example, $D^a_\alpha = \partial x^a/\partial x^\alpha$, $D^a_{\alpha\beta} = \partial^2 x^a/\partial x^\alpha \partial x^\beta$, $\cdots$. To avoid ambiguity, a symbol such as D^1_2 will never be used. Finally, repeated indices imply contraction, that is, summation over the repeated index: $D^a_\alpha D^\alpha_A = \Sigma_\alpha(\partial x^a/\partial x^\alpha)(\partial x^\alpha/\partial x^A) = \partial x^a/\partial x^A = D^a_A$.

Consider a generalized gradient $(\partial_a, \partial_{ab})$. This pair of differential operators has the transformation rule

$$(\partial_\alpha, \partial_{\alpha\beta}) = (\partial_a, \partial_{ab})\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix} = (\partial_a D^a_\alpha,\ \partial_a D^a_{\alpha\beta} + \partial_{ab} D^a_\alpha D^b_\beta).$$

As the indices indicate, the "multiplication" of the entries of the two "matrices" means contraction. The transformation rule given by this generalized Jacobian operator $\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}$ is linear, homogeneous, and transitive; the last means that

$$\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}\begin{pmatrix} D^\alpha_A & D^\alpha_{AB} \\ 0 & D^\alpha_A D^\beta_B \end{pmatrix} = \begin{pmatrix} D^a_A & D^a_{AB} \\ 0 & D^a_A D^b_B \end{pmatrix},$$

which implies, in particular, that

$$\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}\begin{pmatrix} D^\alpha_c & D^\alpha_{cd} \\ 0 & D^\alpha_c D^\beta_d \end{pmatrix} = \begin{pmatrix} \delta^a_c & 0 \\ 0 & \delta^a_c \delta^b_d \end{pmatrix},$$

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where $\delta^a_c = 1$ if $a = c$, and $\delta^a_c = 0$ if $a \neq c$ (Kronecker delta). With the notation

$$K(a, \alpha) = \begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}, \cdots,$$

these formulae may be written as follows:

$$K(a, \alpha) K(\alpha, A) = K(a, A),$$

$$K(a, \alpha) K(\alpha, a) = \text{identity}.$$

It is then natural to call $(\partial_a, \partial_{ab})$ the components of a tensorial operator of type K, and to define a tensor (field) of type K as an equivalence class of sets of real, differentiable, locally defined functions on the manifold (T_a, T_{ab}), $(T_\alpha, T_{\alpha\beta})$, $\cdots$, related by the transformation rules exemplified by

$$(T_\alpha, T_{\alpha\beta}) = (T_a, T_{ab}) \begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}.$$

If the functions and derivatives are evaluated at a point of the manifold, the tensors of type K form a vector space over the real numbers.

Ordinary covariant vectors are said to be of type J, where $J(a, \alpha) = D^a_\alpha$ (the usual Jacobian matrix), and contravariant vectors are of type J^{-1}, where $J^{-1}(a, \alpha) = J(\alpha, a) = D^\alpha_a$. More generally,

$$J^{-2} = D^\alpha_a D^\beta_b, \qquad J^3 = D^a_\alpha D^b_\beta D^c_\gamma, \qquad J^{-2} J^2 = D^\alpha_a D^\beta_b D^c_\gamma D^d_\delta, \cdots.$$

For instance, a Riemann metric is a tensor of type J^{-2}, and the curvature tensor is of type $J^{-1} J^3$.

