

DENSE INVERSE LIMIT RINGS

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We are concerned here with certain ideals with finitely many generators, in linear algebras of a type studied in [1] and [8] and named \mathcal{F} -algebras by Michael [8]. Our main result is this: Let A be an \mathcal{F} -algebra with a unit element 1, and let J be a proper right ideal with finitely many generators, in A . Then there is a continuous homomorphism T of A into a Banach algebra B with unit such that $T(J)$ lies in some proper right ideal of B . This is 4.2 below.

The assertion that B can be selected so as to be a Banach algebra (that is, a *complete* normed linear algebra), rather than merely a normed linear algebra, is important. In fact, the resulting weaker proposition is elementary, and we shall indicate briefly why it does not immediately imply our result 4.2. If T were a continuous homomorphism of A into a normed algebra N , and J_1 were the ideal generated by $T(J)$, then J_1 would certainly be proper, but might be *everywhere-dense*. Thus if N were completed, J_1 might generate the improper ideal. On the other hand, in a Banach algebra with unit, each proper ideal is contained in a closed proper ideal.

In the special case in which A is commutative and J has a single generator (which is to say that J is the principal ideal generated by a single element a), we obtain a proposition which can be immediately deduced from the main theorems of the paper cited, namely [1, 7.1] and [8, 5.2]. The advance of the present work over these earlier papers comes from the technique presented below for controlling the variety of solutions (x_1, \dots, x_N) of the equation

$$1.1 \quad a_1 x_1 + \dots + a_N x_N = 1,$$

where a_1, \dots, a_N are given elements of some ring with unit. For $N > 1$, this variety exists even in the commutative case.

We call a system $\{a_1, \dots, a_N\}$ *right regular* if 1.1 can be solved in the ring in question. If this ring is an algebra A with unit over the complex numbers, and $\lambda_1, \dots, \lambda_N$ are complex numbers, while a_1, \dots, a_N belong to A , then $(\lambda_1, \dots, \lambda_N)$ is said to belong to the *joint right spectrum* $\sigma(a_1, \dots, a_N; A)$ of (a_1, \dots, a_N) if $\{a_1 - \lambda_1, \dots, a_N - \lambda_N\}$ is *not* right regular. Our main result enables us to conclude that $\sigma(a_1, \dots, a_N; A)$ is the union of the joint right spectra $\sigma(T(a_1), \dots, T(a_N); B)$, where T is a continuous homomorphism of A (now assumed to be an \mathcal{F} -algebra with unit) into a Banach algebra B with unit, the union being over a countable family of such pairs (B, T) . If, in addition, A is commutative, there is the following consequence. Let Δ be the class of all continuous homomorphism ζ of A on the complex numbers C (it is known that Δ has a natural one-to-one relation to the class of *closed* maximal ideals.) Then the joint spectrum $\sigma(a_1, \dots, a_N; A)$ coincides with the image in complex N -space C^N of Δ obtained through the mapping

$$\zeta \rightarrow (\zeta(a_1), \dots, \zeta(a_N)).$$

Michael has a discussion [8, 12.5] of the continuity of complex-valued homomorphisms of a commutative \mathcal{F} -algebra with unit; and in this discussion the

solvability of equation 1.1 plays a part. (In fact his assumption I, for the case in which his T is the Δ defined above, is precisely our theorem. To see this, one must correct the misprinted ' \neq ' to ' $=$ '.) His theorem and ours allow the deduction that, if $\{a_1, \dots, a_N\}$ is a system of *rational generators* of A , then there are no discontinuous complex-valued homomorphisms of A (see Section 8 below).

Finally, we deduce a characterization of the \mathcal{F} -algebra $\text{Hol}(\Omega)$ of complex-valued and holomorphic functions defined on any open plane set Ω . This characterization requires one generator and a derivation (see Section 8 below for the details.)

2. INVERSE LIMIT RINGS

It was recognized in [1, p. 455] and demonstrated in [8, 5.1] that \mathcal{F} -algebras are inverse limits of Banach algebras. Most of our present constructions apply to inverse limit rings more general than \mathcal{F} -algebras. Furthermore, we propose to define \mathcal{F} -algebras as inverse limits of Banach algebras (see below), because in this way we can avoid repeating some cumbersome phrasing necessitated by the approach used in [1] and [8].

Accordingly, we proceed to a brief study of inverse limit spaces.

Let V be a partially ordered, directed set, to be used for indexing. Let $\{B_v : v \in V\}$ be a family of topological spaces. Suppose for each pair u, v of elements of V such that $u \leq v$ there is a continuous mapping

$$2.1 \quad \pi_u^v : B_v \rightarrow B_u.$$

Assume also the compatibility property

$$2.2 \quad \text{if } u \leq v \leq w, \text{ then } \pi_u^w \circ \pi_w^v \subset \pi_u^v.$$

(The system of mapping $\{\pi_u^v\}$ is called an *inverse mapping system*.) Form the topological product B of these spaces,

$$2.3 \quad B = \prod_{u \in V} B_u.$$

The component of a point b in B_v is denoted by b_v . The set A of all b in B such that $b_u = \pi_u^v b_v$ for all u, v in V for which $u \leq v$ is the inverse limit of these B_v , or more precisely, of the mapping system involved [6, p. 31]. For each $b \in A$ and each $v \in V$ we denote b_v by $\pi_v(b)$, thus introducing the *natural mapping* of A into B_v .

