SIMPLE WAVES IN THE STEADY ROTATIONAL PLANE SUPERSONIC FLOW OF A POLYTROPIC GAS OF CONSTANT ENTROPY

N. Coburn

1. INTRODUCTION

By use of the intrinsic form of the characteristic relations [1], we study the problem of determining all plane, steady rotational flows possessing ∞^1 straight-line bicharacteristics. We call these flows *simple waves*, and we investigate their properties.

For all γ (where γ is the ratio of the specific heats of the gas), two types of simple waves are shown to exist: (1) flows in which the simple waves are radial lines; (2) flows in which the simple waves are the ∞^1 lines tangent to a circle. These flows are vortex flows. Expressions for the speed of sound, the magnitude of the velocity, and the vorticity are determined. In the case of $\gamma = 3$, the above two types of simple waves are found to be the *only* simple waves. Finally, all flows at Mach number one are shown to be centered simple waves.

2. THE BASIC RELATIONS

The basic relations were derived in a previous paper [1]. Here, we shall summarize them briefly. References to equations in [1] will be starred.

Let x^j $(j = 1, 2, 3)$ denote a Cartesian orthogonal coordinate system in Euclidean three-space, and let us denote partial derivatives by the symbolism

$$\partial_j = \frac{\partial}{\partial x^j} .$$

In a Cartesian orthogonal system, covariant and contravariant quantities are equivalent. However, in order to use the Einstein summation convention of summing on repeated upper and lower indices, we shall use both of these quantities. Further, we introduce the following scalars, vectors and tensors:

- q, the magnitude of the velocity vector;
- c, the local speed of sound;
- $b = (q^2 - c^2)^{1/2}$;
- t^j, the unit tangent vector along a bicharacteristic curve;
- n^j, the unit normal vector of a characteristic surface;
- p^j, a unit vector orthogonal to t^j, n^j and such that the ordered orthogonal triple p^j, t^j, n^j forms a right-handed set;

Received January 20, 1958.

The author is indebted to the Office of Ordnance Research, U. S. Army, for financial aid during this research, which was administered by the Engineering Research Institute of the University of Michigan.

$w^j = p^k \partial_k p^j$, the curvature vector of the p^j-congruence;

$m^j = t^k \partial_k t^j$, the curvature vector of the t^j-congruence;

$u^j = n^k \partial_k n^j$, the curvature vector of the n^j-congruence;

s_{jk}, the second fundamental tensor of the characteristic surfaces;

M^*, the mean curvature of the characteristic surfaces.

Finally, we introduce the notation

$$\frac{\partial}{\partial t} = t^j \partial_j, \quad \frac{\partial}{\partial n} = n^j \partial_j, \quad \frac{\partial}{\partial p} = p^j \partial_j$$

for directional derivatives, and the stagnation enthalpy

$$h_0 = \frac{q^2}{2} + \frac{c^2}{\gamma - 1}\,. \tag{2.1}$$

For constant entropy, the basic equations (4.1)*, (4.5)*, (4.6)*, (3.8)* for $c > 0$ become, respectively,

$$b^2 \frac{\partial}{\partial t} \frac{q^2}{b} = -c^2 b t_k w^k + b^2 c s_{jk} t^j t^k + c^3 M^* + b \frac{\partial h_0}{\partial t}\,, \tag{2.2}$$

$$\begin{aligned} b[(\gamma - 3)q^2 + 4c^2] \frac{\partial b}{\partial n} &= [2q^2 - (\gamma + 3)c^2] b^2 s_{jk} t^j t^k \\ &+ (\gamma - 1) b^2 c^2 M^* - cb[(\gamma - 3)q^2 + 4c^2] t_k u^k \\ &- (\gamma - 1) b^3 c t_k w^k + [(\gamma - 3)q^2 - (\gamma - 5)c^2] \frac{\partial h_0}{\partial n}\,, \end{aligned} \tag{2.3}$$

$$q \frac{\partial q}{\partial p} = -(b^2 m_k + c^2 u_k) p^k + cbK + \frac{\partial h_0}{\partial p}\,, \tag{2.4}$$

$$c \frac{\partial h_0}{\partial n} + b \frac{\partial h_0}{\partial t} = 0\,, \tag{2.5}$$