Dual to the generalized gradient is the pair formed by the acceleration vector and the matrix of squared velocity components, $\begin{pmatrix} d^2x^a/dt^2 \\ (dx^a/dt)(dx^b/dt) \end{pmatrix}$, since the following transformation rule is valid:

$$\begin{pmatrix} d^2x^\alpha/dt^2 \\ (dx^\alpha/dt)(dx^\beta/dt) \end{pmatrix} = \begin{pmatrix} D^\alpha_a & D^\alpha_{ab} \\ 0 & D^\alpha_a D^\beta_b \end{pmatrix} \begin{pmatrix} d^2x^a/dt^2 \\ (dx^a/dt)(dx^b/dt) \end{pmatrix}.$$

More generally, a tensor (field) of type K^{-1} is an equivalence class of sets of real, differentiable, locally defined functions $\begin{pmatrix} T^a \\ T^{ab} \end{pmatrix}$, $\begin{pmatrix} T^\alpha \\ T^{\alpha\beta} \end{pmatrix}$, $\cdots$ connected by the transformation formulae:

$$\begin{pmatrix} T^\alpha \\ T^{\alpha\beta} \end{pmatrix} = K^{-1} \begin{pmatrix} T^a \\ T^{ab} \end{pmatrix} = \begin{pmatrix} D^\alpha_a & D^\alpha_{ab} \\ 0 & D^\alpha_a D^\beta_b \end{pmatrix} \begin{pmatrix} T^a \\ T^{ab} \end{pmatrix}, \cdots.$$

The tensors of type K^{-1}, evaluated at a point, form a real vector space which is dual to the space of tensors of type K evaluated at the same point; the duality is given by the pairing

$$(T_a, T_{ab})\begin{pmatrix} S^a \\ S^{ab} \end{pmatrix} = T_a S^a + T_{ab} S^{ab} \quad \text{is a real number}.$$

This pairing is independent of the coordinate system; in other words, it is a well-defined pairing of equivalence classes, because $K^{-1}(a, \alpha) = K(\alpha, a)$ and $K(a, \alpha)K(\alpha, a) =$ identity.

If T_a is a tensor of type J, then $(T_a, \partial_b T_a)$ is a tensor of type K. Dually, if a curve in the space of tensors of type J^{-1} is given by $S^a = S^a(t)$, then $\begin{pmatrix} dS^a/dt \\ S^a dx^b/dt \end{pmatrix}$ is a tensor of type K^{-1}, since

$$dS^\alpha/dt = d(S^a D^\alpha_a)/dt = (dS^a/dt)D^\alpha_a + S^a D^\alpha_{ab}(dx^b/dt).$$

If a manifold is given, together with a distinguished tensor of type K, (G_a, G_{ab}), a metric generalization of Riemann space is obtained from the invariant

$$\left(\frac{dS}{dt}\right)^2 = G_a \frac{d^2x^a}{dt^2} + G_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt},$$

(where $x(t)$ is a curve in M and (G_a, G_{ab}) is chosen so that the expression on the right is nonnegative) which reduces to the Riemann metric when $G_a = 0$ ($G_a = 0$ is an invariant property, since $G_\alpha = G_a D^a_\alpha$). If G_a could be identified with an electromagnetic potential, and G_{ab} with a gravitational potential, then it might be argued that an electromagnetic "shield" is conceivable (since $G_a = 0$ is an invariant property), while a gravitational "shield" is inconceivable (since $G_{ab} = 0$ is not an invariant property).

If the Riemann metric is regarded as the special tensor $(0, G_{ab})$ of type K, then its counterpart in the space of tensors of type K^{-1} is the special tensor $\begin{pmatrix} H^a \\ 0 \end{pmatrix}$; for $H^{ab} = 0$ is an invariant property, because $H^{\alpha\beta} = H^{ab}D^\alpha_a D^\beta_b$; and $(0, G_{ab})$ is "orthogonal" to $\begin{pmatrix} H^a \\ 0 \end{pmatrix}$ under the "inner product" given by

$$(G_a, G_{ab})\begin{pmatrix} H^a \\ H^{ab} \end{pmatrix} = G_a H^a + G_{ab} H^{ab}.$$

It thus appears that the class of Riemann manifolds is dual to the class of manifolds in which a contravariant vector field is distinguished.