Let us call the mapping system *dense* if the image in 2.1 (which is not required to exhaust B_u) is at least dense in B_u , for all u, v such that $u \leq v$. Let us also call A the dense inverse limit of the B_v . This will not generally insure that $\pi_v(A)$ is dense in B_v (see [4]). However, when $\pi_v(A)$ is dense in B_v for every v , we shall say that A is a *strongly dense inverse limit*.

2.4 THEOREM. *Let $\{B_0, B_1, B_2, \dots\}$ be a sequence of complete metric spaces, and let mappings*

$$2.41 \quad \pi_n^{n+1} : B_{n+1} \rightarrow B_n \quad (n = 0, 1, 2, \dots)$$

have dense range. Then these generate a dense mapping system with index class $\{0, 1, 2, \dots\}$. The inverse limit A is a complete metric space, and its natural image in each B_n is dense.

Proof. The construction and the denseness of the mapping system are rather obvious, and may be passed over.

In the present circumstances, the product space B is metrizable; indeed we have in mind the metric

$$2.42 \quad \rho(b, b') = \sup_n \max(\rho_n(b_n, b'_n), 2^{-n}),$$

where ρ_0, ρ_1, \dots are the metrics in B_0, B_1, \dots , respectively. With this metric 2.42, or any other uniformly equivalent with it, the completeness of A is readily shown.

We turn to the proof that $\pi_n(A)$ is dense in B_n for each n . It is actually sufficient to establish this for $n = 0$, and we proceed to that task.

Let b_0 be a point of B_0 , and let ε be a positive number. We shall construct an $a \in A$ such that

$$2.43 \quad \rho_0(b_0, a_0) \leq \varepsilon.$$

In the construction we shall omit the indices on the various metrics, and also on the mappings 2.41. Moreover, we shall denote the composite of any m successive mappings of the set 2.41 by π^m .

Select positive numbers $\varepsilon_1, \varepsilon_2, \dots$ for which $\varepsilon_1 + \varepsilon_2 + \dots \leq \varepsilon$. An element b_1 exists in B_1 such that

$$\rho(b_0, \pi b_1) < \varepsilon_1.$$

Having chosen b_1 , we can find a b_2 in B_2 such that

$$\rho(b_1, \pi b_2) < \varepsilon_2, \quad \rho(\pi b_1, \pi^2 b_2) < \varepsilon_2.$$

The latter of these is possible because π is continuous. Continuing in this, we arrive at $b_1, b_2, \dots, b_n, \dots$ ($b_n \in B_n$) such that

$$\rho(\pi^k b_n, \pi^{k+1} b_{n+1}) < \varepsilon_{n+1}$$

for all $k < n$. For each n , consider the sequence

$$c_{n,p} = \pi^{p-n} b_p \quad (p = n, n + 1, n + 2, \dots).$$

This is a Cauchy sequence, because

$$\rho(c_{n,p+1}, c_{n,p}) < \varepsilon_{p+1}.$$

Let the limit in B_n of this sequence be called a_n . For example, $a_n = \lim \pi^p b_p$. From $\pi c_{n+1,p} = c_{n,p}$ we can deduce $\pi a_{n+1} = a_n$; and this means that $\{a_n\}$ is a point of A . It remains to show 2.43. Using the metric in B_0 , we have

$$\rho(b_0, \pi^p b_p) < \rho(b_0, \pi b_1) + \dots + \rho(\pi^{p-1} b_{p-1}, \pi^p b_p) < \varepsilon.$$

Taking the limit as $p \rightarrow \infty$, we obtain 2.43.

We have thus established that a dense inverse limit of a sequence of complete metric spaces is strongly dense.

We now consider algebraic structures in the B_v . Consider the case in which each B_v is a topological group, and each mapping of the system (2.1) is a homomorphism. Then A receives a topological group structure (this is not the *projective* limit of [10], where the mappings 2.1 are required to be "onto").

For later use, we wish to establish the following.

2.6 THEOREM. *Let A be a strongly dense inverse limit group. Let T be a continuous homomorphism of A into a Banach space K . Then there is an index v such that*

$$2.61 \quad T = T_v \circ \pi_v,$$

where T_v is a continuous homomorphism of B_v into K . For each v there is at most one T_v such that 2.61 holds.

Proof. First we make a general remark on the topology of inverse limit spaces, using the notation of [1] introduced at the beginning of this section. Let $e \in A$, and let u be a neighborhood of e_v , for some v . Consider

$$2.62 \quad \{a: a_v \in u\}.$$

This is a neighborhood of e . By varying v and u in all possible ways, we obtain a neighborhood basis for e in A (but not in B)!