where

$$K = n^j p^k (\partial_j t_k - \partial_k t_j)\,. \tag{2.6}$$

We shall study *plane flows*. This means that (1) q, c, h_0 are functions of x and y only, where x, y, z are orthogonal Cartesian coordinates; (2) t^j and n^j lie in planes perpendicular to the z-axis and are independent of z; (3) p^j is parallel to the z-axis. Thus, $w^j = 0$, and the characteristic surfaces are cylinders with generators parallel to the z-axis. One principal direction lies along these generators (the p^j-direction). Since the principal normal of t^j is n^j, it follows that t^j is the other principal direction and that

$$M^* = s_{jk} t^j t^k = -k\,, \tag{2.7}$$

where k is the curvature of the t^j-congruence. The ordered triad consisting of the tangent vector to a curve in the xy-plane, its principal normal, and the vector p^j form a right-handed set in the usual convention. Hence, we may write

$$t^k \partial_k t^j = m^j = kn^j, \tag{2.8}$$

$$n^k \partial_k n^j = u^j = -'kt^j, \tag{2.9}$$

where $'k$ is the curvature of the n^j-congruence.

Now we introduce the orthogonal curvilinear coordinates α, β, in the xy-plane, where α = variable (or β = constant) are the bicharacteristic curves with unit tangent vector t^j, and where β = variable (or α = constant) are the curves of the n^j-congruence. We denote the square of the element of arc in the $\alpha\beta$-system by

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2, \tag{2.10}$$

where A and B are the metric coefficients. If $\theta(\alpha, \beta)$ is the angle between the x-axis and the vector t^j, along a bicharacteristic α = variable, then the components of t^j and n^j in the xy-system are

$$t^j, (\cos\theta, \sin\theta) \quad \text{and} \quad n^j, (-\sin\theta, \cos\theta). \tag{2.11}$$

From (2.11), we find that

$$\frac{\partial x}{\partial \alpha} = A\cos\theta, \quad \frac{\partial y}{\partial \alpha} = A\sin\theta, \quad \frac{\partial x}{\partial \beta} = -B\sin\theta, \quad \frac{\partial y}{\partial \beta} = B\cos\theta. \tag{2.12}$$

Forming the scalar product of (2.8) with n_j, and noting that

$$t^k \partial_k t^j = A^{-1}\frac{\partial t^j}{\partial \alpha},$$

we find by use of (2.11) that

$$k = A^{-1}\frac{\partial \theta}{\partial \alpha}. \tag{2.13}$$

Similarly, (2.9) furnishes

$$'k = B^{-1}\frac{\partial \theta}{\partial \beta}. \tag{2.14}$$

By differentiation of (2.12) and use of

$$\frac{\partial^2 x}{\partial \alpha\, \partial \beta} = \frac{\partial^2 x}{\partial \beta\, \partial \alpha}, \quad \frac{\partial^2 y}{\partial \alpha\, \partial \beta} = \frac{\partial^2 y}{\partial \beta\, \partial \alpha},$$

we find that

$$A\frac{\partial \theta}{\partial \beta} = \frac{\partial B}{\partial \alpha}, \quad B\frac{\partial \theta}{\partial \alpha} = \frac{\partial A}{\partial \beta}. \tag{2.15}$$

Thus (2.13) and (2.14) may be written in the form

$$k = -\frac{1}{AB}\frac{\partial A}{\partial \beta}, \qquad 'k = \frac{1}{AB}\frac{\partial B}{\partial \alpha}. \tag{2.16}$$

Now we express (2.2) to (2.5) in terms of derivatives with respect to α and β. First we note that

$$t^j\partial_j = \frac{\partial}{\partial t} = A^{-1}\frac{\partial}{\partial \alpha}, \qquad n^j\partial_j = \frac{\partial}{\partial n} = B^{-1}\frac{\partial}{\partial \beta}.$$

Further, from the fact that the p^j form a congruence of straight lines, and by use of (2.7), (2.13), (2.14), (2.16), we find that

$$t^k w_k = K = 0, \qquad t^k u_k = -'k = \frac{1}{AB}\frac{\partial B}{\partial \alpha} = \frac{-1}{B}\frac{\partial \theta}{\partial \beta}, \tag{2.17}$$

$$s_{jk}t^j t^k = M^* = -k = \frac{1}{AB}\frac{\partial A}{\partial \beta} = -\frac{1}{A}\frac{\partial \theta}{\partial \alpha}. \tag{2.18}$$