The tensors of type K^{-1} may also be used to give an invariant definition of a linear, second-order, partial differential equation on a manifold. For if this equation is written as

$$T^a \partial_a u + T^{ab}\partial_{ab} u = (\partial_a, \partial_{ab})\begin{pmatrix} T^a \\ T^{ab} \end{pmatrix}(u) = F,$$

where F is an invariant real function on the manifold, then requiring the equation to be invariant is equivalent to requiring that $\begin{pmatrix} T^a \\ T^{ab} \end{pmatrix}$ constitute a tensor of type K^{-1}. The usual classification of such equations into the parabolic, elliptic and hyperbolic cases according to the sign of the determinant of T^{ab} is likely to be insufficient for an invariant study, since it ignores the term T^a, whose transformation formula involves the second derivative, $D^a_{\alpha\beta}$, whereas the term T^{ab} uses only the first derivatives $D^a_\alpha D^b_\beta$ in its transformation law.

How are ordinary vectors to be identified within the framework of tensors of type K and K^{-1}? It turns out that an ordinary contravariant vector may be identified with a special kind of tensor of type K^{-1}, while an ordinary covariant vector may be identified with a coset of tensors of type K. This is implied by the following decomposition theorem.

THEOREM 1. *Let*

T = *space of tensors of type* K,

T^* = *space of tensors of type* K^{-1},

V = *space of covariant vectors*,

V^* = *space of contravariant vectors*,

M = *space of tensors of type* J^2,

M^* = *space of tensors of type* J^{-2},

H = *space of tensors of type* K *which have the form* $(0, S_{ab})$,

H^* = *space of tensors of type* K^{-1} *which have the form* $(S^a, 0)$.

Then the following isomorphisms are valid:

$$H \approx M, \qquad H^* \approx V^*, \qquad T/H \approx V, \qquad T^*/H^* \approx M^*.$$

Proof. If (S_a, S_{ab}) is of type K, and $S_a = 0$ in one coordinate system x^a, then $S_\alpha = 0$ in every other coordinate system x^α, since $S_\alpha = S_a D^a_\alpha$. Hence the space H is well-defined. Moreover, if $S_a = 0$, the transformation rule of S_{ab} reduces to $S_{\alpha\beta} = S_{ab} D^a_\alpha D^b_\beta$, so that S_{ab} is in M. Hence $\phi[(0, S_{ab})] = S_{ab}$ defines an isomorphism of H onto M. If $\begin{pmatrix} S^a \\ S^{ab} \end{pmatrix}$ is of type K^{-1}, then $S^{\alpha\beta} = S^{ab} D^\alpha_a D^\beta_b$, so that the vanishing of S^{ab} is a property independent of the coordinate system, and the space H^* is well-defined. Moreover, if $S^{ab} = 0$, the transformation rule of S^a becomes $S^\alpha = S^a D^\alpha_a$, so that S^a is in V^*. Hence $\phi\left[\begin{pmatrix} S^a \\ 0 \end{pmatrix}\right] = S^a$ defines an isomorphism of

H^* onto V^*. Finally, $\phi[(S_a, S_{ab})] = S_a$ defines a linear mapping of T onto V with kernel H, and $\psi\left[\begin{pmatrix} S^a \\ S^{ab} \end{pmatrix}\right] = S^{ab}$ defines a linear mapping of T^* onto M^* with kernel H^*, so that $T/H \approx V$ and $T^*/H^* \approx M^*$.

The study of arbitrary tensors of types K and K^{-1} may be reduced to the study of symmetric tensors of types K and K^{-1} and antisymmetric tensors of types J^2 and J^{-2} by the following theorem.

THEOREM 2. *If*

$$T = \textit{space of tensors of type } K,$$

$$A = \textit{space of antisymmetric tensors of type } J^2,$$

$$S = \textit{space of symmetric tensors of type } K,$$

$$T^* = \textit{space of tensors of type } K^{-1},$$

$$A^* = \textit{space of antisymmetric tensors of type } J^{-2},$$

$$S^* = \textit{space of symmetric tensors of type } K^{-1},$$

then $T = A \oplus S$ *and* $T^* = A^* \oplus S^*$.