Now consider that T is continuous at the identity e in A . There is therefore a neighborhood W of e of the form 2.62 such that $a \in W$ implies $\|T(a)\| \leq 1$. Let us show that $T(a)$ depends only on a_v , the index v here being limited to that one occurring in the chosen representation of W by 2.62. It suffices to show that $a_v = e_v$ implies $T(a) = 0$. Suppose, therefore, that $a_v = e_v$. Then $(a^n)_v \in u$ for every positive n , so that $\|T(a^n)\| \leq 1$. But $T(a^n) = nT(a)$, since T is a homomorphism. Hence $T(a) = 0$.

Consequently we may define $T_v(b) = T(a_v)$ for each b in B_v of the form a_v ($a \in A$). This T_v may be extended to all of B_v , because $\pi_v(B)$ is dense, and K is complete. These considerations are sufficient to prove 2.6.

We mention that the algebras of [1] and [8] are examples of strongly dense inverse limit algebras. In these examples, each B_v is a Banach algebra. While $\pi_v(A)$ is dense in B_v , it does not generally exhaust B_v . Why not "throw away" the other points of B_v , and exhibit A as a *projective* limit of normed linear algebras? The answer is that we wish to reduce various questions about A to questions about Banach algebras (which are more tractable than normed algebras).

3. RIGHT REGULAR SYSTEMS IN TOPOLOGICAL RINGS

Let B be a ring with unit. Let (b_1, \dots, b_N) be an ordered n -tuple of elements of B . We say that (b_1, \dots, b_N) is a *right regular system*, if the equation

$$3.1 \quad b_1 x_1 + \dots + b_N x_N = 1 \quad (x_1, \dots, x_N \in B)$$

can be solved.

3.2 THEOREM. *Let (b_1, \dots, b_N) be a right regular system, and suppose that (x_1, \dots, x_N) is a solution of 3.1. Choose t_1, \dots, t_N arbitrarily in B , and define*

$$3.21 \quad z_i = t_i + x_i(1 - \sum b_k t_k) \quad (i = 1, \dots, N).$$

Then (z_1, \dots, z_N) is a solution of 3.1, and every solution of 3.1 can be obtained in this way from the particular solution (x_1, \dots, x_N) .

It will suffice to present a demonstration of the last clause of 3.2. Let (x_1, \dots, x_N) be one solution of 3.1, and (z_1, \dots, z_N) another. Define $t_i = z_i - x_i$. Then 3.21 becomes an identity. (This is not the only possible choice for t_1, \dots, t_N .)

3.3 COROLLARY. *Suppose that B_1, B_2 are topological rings with a homo-*
morphism

$$\pi: B_2 \rightarrow B_1$$

such that $\pi(B_2)$ is dense in B_1 . Suppose (b_1, \dots, b_N) is a right regular system in B_2 (so that $(\pi b_1, \dots, \pi b_N)$ is such a system in B_1), and suppose y_1, \dots, y_N are elements of B_1 such that

$$3.31 \quad (\pi b_1)y_1 + \dots + (\pi b_N)y_N = 1.$$

Then elements z_1, \dots, z_N can be found in B_2 satisfying

$$3.32 \quad b_1 z_1 + \dots + b_N z_N = 1,$$

and such that for each i ($i = 1, \dots, N$), πz_i lies in a preassigned neighborhood of y_i in B_1 .

Proof. It is possible to find x_1, \dots, x_N in B_2 such that $b_1 x_1 + \dots + b_N x_N = 1$. Now form

$$z_i = t_i + x_i(1 - \sum b_k t_k) \quad (i = 1, \dots, N),$$

selecting t_i so that πt_i is very close to y_i in B_1 . These z_i satisfy 3.32. Moreover, $\sum \pi b_k \pi t_k$ is close to $\sum (\pi b_k)y_k$, which is 1, so that $\pi(1 - \sum b_k t_k)$ is close to 0, whence πz_i is close to πt_i , which is close to y_i .

We next apply these ideas to dense inverse limit rings.

4. RIGHT REGULAR SYSTEMS IN INVERSE LIMIT RINGS

Let A be an inverse limit of topological rings B_v . Let a_1, \dots, a_N be a right regular system in A . Then naturally $\pi_v a_1, \dots, \pi_v a_N$ is right regular in B_v , for every v . Since [1, p. 462] was written, we have found that the converse is also true, at least for dense inverse limits of sequences of complete metric rings. From here to the end of Section 4, A is supposed to be a dense inverse limit of a sequence of complete metric rings B_0, B_1, \dots .

4.2 THEOREM. *Suppose (a_1, \dots, a_N) is a finite set of elements of A such that*

$$\pi_n a_1, \dots, \pi_n a_N$$

is a right regular system in B_n , for each n . Then (a_1, \dots, a_N) is a right regular system in A .