Thus, (2.2) to (2.5) become

$$b^2\frac{\partial}{\partial \alpha}\frac{q^2}{b} = -cq^2\frac{\partial \theta}{\partial \alpha} + b\frac{\partial h_0}{\partial \alpha}, \tag{2.19}$$

$$[(\gamma - 3)q^2 + 4c^2]\frac{\partial b^2}{\partial \beta} = 4b^2[q^2 - 2c^2]\frac{\partial}{\partial \beta}\log A \tag{2.20}$$

$$+ 2cb[(\gamma - 3)q^2 + 4c^2]\frac{\partial \theta}{\partial \beta} + 2[(\gamma - 3)q^2 - (\gamma - 5)c^2]\frac{\partial h_0}{\partial \beta},$$

$$\frac{c}{B}\frac{\partial h_0}{\partial \beta} + \frac{b}{A}\frac{\partial h_0}{\partial \alpha} = 0. \tag{2.21}$$

Note that (2.4) is an identity and that h_0 can be expressed in terms of c and q by use of (2.1).

3. SIMPLE WAVES IN PLANE ROTATIONAL FLOWS

If the bicharacteristics form a family of ∞^1 straight lines, then the flow is said to consist of ∞^1 simple waves (the straight-line bicharacteristics). Thus $k = 0$ (see (2.8)), and (2.16) shows that $A = A(\alpha)$. By proper choice of scale factor (taking α to be arc length) along the curves α = variable, we can assume that

$$A = 1. \tag{3.1}$$

Hence the second equation of (2.15) leads to $\theta = \theta(\beta)$, and by proper choice of scale factor along the curves β = variable, we can choose

$$\theta = \beta. \tag{3.2}$$

The special case $\theta = 1$ implies that the ∞^1 bicharacteristics are parallel lines. From (3.1), (3.2) and the first equation of (2.15), we find

$$B = \alpha + g(\beta), \tag{3.3}$$

where $g(\beta)$ is an arbitrary function of β. The special case $\theta = 1$ leads to $B = 1$.

By substituting (3.1), (3.2), (3.3) into (2.12) and integrating, we find

$$x = \alpha \cos \beta - \int g \sin \beta \, d\beta, \qquad y = \alpha \sin \beta + \int g \cos \beta \, d\beta. \tag{3.4}$$

Two special cases will be studied: (1) $g(\beta) = 0$; (2) $g(\beta) = a\beta$, where a is an arbitrary nonvanishing constant.

Case 1: $g(\beta) = 0$. Here (3.4) becomes

$$x = \alpha \cos \beta, \qquad y = \alpha \sin \beta. \tag{3.5}$$

Thus $r = \alpha$ and $\theta = \beta$, where r and θ are the usual polar coordinates. *The bicharacteristics,* θ = constant, *are radial straight lines.*

Case 2: $g(\beta) = a\beta$. Here (3.4) becomes

$$x = (\alpha + a\beta) \cos \beta - a \sin \beta, \qquad y = (\alpha + a\beta) \sin \beta + a \cos \beta. \tag{3.6}$$

We shall show that *the locus of the bicharacteristics,* β = constant, *consists of the* ∞^1 *lines tangent to the circle* $x^2 + y^2 = a^2$.

Since $\theta = \beta$ is the angle from the x-axis to any bicharacteristic, it follows that $\beta + \pi/2$ is the angle between the x-axis and any line perpendicular to a bicharacteristic. The coordinates of any point P on the circle $x^2 + y^2 = a^2$ are

$$a \cos\left(\beta + \frac{\pi}{2}\right), \qquad a \sin\left(\beta + \frac{\pi}{2}\right).$$

Thus the coordinates of a point Q on the line tangent to $x^2 + y^2 = a^2$ at P and at a distance R from the point P are

$$'x = a \cos\left(\beta + \frac{\pi}{2}\right) + R \cos \beta, \qquad 'y = a \sin\left(\beta + \frac{\pi}{2}\right) + R \sin \beta. \tag{3.7}$$

Evidently, if $R = \alpha + a\beta$, then from (3.6) and (3.7), we see that

$$x = {'x}, \qquad y = {'y},$$

and β = constant are the ∞^1 lines tangent to $x^2 + y^2 = a^2$.