Proof. It suffices to show that the following sums are invariant, that is, do not depend on the coordinate system.

$$(P_a, P_{ab}) = \left(0, \frac{1}{2}(P_{ab} - P_{ba})\right) + \left(P_a, \frac{1}{2}(P_{ab} + P_{ba})\right),$$

$$\begin{pmatrix} P^a \\ P^{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}(P^{ab} - P^{ba}) \end{pmatrix} + \begin{pmatrix} P^a \\ \frac{1}{2}(P^{ab} + P^{ba}) \end{pmatrix}.$$

That $\left(0, \frac{1}{2}(P_{ab} - P_{ba})\right)$ is well-defined and may be identified with a tensor in A follows from

$$\frac{1}{2}(P_{\alpha\beta} - P_{\beta\alpha}) = \frac{1}{2}(P_{ab} D^a_\alpha D^b_\beta + P_a D^a_{\alpha\beta} - P_{ba} D^b_\beta D^a_\alpha - P_b D^b_{\beta\alpha}) = \frac{1}{2}(P_{ab} - P_{ba}) D^a_\alpha D^b_\beta.$$

Note that antisymmetry in the second component requires the vanishing of the first component, so that antisymmetry in the second component is not in general an invariant property. Since

$$\frac{1}{2}(P_{\alpha\beta} + P_{\beta\alpha}) = \frac{1}{2}(P_{ab} D^a_\alpha D^b_\beta + P_a D^a_{\alpha\beta} + P_{ba} D^b_\beta D^a_\alpha + P_b D^b_{\beta\alpha})$$
$$= \frac{1}{2}(P_{ab} + P_{ba}) D^a_\alpha D^b_\beta + P_a D^a_{\alpha\beta},$$

it follows that the second term in the first sum is well-defined and in S. In fact, symmetry in the second component of any (P_a, P_{ab}) is an invariant property. For the second sum, notice that $P^\alpha = P^a D^\alpha_a + P^{ab} D^\alpha_{ab}$, so that the vanishing of the first component is not an invariant property unless the second component is antisymmetric, which is the case, however, in the tensor $\begin{pmatrix} 0 \\ \frac{1}{2}(P^{ab} - P^{ba}) \end{pmatrix}$. Since antisymmetry is an invariant property of every tensor of type K^{-1}, it follows that $\begin{pmatrix} 0 \\ \frac{1}{2}(P^{ab} - P^{ba}) \end{pmatrix}$ is well-defined and may be identified with $\frac{1}{2}(P^{ab} - P^{ba})$, which is in A*. Finally, the second term is well-defined, since

$$P^\alpha = P^a D^\alpha_a + P^{ab} D^\alpha_{ab} = P^a D^\alpha_a + \frac{1}{2}(P^{ab} + P^{ba}) D^\alpha_{ab}.$$

In fact, symmetry is an invariant property of every tensor of type K^{-1}. Therefore the reduction theorems are established.

The only natural way of combining the ordinary Jacobians so as to yield new linear, homogeneous, transitive transformation operators seems to be through iterated tensor multiplication of J and J^{-1}, for example, $J^2 J^{-3} = D^a_\alpha D^b_\beta D^\gamma_c D^\delta_d D^\rho_r$. There are of course more possibilities if K and K^{-1} are considered along with J and J^{-1}. Some of these are illustrated below.

If δ^c_a is the Kronecker tensor and Γ^c_{ab} is an affine connection, that is, $\Gamma^\gamma_{\alpha\beta} = D^a_{\alpha\beta} D^\gamma_a + \Gamma^c_{ab} D^a_\alpha D^b_\beta D^\gamma_c$, then

$$(\delta^\gamma_\alpha, \Gamma^\gamma_{\alpha\beta}) = (\delta^c_a, \Gamma^c_{ab}) \begin{pmatrix} D^a_\alpha D^\gamma_c & D^a_{\alpha\beta} D^\gamma_c \\ 0 & D^a_\alpha D^b_\beta D^\gamma_c \end{pmatrix},$$

so that $(\delta^c_a, \Gamma^c_{ab})$ is a tensor whose type is the tensor product of K and J^{-1}

$$KJ^{-1} = \begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix} D^\gamma_c = \begin{pmatrix} D^a_\alpha D^\gamma_c & D^a_{\alpha\beta} D^\gamma_c \\ 0 & D^a_\alpha D^b_\beta D^\gamma_c \end{pmatrix}.$$

Since δ^c_a is a constant tensor, it is natural to identify the affine connection Γ^c_{ab} with the pair $(\delta^c_a, \Gamma^c_{ab})$. Then the affine connections clearly form a convex subset of the vector space of all tensors of type KJ^{-1}.