In our proof, we shall deal with many N -tuples of elements, some in B_n and some in A . We shall speak of $a \in A^N$ (the topological product) and of $b \in B_n^N$ and mean thereby (a_1, \dots, a_N) ($a_k \in A$) and (b_1, \dots, b_N) ($b_k \in B_n$), respectively. It is convenient to have a metric in these product spaces. In order to be specific, we define

$$\rho(b, c) = \max_k \rho(b_k, c_k),$$

where on the right we have the metric in the factor space. We shall also write (b, c) for $b_1c_1 + \dots + b_Nc_N$. This is of course an element of the ring concerned. The symbol π_n will be used for the canonical homomorphism of A into B_n , and also for that of A^N into B_n^N . For π_n^{n+1} (see 4.1) we use π , and for the composite of any m successive ones we use π^m . These notations will be used also for B_n^N . Various elements in B_n^N we distinguish by superscripts which do not indicate powers.

In order to find $x \in A^N$ such that

$$4.21 \quad (a, x) = 1,$$

we begin by observing that $y \in B_0^N$ can be found such that

$$4.22 \quad (\pi_0 a, y) = 1.$$

Let $\varepsilon_1 + \varepsilon_2 + \dots$ be any convergent infinite series of positive numbers. Using our main tool 3.3, we can find $y^1 \in B_1^N$ such that

$$4.23 \quad (\pi_1 a, y^1) = 1$$

and

$$4.24 \quad \rho(\pi y^1, y^0) < \varepsilon_1 \quad (\text{in } B_0^N).$$

By an evident induction process we obtain $y^2 \in B_2^N$, $y^3 \in B_3^N$, \dots such that

$$4.25 \quad (\pi_n a, y^n) = 1 \quad (n = 1, 2, \dots)$$

and

$$4.26 \quad \rho(\pi^m y^n, \pi^{m-1} y^{n-1}) < \varepsilon_n \text{ for all } m < n.$$

Let

$$c(k)_n = \pi^{n-k} y^n \quad (n = k, k+1, \dots).$$

Then

$$4.27 \quad c(k)_n = \pi c(k+1)_n \quad (n > k),$$

and (from 4.26)

$$4.28 \quad \rho(c(n-m)_n, c(n-m)_{n-1}) < \varepsilon_n \quad (n > m).$$

Inserting π^{n-k} into 4.25 yields

$$4.29 \quad (\pi_k a, c(k)_n) = 1 \quad (n > k).$$

By 4.28, $c(k)_n$ forms a Cauchy system in B_k^N , convergent to some n -tuple $c(k)$. By 4.29, $(\pi_k a, c(k)) = 1$; by 4.27, $c(k) = \pi c(k+1)$; that is $c(\cdot)$ defines an element x of the inverse limit algebra, evidently satisfying 4.21. This proves 5.2 completely.

We need not mention units in 4.2; for if A has none, then there are no regular systems. We tacitly used the fact that if B_{n+1} has a unit 1, then $\pi(1)$ is a (the) unit for B_n ; and if every B_n has a unit, so has A .

If A has no unit, then B_n has no unit for n sufficiently large, say for $n > n_0$. One could discard all B_n with $n < n_0$, and adjoin units to all the others and thus to A . A system (a_1, \dots, a_N) which then becomes regular could be called quasi-regular in its original situation; and this would lead to a version of 4.2 having content even when there is no unit. We do not pursue this idea in this paper.

Let us call an ideal M in a topological algebra A (over a topological field K) *co-finite* if it is closed and A/M is finite-dimensional.

In the following theorem, A is supposed to be as agreed upon at the start of this section; moreover, each B_n is to be an algebra with unit over a topological field K , such that (see 2.41)

$$4.3 \quad \pi_n^{n+1}(kb) = k\pi_n^{n+1}(b) \quad (k \in K, b \in B_{n+1}; n = 0, 1, 2, \dots).$$

4.4 THEOREM. *Let each B_n have the property that every proper, finitely generated right ideal lies in some co-finite maximal right ideal. Then A has this property also.*

Recall that in a Banach algebra with unit element, or more generally in a "Q-ring" with unit element [8, E], every proper right ideal is contained in a closed ideal.

Indeed, let a_1, \dots, a_N generate the ideal J in question. Since J is proper, (a_1, \dots, a_N) forms a nonregular system in A , and hence there is an n such that the elements $\pi_n a_1, \dots, \pi_n a_N$ lie in some finitely generated proper ideal J_n of B_n . Expand J_n to a maximal co-finite right ideal M_n of B_n . There is a mapping of A into B_n/M_n , wherein A maps on a dense set, which implies that A maps onto B_n/M_n . The kernel M of that mapping is therefore a co-finite maximal closed ideal, and it contains J .

