# IMPROPER AFFINE HYPERSPHERES OF CONVEX TYPE AND A GENERALIZATION OF A THEOREM BY K. JÖRGENS

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## 1. INTRODUCTION

The purpose of this article is to derive certain inequalities satisfied by the elliptic solutions of the nonlinear partial differential equation

$$\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 1 \tag{1.1}$$

with n independent variables $(x) = (x^1, \cdots, x^n)$. An elliptic solution of (1.1) is a function $u = f(x) = f(x^1, \cdots, x^n)$ whose Hessian matrix (symmetric matrix of second derivatives) is definite at each point; that is to say, u is locally either a convex or a concave function (the latter can occur only if n is even) of (x): without loss of generality we shall consider only convex solutions.

The motivation for studying (1.1) is that it represents a special but significant case of the more general equation

$$\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = \phi(x) , \tag{1.1a}$$

where $\phi(x)$ is a given, positive-valued function; this last equation occurs, for instance, as a result of an elementary transformation of the one arising from the Minkowski problem on closed, convex hypersurfaces in Euclidean (n + 1)-space. The outstanding question about the regularity of weak solutions of the Minkowski problem is reduced to the question, if the kth partial derivatives of $\phi(x)$ in (1.1a) satisfy a Hölder condition, whether there exists a convex solution u of (1.1a), locally at least, and whether any such solution has all of its (k + 2)nd partial derivatives satisfying a Hölder condition.

It has been shown that (1.1a) has at most one convex solution u in a bounded domain D, if the boundary values of u are prescribed; however a necessary condition for the existence of such a solution with arbitrarily given, but smooth, boundary values is that the domain D be strictly convex. Aside from this, in order to establish the existence of a solution of the boundary value problem by the method of continuity, one requires some *a priori* estimates of the bounds for u, for its second partial derivatives, and for their Hölder constants, in terms of the domain D, the function $\phi(x)$, the boundary data, and their Hölder constants.

The main results of this article (wishfully the first of a series), viewed in the context of the more general problem, provide information on the *a priori* bounds, irrespective of the boundary values of u; the dependence of the bounds on the properties of $\phi(x)$ is set aside for the time being, by considering equation (1.1) instead of the more general one (1.1a). Stated briefly, the inequalities established in the present

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paper provide an *a priori* majorant for the third derivatives of a convex solution of (1.1) in a domain D at an interior point $(x) \in D$ in terms of any uniform bound for the second derivatives in any neighborhood of (x), and in terms of the distance of (x) from the boundary of D: this reduces the problem of the existence of a solution u for the boundary value problem associated with (1.1) to estimating uniform bounds for the second derivatives in D. Better still, when the number n of independent variables does not exceed 5, Theorem 2, which is a generalization of the same statement for 2 variables by K. Jörgens [6] (related to Bernstein's theorem on minimal surfaces), enables us to dispense with uniform bounds for the second derivatives in a neighborhood of (x): the bounds for the third derivatives at (x) depend only on the bounds for the second derivatives at (x) and on the distance of (x) from the boundary of D.

Although most of the results in this paper can be formulated and proved in the language of partial differential equations, it would seem unfair to the informed reader to hide the fact that the heuristic reasoning behind them is decidedly geometrical, pertaining more specifically to affine differential geometry; on the other hand I am reluctant to make a display of concepts unfamiliar to many nonspecialists, when they are not essential in understanding the article. For this reason I adopt a compromise course, using only the well known formal techniques of the Ricci calculus (as in L. P. Eisenhart's *Riemannian Geometry* or in [3]), but interjecting an occasional paragraph or restatement of a result, interpreting the context in geometrical terms with special reference to [1].