Now we consider the system (2.19), (2.20$_1$, (2.21) for the case of simple waves ((see (3.1), (3.2), (3.3)). First (2.19) reduces to

$$b\frac{\partial}{\partial \alpha}\frac{q^2}{b} = \frac{\partial h_0}{\partial \alpha}. \tag{3.8}$$

By use of (2.1), this equation becomes

$$(1 - \gamma)c^2\frac{\partial q^2}{\partial \alpha} + [(\gamma - 3)q^2 + 2c^2]\frac{\partial c^2}{\partial \alpha} = 0. \tag{3.9}$$

Integrating this homogeneous equation, we obtain

(3.10) $$q^2 = c^2 + f^2 c^n,$$

where f is an arbitrary function of β and

(3.11) $$n = 2(\gamma - 3)/(\gamma - 1).$$

From (3.10), we see that h_0 of (2.1) reduces to

(3.12) $$2(\gamma - 1)h_0 = (\gamma + 1)c^2 + (\gamma - 1)f^2 c^n.$$

Substituting (3.10), (3.12) into (2.20), (2.21) and solving for $\partial c^2/\partial\alpha$ and $\partial c^2/\partial\beta$, we obtain two very complicated relations whose consistency must be studied. Since no complete results have been obtained for the general γ, we shall study two γ's: (1) the general theory for $\gamma = 3$; (2) the cases $B = \alpha$ and $B = \alpha + a\beta$ (see (3.5), (3.6)) for arbitrary γ.

First we consider the case $\gamma = 3$. From (3.11), we find that $n = 0$, and (3.10) becomes

(3.13) $$q^2 = c^2 + f^2.$$

Thus (3.12) reduces to

(3.14) $$2h_0 = 2c^2 + f^2.$$

By use of (3.14) and (3.13), we find that (2.20) furnishes

(3.15) $$\frac{\partial c^2}{\partial \beta} = ff' - 2cf.$$

On substitution of (3.14) and (3.15) into (2.21), the latter becomes, through use of (3.1) and (3.3),

(3.16) $$\frac{\partial c}{\partial \alpha} - \frac{c}{\alpha + g} = -\frac{f'}{\alpha + g}.$$

Integrating the linear equation (3.16), we obtain

(3.17) $$c = f' + (\alpha + g)k,$$

where k is an arbitrary function of β. If we substitute (3.17) into (3.15) and require that the resulting quadratic in α be an identity, we find that f *and* k *are arbitrary constants but*

(3.18) $$g = a\beta,$$

where a *is an arbitrary constant. Thus, for* $\gamma = 3$, *the only simple waves are* (1) *a family of radial straight lines* (see (3.5) for $a = 0$), (2) *a family of lines tangent to the circle* $x^2 + y^2 = a^2$ (see (3.6)). From (3.5), (3.7) and (3.18), we see that (3.17) leads to

(3.19) $$c = kR, \qquad R = \alpha + a\beta,$$

where R is (1) the distance from the point of intersection of the radial lines in the first case; (2) the distance from a point P of $x^2 + y^2 = a^2$ to any point Q of the line tangent to $x^2 + y^2 = a^2$ and passing through P, in the second case. Finally, from (3.13), (3.14) and (3.19), we see that for $\gamma = 3$,

$$q^2 = k^2R^2 + f^2, \qquad 2h_0 = 2k^2R^2 + f^2. \tag{3.20}$$

The streamlines are R = constant; the flows are vortex flows (see [4]; Prim shows that the only plane rotational flows with an isometric streamline pattern are vortex and parallel flows). From (3.3)* and (2.17), we see that the vorticity vector lies in the z-direction and has magnitude 2k.

Now we consider the possibility of solutions for c, q, h_0 in terms of R, where $A = 1$, $R = B = \alpha + a\beta$, in the case of arbitrary γ. This implies that the simple waves are vortex flows [4] of type (3.5) or (3.6). Since $h_0 = h_0(c)$, equation (2.21) leads, by use of (3.10), to

$$c^2 = c_0^2\left(\frac{R}{R_0}\right)^{\gamma-1}, \qquad q^2 = c_0^2\left[\left(\frac{R}{R_0}\right)^{\gamma-1} + \left(\frac{R}{R_0}\right)^{\gamma-3}\right], \tag{3.21}$$

and hence by (3.12) to

$$2(\gamma - 1)h_0 = c_0^2\left[(\gamma + 1)\left(\frac{R}{R_0}\right)^{\gamma-1} + (\gamma - 1)\left(\frac{R}{R_0}\right)^{\gamma-3}\right], \tag{3.22}$$

where a is an arbitrary negative constant, f is an arbitrary function of β and

$$R_0 = -a \qquad f = c_0^{2/(\gamma-1)}. \tag{3.23}$$

It remains to show that (2.20) is satisfied. By substituting (3.21) and (3.22) into (2.20), we obtain a fourth-degree polynomial in R, whose coefficients are zero if c_0 is constant. Again, from (3.3)*, (2.17) and (3.21), we find that the vorticity vector is parallel to the z-axis and has magnitude

$$\omega = \frac{c_0}{R_0}\,\frac{\gamma + 1}{2}\left(\frac{R}{R_0}\right)^{(\gamma-3)/2}. \tag{3.24}$$