For infinite systems (a_1, \dots, a_N, \dots) , Theorem 4.2 does not hold, as the following example shows. Let B_n be the Banach algebra of complex functions on the integers $\{0, 1, 2, \dots, n\}$. Thus, B_n is essentially C^{n+1} with the operations defined coordinate-wise. The mappings (2.41) are the natural ones:

$$\pi_n^{n+1}(\lambda_0, \dots, \lambda_{n+1}) = (\lambda_0, \dots, \lambda_n).$$

A is the space (s) of sequences $(\lambda_0, \lambda_1, \lambda_2, \dots)$. Let a_N be the sequence of N ones followed by zeros ($N = 1, 2, \dots$). Then $F = \{a_1, a_2, \dots\}$ generates a *proper* ideal in A , but (a_1, \dots, a_N) is already a regular system in B_{N-1} . This illustrates the following theorem, which is easy to prove: *If F in the A of 4.2 is such that $\pi_n F$ always generates an everywhere-dense ideal in B_n , then F generates such an ideal in A .*

5. JOINT SPECTRA

Let B and K be topological rings. Let $\Delta(B, K)$ be the class of continuous homomorphisms of B into K , exclusive of the 0-homomorphism if K has a unit.

Let N be a positive integer, and let (b_1, \dots, b_N) be an ordered N -tuple of elements of B . By the joint K -spectral image for (b_1, \dots, b_N) relative to B we mean

$$5.1 \quad \{(\zeta(b_1), \dots, \zeta(b_N)) : \zeta \in \Delta(B, K)\}.$$

This is a subset of K^N , and we denote it by $\Delta(b_1, \dots, b_N; B, K)$.

Now let A be an inverse limit of rings B_v . Since every continuous homomorphism ζ of any B_v induces a homomorphism $\zeta \circ \pi_v$ of A , we have

$$5.2 \quad \Delta(a_1, \dots, a_N; A, K) \supset \bigcup_v \Delta(\pi_v a_1, \dots, \pi_v a_N; B_v, K),$$

for every N -tuple (a_1, \dots, a_N) in A .

When K is a Banach algebra (for example, the real or complex number field) and A is a *strongly dense* inverse limit, the inclusion in 5.2 becomes an equality. This can be traced back to a more fundamental relation.

5.21 THEOREM. *If K is a Banach algebra and A is a strongly dense inverse limit of topological rings B_v , then (in a natural sense)*

$$5.22 \quad \Delta(A, K) = \bigcup_v \Delta(B_v, K).$$

To show 5.21, we need first to imbed $\Delta(B_u, K)$ in $\Delta(A, K)$ in a natural way. For ζ in the former, $\zeta \circ \pi_u$ lies in the latter; and since $\pi_u A$ is dense in B_u , different ζ 's make for different $\zeta \circ \pi_u$'s. Thus we can imbed as desired. In fact, after imbedding, we obtain $\Delta(B_u, K) \subset \Delta(B_v, K)$ for $u \leq v$. Theorem 2.6 shows that the left side of 5.22 is exhausted by the union on the right. (More precisely, the left side of 5.22 is an injective, direct limit, with mappings being the duals of those in the inverse mapping system.)

The example given at the end of Section 4 illustrates the inclusion relation $\Delta(B_n, C) \subset \Delta(B_{n+1}, C)$ (here $K = C$, the complex number field). Indeed, after a natural identification, $\Delta(B_n, C)$ is the set of integers $\{0, 1, \dots, n\}$.

We now turn to another concept. We consider a ring B with unit, in which a subring K has been distinguished.

Let $\{b_1, \dots, b_N\}$ be a set of elements of B . Consider the class of ordered N -tuples (k_1, \dots, k_N) of elements in K such that $\{b_1 - k_1, \dots, b_N - k_N\}$ generates a *proper* right ideal in B . This class of N -tuples is the *joint right K -spectrum* of (b_1, \dots, b_N) relative to B . It is, of course, a subset of K^N , and is to be denoted by $\sigma(b_1, \dots, b_N; B, K)$. There is a relation to the joint spectral image,

$$5.3 \quad \sigma(b_1, \dots, b_N; B, K) \supset \Delta(b_1, \dots, b_N; B, K)$$

which rests on the fact that the kernel of each element of $\Delta(B, K)$ is a proper ideal.

We add that $\sigma(F; B, K)$ can be easily defined for an infinite family F of elements. See [9].

If this idea is to be applicable to an inverse limit, each B_v must contain K (or perhaps an algebra isomorphic to it), and the mappings (2.1) must preserve K ; and then A will contain K , and the π_v will also preserve it. (These conditions are usually met when all the B_v are algebras over a field K .) Now suppose (k_1, \dots, k_N) belongs to $(\pi_v a_1, \dots, \pi_v a_N; B_v, K)$ for some v . Then $(\pi_v(a_1 - k_1), \dots, \pi_v(a_N - k_N))$ is not regular in B_v , and hence $(a_1 - k_1, \dots, a_N - k_N)$ is not regular in A . This shows that

$$5.34 \quad \sigma(a_1, \dots, a_N; A, K) \supset \bigcup_v \sigma(\pi_v a_1, \dots, \pi_v a_N; B_v, K).$$

For the opposing inclusion, we have only 4.2 to help us, so we assume its hypotheses, as well as those just above.