It is convenient to consider transposition about the secondary diagonal in the transformation operators K and K^{-1}. With the notation $K^*(a, \alpha) = \begin{pmatrix} D^a_\alpha D^b_\beta & D^a_{\alpha\beta} \\ 0 & D^a_\alpha \end{pmatrix}$, it is immediate that $K^*(a, A) = [K(a, \alpha) K(\alpha, A)]^* = K^*(\alpha, A) K^*(a, \alpha)$.

If T^a is a tensor of type J^{-1}, then $\partial_\beta T^\alpha = T^a D^\alpha_{ab} D^b_\beta + \partial_b T^a D^\alpha_a D^b_\beta$, so that

$$(T^{\alpha}, \partial_{\beta}T^{\alpha}) = (T^{a}, \partial_{b}T^{a})\begin{pmatrix} D^{\alpha}_{a} & D^{\alpha}_{ab}D^{b}_{\beta} \\ 0 & D^{\alpha}_{a}D^{b}_{\beta} \end{pmatrix},$$

and it is easily verified that this operator is transitive, in other words, that

$$\begin{pmatrix} D^{\alpha}_{a} & D^{\alpha}_{ab}D^{b}_{\beta} \\ 0 & D^{\alpha}_{a}D^{b}_{\beta} \end{pmatrix}\begin{pmatrix} D^{A}_{\alpha} & D^{A}_{\alpha\beta}D^{\beta}_{B} \\ 0 & D^{A}_{\alpha}D^{\beta}_{B} \end{pmatrix} = \begin{pmatrix} D^{A}_{a} & D^{A}_{ab}D^{b}_{B} \\ 0 & D^{A}_{a}D^{b}_{B} \end{pmatrix}.$$

Hence $(T^{a}, \partial_{b} T^{a})$ is a tensor of type

$$K^{-1*}\cdot J = \begin{pmatrix} D^{\alpha}_{a}D^{\beta}_{b} & D^{\alpha}_{ab} \\ 0 & D^{\alpha}_{a} \end{pmatrix}\cdot D^{b}_{\beta} = \begin{pmatrix} D^{\alpha}_{a} & D^{\alpha}_{ab}D^{b}_{\beta} \\ 0 & D^{\alpha}_{a}D^{b}_{\beta} \end{pmatrix}.$$

Finally, the partial derivative of any ordinary tensor is a generalized tensor whose type is compounded of the simple types that have been illustrated. More precisely, the following theorem holds.

THEOREM 3. *If* $T^{ab\cdots m}_{np\cdots r}$ *is a tensor of type* $J^{-h}J^{k}$, *then* $(T^{ab\cdots m}_{np\cdots r}, \partial_{s} T^{ab\cdots m}_{np\cdots r})$ *is a tensor of type* $D(J^{-h}J^{k})$, *where*

$$D(J^{-h}J^{k}) = D(J^{-1})J^{-h+1}J^{k} + \cdots + J^{-h+1}D(J^{-1})J^{k} + J^{-h}D(J)J^{k-1} + \cdots$$

$$+ J^{-h}J^{k-1}D(J) - (k+h)\begin{pmatrix} J^{-h}J^{k} & 0 \\ 0 & J^{-h}J^{k+1} \end{pmatrix},$$

and where $D(J) = K$ *and* $D(J^{-1}) = K^{-1*}\cdot J$. Note that the expansion of $D(J^{-h}J^{k})$ is quite like the product rule for ordinary differentiations, and that each term in the expansion is itself a linear, homogeneous, transitive transformation operator.