4. ROTATIONAL PLANE FLOWS AT MACH NUMBER ONE

We shall show that *all plane rotational flows at Mach number one* ($q = c$) *are vortex flows with centered simple waves (the streamlines are orthogonal to the straight-line bicharacteristics).*

To prove this result, we shall consider (2.21)*, (3.6)*, (3.7)* and (3.9)* which are valid for $q = c$ or $b = 0$. These relations become, by use of (2.16), (2.17), and (2.18),

$$M^* = -k = 0, \tag{4.1}$$

$$\frac{\partial h_0}{\partial \alpha} = c\frac{\partial c}{\partial \alpha} + \frac{Ac^2}{B}\frac{\partial \theta}{\partial \beta}, \tag{4.2}$$

$$\frac{\partial h_0}{\partial \beta} = 0. \tag{4.3}$$

From (4.1) and (2.8), it follows that the bicharacteristics are straight lines. Thus, by use of (3.1), (3.2), (3.3), the equation (4.2) becomes

$$\frac{dh_0}{d\alpha} = c\frac{\partial c}{\partial \alpha} + \frac{c^2}{\alpha + g(\beta)}. \tag{4.4}$$

For $q = c$, the relation (2.1) reduces to

$$h_0 = \frac{\gamma + 1}{2(\gamma - 1)} c^2. \tag{4.5}$$

In view of (4.3), it follows that c is independent of β. Hence, by (4.4), $g(\beta)$ must be a constant. By use of (3.4), we see that this constant can be chosen to be zero. *The bicharacteristics are radial straight lines* (see 3.5), *and* $r = \alpha$ (where r is the radial distance in polar coordinates). Integration of (4.4) by means of (4.5) leads to

$$c^2 = c_0^2\left(\frac{r}{r_0}\right)^{\gamma - 1}, \tag{4.6}$$

where c_0 and r_0 are arbitrary constants (see 3.21). The magnitude of the vorticity vector is (see (3.3)*)

$$\omega = \frac{c_0}{r_0}\frac{\gamma + 1}{2}\left(\frac{r}{r_0}\right)^{(\gamma - 3)/2}. \tag{4.7}$$

5. SIGNIFICANCE OF THE RESULTS

R. Courant and K. O. Friedrichs have shown that the irrotational flow in a region adjacent to a region of constant state is a simple wave [2, p. 61]. For $\gamma = 3$, the only rotational simple waves are waves of the two types discussed in Section 3. In both of these waves, the vorticity vector has magnitude $2k$, where $c = kR$. Thus *no bicharacteristic can be the boundary line between a region of irrotational flow and a region of rotational flow, when the latter region is spanned by a simple wave.*

To determine *all* simple waves in the cases where γ is arbitrary, we use (3.10) and (3.12) to eliminate q and h_0 in (2.20) and (2.21), solve these last equations for $\partial c^2/\partial\alpha$ and $\partial c^2/\partial\beta$, and then require that

$$\frac{\partial^2 c^2}{\partial\alpha\,\partial\beta} = \frac{\partial^2 c^2}{\partial\beta\,\partial\alpha}. \tag{5.1}$$

By eliminating $\partial c^2/\partial\alpha$ and $\partial c^2/\partial\beta$ in (5.1), we obtain an algebraic equation in c^2, α, β, which must be consistent with the solutions of c^2 obtained by integrating the equations for $\partial c^2/\partial\alpha$ and $\partial c^2/\partial\beta$ [3, pp. 15, 16]. This is a complicated procedure and probably will not furnish any solutions in addition to (3.21), (3.22). Since the vorticity does not vanish along any simple wave of this type (see 3.24), it appears plausible that the result of the previous paragraph is valid for all γ.

Finally, we note that the only possible supersonic, rotational flows at Mach number one are the centered simple waves discussed in Section 4.

REFERENCES

1. N. Coburn, *Intrinsic form of the characteristic relations in the steady supersonic flow of a compressible fluid,* Quart. Appl. Math. 15 (1957), 237-248.

2. R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience Publishers, Inc. New York, 1948.

3. T. Levi-Civita, *The absolute differential calculus (calculus of tensors);* translated by Miss M. Long; Blackie and Son, London, 1929.

4. R. C. Prim, *Steady rotational flow of an ideal gas*, J. Rational Mech. Anal. 1 (1952), 425-496.

The University of Michigan