5.4 THEOREM. *Let A be a dense inverse limit of complete metric rings with unit B_0, B_1, \dots , each containing a subring K such that*

$$5.41 \quad \pi_n^{n+1}(k) = k \quad (k \in K).$$

Then, for $a_1, \dots, a_N \in A$, 5.34 becomes an equality.

The proof consists in recognizing that 4.2 says that if $(0, \dots, 0)$ belongs to no term on the right of 5.34, then it does not belong to that on the left.

6. RELATION OF JOINT SPECTRUM AND SPECTRAL IMAGE

We assume

6.1 *A is a dense inverse limit of a sequence of complete metric rings with unit: B_0, B_1, \dots ;*

6.11 *each B_n contains a ring K such that 5.41 holds;*

6.2 *K is a Banach space;*

and finally

6.21 *the relation of K to each B_n is such that when the subset $\{b_1, \dots, b_N\}$ of B_n is contained in some proper right ideal of B_n , then there is a continuous homomorphism f of B_n into K such that $f(b_i) = 0$ ($i = 1, \dots, n$).*

6.3 THEOREM. *If 6.1 to 6.21 hold, then, for any subset $\{a_1, \dots, a_N\}$ of A which is contained in some proper right ideal of A , there is a continuous homomorphism f of A into K such that $f(a_i) = 0$ ($i = 1, \dots, n$); and*

$$6.31 \quad \sigma(a_1, \dots, a_N; A, K) = \Delta(a_1, \dots, a_N; A, K).$$

In view of the inclusion $\sigma \supset \Delta$ (5.3), the two parts of 6.3 say the same thing. The inclusion $\sigma \subset \Delta$ is provided by 6.21 and the fact that kernels of homomorphisms are ideals.

We have no application at hand which exploits the possibility (left open in 6.3) that A might be noncommutative.

To prevent misunderstanding, we state 6.3 expressly for \mathcal{F} -algebras. We recall that the field of scalars in an \mathcal{F} -algebra is the complex number field \mathbb{C} .

6.32 THEOREM. *Let A be a commutative \mathcal{F} -algebra with unit. If the set $\{a_1, \dots, a_n\}$ is included by some proper ideal, then there is a continuous homomorphism f of A on \mathbb{C} such that $f(a_i) = 0$ ($i = 1, \dots, n$).*

Proof. Since the B_n are here (necessarily) commutative Banach algebras, and $K = \mathbb{C}$, we see that 6.21 holds, because every ideal in B_n is contained in at least one maximal ideal, all maximal ideals in B_n are closed, and the quotient algebras are always isomorphic with \mathbb{C} . These are of course the basic facts of I. M. Gelfand's theory.

7. CONTINUITY OF MULTIPLICATIVE LINEAR FUNCTIONALS

Michael [8, 12.1] calls a topological algebra A *functionally continuous* if every homomorphism F of A onto the complex numbers is continuous. He establishes this property for commutative symmetric \mathcal{F} -algebras (see [8, 12.6]) and for some special algebras. Our present results enable us to extend Michael's considerably, because 6.3 implies the validity of a condition [8, 12.5 (I)]. At this point, we quote a definition from [8]. An element $y \in A$ is called *bounded* if the set of complex numbers $\{\zeta(y) : \zeta \in \Delta(A, \mathbb{C})\}$ is bounded.

7.1 THEOREM. *Let A be a commutative \mathcal{F} -algebra with unit. Then A is functionally continuous if either 7.12 or 7.13 and 7.14, below, hold.*

7.12 *There are elements g_1, \dots, g_N in A such that the inverse image in $\Delta(A, \mathbb{C})$ of each point in $\Delta(g_1, \dots, g_N; A, \mathbb{C})$ is a compact set in Δ (compact in the weak topology).*

7.13 *For each $x \in A$ there is a bounded element $y \in A$ such that $\zeta(x) = 0$ precisely whenever $\zeta(y) = 0$ (all $\zeta \in \Delta(A, \mathbb{C})$);*

7.14 *if y_1, y_2, \dots belong to A and are bounded, and $s(\zeta) = \zeta(y_1) + \zeta(y_2) + \dots$ converges uniformly on $\Delta(A, \mathbb{C})$, then there is an $a \in A$ such that $s(\zeta) = \zeta(a)$ for all $\zeta \in \Delta(A, \mathbb{C})$.*

The proof can easily be constructed by examining [8, 12.5]. Actually, Michael uses a weaker form of 7.13 and 7.14; but we submit that 7.13 and 7.14 are the conditions one will attempt to verify in any concrete case.

An easy way to ensure 7.12 is to assume that A is finitely generated; in other words, that there are g_1, \dots, g_N whose polynomials are dense in A . Then a continuous homomorphism is determined by its values on g_1, \dots, g_N . This makes $\Delta(g_1, \dots, g_N; A, \mathbb{C})$ a one-to-one image of $\Delta(A, \mathbb{C})$. The same conclusion can be reached in the case in which A is generated by g_1, \dots, g_N together with the inverses of those polynomials (in g_1, \dots, g_N) that *have* inverses in A (g_1, \dots, g_N "rational generators").