Proof. Since

$$\partial_{\sigma}T^{\alpha\beta\cdots\mu}_{\nu\pi\cdots\rho} = T^{ab\cdots m}_{np\cdots r}D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho} + \partial_{s}T^{ab\cdots m}_{np\cdots r}\Big(D^{\alpha}_{as}D^{s}_{\sigma}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho}$$

$$+ \cdots + D^{\alpha}_{a}\cdots D^{\mu}_{ms}D^{s}_{\sigma}D^{n}_{\nu}\cdots D^{r}_{\rho} + D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu\sigma}\cdots D^{r}_{\rho} + \cdots + D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho\sigma}\Big),$$

it follows that

$$(T^{\alpha\beta\cdots\mu}_{\nu\pi\cdots\rho}, \partial_{\sigma}T^{\alpha\beta\cdots\mu}_{\nu\pi\cdots\rho}) = (T^{ab\cdots m}_{np\cdots r}, \partial_{s}T^{ab\cdots m}_{np\cdots r}) \text{ times}$$

$$\begin{pmatrix} D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho}, & D^{\alpha}_{as}D^{s}_{\sigma}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho} + \cdots \\ & + D^{\alpha}_{a}\cdots D^{\mu}_{ms}D^{s}_{\sigma}D^{n}_{\nu}\cdots D^{r}_{\rho} \\ & + D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu\sigma}\cdots D^{r}_{\rho} + \cdots \\ & + D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho\sigma} \\ 0 & , D^{\alpha}_{a}\cdots D^{\mu}_{m}D^{n}_{\nu}\cdots D^{r}_{\rho}D^{s}_{\sigma} \end{pmatrix},$$

The transformation operator on the right is equal to the following sum, where the symbol ^ denotes omission.

$$\begin{pmatrix} D_a^\alpha & D_{as}^\alpha D_\sigma^s \\ 0 & D_a^\alpha D_\sigma^s \end{pmatrix} \hat{D}_a^\alpha \cdots D_\rho^r + \cdots + \begin{pmatrix} D_m^\mu & D_{ms}^\mu D_\sigma^s \\ 0 & D_m^\mu D_\sigma^s \end{pmatrix} D_a^\alpha \cdots \hat{D}_m^\mu \cdots D_\rho^r$$

$$+ \begin{pmatrix} D_\nu^n & D_{\nu\sigma}^n \\ 0 & D_\nu^n D_\sigma^s \end{pmatrix} D_a^\alpha \cdots \hat{D}_\nu^n \cdots D_\rho^r + \cdots + \begin{pmatrix} D_\rho^r & D_{\rho\sigma}^r \\ 0 & D_\rho^r D_\sigma^s \end{pmatrix} D_a^\alpha \cdots \hat{D}_\rho^r$$

$$- (k + h) \begin{pmatrix} D_a^\alpha \cdots D_\rho^r & 0 \\ 0 & D_a^\alpha \cdots D_\rho^r D_\sigma^s \end{pmatrix}$$

$$= D(J^{-1}) J^{-h+1} J^k + \cdots + J^{-h+1} D(J^{-1}) J^k + J^{-h} D(J) J^{k-1} + \cdots + J^{-h} J^{k-1} D(J)$$

$$- (k + h) \begin{pmatrix} J^{-h} J^k & 0 \\ 0 & J^{-h} J^{k+1} \end{pmatrix} = D(J^{-h} J^k) .$$

The transitivity of $D(J^{-h}J^k)$ clearly reduces to the transitivity of each term in the sum. Since the tensor product of transitive operators is transitive, and since it has already been verified that $D(J) = K$ and $D(J^{-1}) = K^{-1} * \cdot J$ are transitive, the transitivity of $D(J^{-h} J^k)$ follows. Thus the theorem is proved.

I shall later extend these considerations and, in particular, give a reduction of covariant differentiation to tensor contractions, and construct differential cohomology rings which are not equivalent to the de Rham cohomology rings.

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