8. A CHARACTERIZATION OF THE ALGEBRA OF FUNCTIONS ON A PLANE OPEN SET

Let Ω be an open set in the complex plane; and let $\text{Hol}(\Omega)$ be the algebra of holomorphic functions on Ω . We shall exhibit it as an \mathcal{F} -algebra. Select compact subsets K_0, K_1, \dots of Ω such that $K_n \subset K_{n+1}$ and $\bigcup K_n = \Omega$. For $f \in \text{Hol}(\Omega)$, let

$$8.1 \quad \|f\|_n = \max |f(K_n)|.$$

These seminorms define a topology in $\text{Hol}(\Omega)$ which is independent of how the K_n are selected.

Let B_n be the uniform closure, under uniform convergence on K_n , of the functions in $\text{Hol}(\Omega)$ restricted to K_n , for $n = 0, 1, \dots$. Then $\text{Hol}(\Omega)$ is the dense inverse limit of these Banach algebras.

For our discussion, it is convenient to mention explicitly that there is a sequence of positive numbers r_0, r_1, r, \dots such that

$$8.2 \quad K_{n+1} \text{ includes the } r_n\text{-neighborhood of } K_n, \text{ for } n = 0, 1, 2, \dots.$$

Finally, we mention the special role of the function z ($z(\lambda) = \lambda$ for all $\lambda \in C$).

8.3 *The algebra is generated by z together with the inverses of those polynomials in z which have inverses in the algebra.*

This follows from Runge's theorem [2, p. 300]. The "Bereich" there is supposed to be connected; but this causes no difficulty, since each K_n lies in at most finitely many components.

The properties already considered do not characterize $\text{Hol}(\Omega)$. A competing algebra from which $\text{Hol}(\Omega)$ has to be distinguished is $\mathcal{C}^\infty [0, 1]$ which shares with $\text{Hol}(\Omega)$ all these properties, and also the following:

8.41 *There is a continuous derivation D in the algebra such that $Dz = 1$.*

A derivation is a linear transformation of the algebra in question into itself such that $D(fg) = f \cdot Dg + Df \cdot g$. Clearly $D^k = DD \dots D$ is also continuous, but this observation (valid in \mathcal{C}^∞ as well) does not suffice to establish the following property of $\text{Hol}(\Omega)$:

8.42 There are positive real numbers r_0, r_1, r, \dots such that

$$\|D^k f\|_n \leq k! r_n^{-k} \|f\|_{n+1} \quad (n = 0, 1, 2, \dots).$$

(From 8.41 alone one could hope at most for something like

$$\|D^k f\|_n \leq s_{n,k} \|f\|_{n+k}.)$$

The necessity of 8.42 arises out of 8.2 and Cauchy's integral for the k th derivative on K_n , taken over circles in K_{n+1} .

8.5 THEOREM. *Let A be an \mathcal{F} -algebra with a unit, a generator z in the sense of 8.3, and a derivation satisfying 8.41, 8.42. Then there is a closed semisimple subalgebra H of A such that (N being the radical of A)*

$$8.501 \quad A = H \dot{+} N \quad (\text{vector space direct sum})$$

and H is $\text{Hol}(\Omega)$ for some nonvoid open plane set Ω .

Before proceeding to the proof, we remark that we have not succeeded in making an example where N is nontrivial.

To proceed to the proof, we suppose A is the dense inverse limit of Banach algebras B_0, B_1, \dots . These algebras are commutative, by 8.3. Let K_n be the (compact) spectrum of z in B_n . Let Ω be the union of the K_n . The set Ω can be identified with $\Delta(A, C)$. We turn to showing the crucial fact 8.2.

For any complex number t ($|t| < r_n$) and any $f \in A$, the series

$$8.51 \quad \sum_{k=0}^{\infty} \frac{t^k D^k}{k!} f = T_t f$$

converges in the norm of B_n , and

$$\|T_t f\|_n \leq (1 - r_n^{-1} |t|)^{-1} \|f\|_{n+1}.$$

This comes from 8.42. Hence T_t can be extended to an operator of B_{n+1} into B_n , and

$$8.52 \quad \|T_t\| \leq (1 - r_n^{-1} \|t\|)^{-1}, \quad T_{t_1 + t_2} = T_{t_1} T_{t_2}.$$

It is moreover a homomorphism, as one can deduce from 8.51 and Leibniz' rule. Now $T_t(z) = z + t$. For $\zeta \in \Delta(B_n, C)$ (which we have identified with K_n), $\zeta \circ T_t$ is an element of K_{n+1} , provided $|t| < r_n$. Its position in the plane is $\zeta(T_t z) = \zeta(z) + t$. This proves 8.2. For $f \in A$ and $\zeta \in \Omega$, we set $\zeta(f) = \hat{f}(\zeta)$. If f is a polynomial or rational function in z , then \hat{f} is in $\text{Hol}(\Omega)$, and likewise for the limits, because they are uniform limits on each K_n which, according to 8.2, lies inside Ω , and is compact. Moreover, this homomorphism of A into $\text{Hol}(\Omega)$ is continuous.

We now study the homomorphism of $\text{Hol}(\Omega)$ into A , as given by the classical formula [7, 78]

$$8.53 \quad F \rightarrow \frac{1}{2\pi i} \int_{\Gamma} \frac{F(\lambda)}{\lambda - z} d\lambda = a_F \quad (F \in \text{Hol}(\Omega)).$$

Γ must be drawn inside Ω . If Γ is drawn so as to enclose, but not touch, K_n , then a_F is an element of B_n .

8.54 *This element does not change if Γ is varied, as long as Γ satisfies the conditions imposed.*

The proof is by linear functionals, and does not require semisimplicity. Gradual displacement of Γ away from K_0, K_1, \dots defines a sequence of elements, selected in turn from B_0, B_1, \dots , of such type as to form an element of A . Of course $a_{F+G} = a_F + a_G$; and $a_{FG} = a_F a_G$. The latter is obvious with semisimplicity, but we can give a proof without it, as follows. Let two contours enclosing K_n be chosen, and define a_F, a_G in B_n , using 8.53. By a deformation we can provide that

$$8.55 \quad \text{the contour for } a_F \text{ lies inside that for } a_G.$$

We use μ as the variable of integration in the case of a_G , so that we avoid the necessity of mentioning the contours. The identity

$$-4\pi^2 a_F a_G = \int F(\lambda) (\lambda - z)^{-1} \gamma(\lambda) d\lambda - \int G(\mu) (\mu - z)^{-1} \phi(\mu) d\mu ,$$

where

$$\gamma(\lambda) = \int G(\mu) (\mu - \lambda)^{-1} d\mu, \quad \phi(\mu) = \int F(\lambda) (\mu - \lambda)^{-1} d\lambda ,$$

is readily verified. But by 8.55, $\gamma(\lambda) = 2\pi i G(\lambda)$, and $\phi(\mu) = 0$; and we obtain $(2\pi i)^2 a_{FG}$. With these facts established, we settle down to specific contours $\Gamma_1, \Gamma_2, \dots$, where Γ_n surrounds K_n , is disjoint from K_n , and lies in K_{n+1} . It follows from 8.53 that

$$8.56 \quad \| a_F \|_n \leq c_n \max | F(K_{n+1}) | ,$$

where c_n depends on the length of Γ_n , and the behavior of $(\lambda - z)^{-1}$ on Γ_n , but not on F . It also follows from 8.53 that $\widehat{a}_F = F$, and so

$$8.561 \quad \max | F(K_n) | \leq \| a_F \|_n .$$

Let H be the subalgebra of all $a_F, F \in \text{Hol}(\Omega)$. Then H is isomorphic to $\text{Hol}(\Omega)$, algebraically (8.53) and topologically (8.56, 8.561). Moreover, $f \rightarrow a_f$ is a continuous projection of A on H (it can be shown that $a_1 = 1 \in A$). Also, $f - a_f$ is always in the radical. This concludes the proof of 8.5.

Evidently, if we adjoin to the conditions of 8.5 any condition ensuring semi-simplicity, then we have a characterization of $\text{Hol}(\Omega)$.

Helmer [3, Thm. 9] shows that if f_1, \dots, f_N are entire functions, then the set $\{f_1, \dots, f_N\}$ generates a principal ideal in the algebra of entire functions. We desire to generalize this result here.

8.6 THEOREM. *Let Ω be an open set in the plane. Let $\{f_1, \dots, f_N\}$ be a finite subset of $\text{Hol}(\Omega)$. Then the ideal generated by $\{f_1, \dots, f_N\}$ is a principal ideal.*

To prove 8.6 we first observe, as is done in [3], that it is sufficient to treat the case in which f_1, \dots, f_N have no common zeros on Ω . Next, we observe that Ω corresponds naturally to $\Delta(\text{Hol}(\Omega), C)$ (see [5]). Then we may apply 4.2, according to which $\{f_1, \dots, f_N\}$ generates (1).

Finally, we wish to describe an algebra of holomorphic functions which is an \mathcal{F} -algebra, but in which the seminorms cannot generally be chosen to be *maximae modulorum*. Let D_0, D_1, \dots be a sequence of closed discs in the plane, of radius r_0, r_2, \dots and center ζ_0, ζ_1, \dots , respectively. Let A be the class of functions that are defined on the union of D_0, D_1, \dots and have at each ζ_n a Taylor expansion that converges absolutely on D_n . For $f \in A$, and each n , define

$$8.7 \quad p_n(f) = \sum (k!)^{-1} r_n^k | f^{(k)}(\zeta_n) | .$$

These seminorms p_1, p_2, \dots define an \mathcal{F} -algebra-structure in A . The space $\Delta(A, C)$ turns out to be the union of D_0, D_1, \dots . We can therefore assert that if f_1, \dots, f_N are functions in A that have no common zeros, then there are g_1, \dots, g_N in A such that $f_1 g_1 + \dots + f_N g_N = 1$.

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