It is clear that equation (1.1) is invariant under unimodular linear transformations of the independent variables $x^1, \cdots, x^n$; therefore it should not be too surprising if most of the analysis to follow should be based on the affine invariants of the graph of $u = f(x)$ $((x) = (x^1, \cdots, x^n))$ in the $(n+1)$-dimensional affine space $E_{n+1}$, with this graph considered as a differentiable hypersurface $V_n$, immersed in $E_{n+1}$. As long as we allow only linear transformations of the independent variables (x), the array of all partial derivatives of u of any given order k can be interpreted as the components of a covariant tensor of valence k, symmetric in all pairs of indices: we are chiefly interested in the tensors defined by the second and third derivatives,

$$g_{ij}(x) = \frac{\partial^2 u}{\partial x^i \partial x^j} \qquad (i, j, \cdots = 1, 2, \cdots, n) \tag{1.2}$$

and

$$A_{ijk}(x) = -\frac{1}{2}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}. \tag{1.3}$$

Since u is convex and satisfies (1.1), it follows that the symmetric tensor $g_{ij}$ defined in (1.2) is positive definite; it therefore determines, in the domain of definition of u, a Riemannian metric

$$ds^2 = g_{ij}\, dx^i dx^j, \tag{1.4}$$

associated with the given function in a natural way, as well as the contravariant tensor $g^{ij}$ defined by the inverse of the matrix of $g_{ij}$. We observe immediately from (1.1) that the determinant g of $g_{ij}$ is identically equal to 1; therefore by differentiation we obtain the relation

$$g^{ij} A_{ijk} = 0 \qquad (k = 1, \cdots, n). \tag{1.5}$$

[Both tensors $g_{ij}$ and $A_{ijk}$ are affine invariants of the graph $V_n$ of u in $E_{n+1}$. For the special case of a solution of (1.1), the metric (1.4) is known as the Schwarz-Pick metric of $V_n$, and the cubic differential form $\psi = A_{ijk}dx^i dx^j dx^k$ is known as the Darboux-Fubini-Pick form. Both the quadratic and the cubic differential forms are defined in the general case of a differentiable hypersurface in $E_{n+1}$, but only in the case of the graph of a solution of (1.1) does their expression take the simple form of (1.2) and (1.3). The graphs of the solutions of (1.1) characterize by means of certain affinely invariant, differential geometric properties the so-called *improper affine hyperspheres*. For a complete exposition of this subject, see [1], with whose notation we have tried to conform.]

If the values of the second derivatives $g_{ij}$ of u are known at a point (x), then a natural norm for the third derivatives at (x) is given by the Riemannian absolute value of the tensor $A_{ijk}$, namely by

$$|\psi| = (A^{ijk}A_{ijk})^{1/2}, \tag{1.6}$$

where $A^{ijk} = g^{ia}g^{jb}g^{kc}A_{abc}$. Similarly the most natural yardstick for measuring in some sense the distance of (x) from the ideal boundary of the domain of existence of u is in terms of the geodesic distance defined by (1.4). Let D denote the interior of the domain of existence of a convex solution u of (1.1) (we remark here that, for the purpose of this paper, D need not be a schlicht domain, nor u single-valued). For any point $(x)_0 \in D$, we define $\mathscr{S}_{x_0}$ to be the set of all rectifiable, parametrized arcs C in D, images of the semiclosed interval $\{0 \leq t < 1\}$, such that the initial point $t = 0$ corresponds to $(x)_0$ and the arc itself is a relatively closed set in the topology of D: in other words $\mathscr{S}_{x_0}$ is the set of rectifiable arcs C in D joining $(x)_0$ with the ideal boundary of D.

DEFINITION 1. *Let* u *be a convex solution of* (1.1) *in an open domain* D. *For each point* $(x)_0 \in D$, *we define the geodesic distance* $\gamma(x)_0$ *of* $(x)_0$ *from the boundary of* D (*relative to* u) *as the quantity*

$$\gamma(x)_0 = \inf_{C\in \mathscr{S}_{x_0}}\left(\int_C (g_{ij}dx^i dx^j)^{1/2}\right), \tag{1.7}$$

*where* $g_{ij}$ *is defined by* (1.2), *and the values of* $\gamma(x)_0$ *can range over the interval* $0 < \gamma(x)_0 \leq \infty$. *Similarly we define the* affine distance $\delta(x)_0$ *of* $(x)_0$ *from the boundary of* D (*always relative to* u) *as the quantity*

$$\delta(x)_0 = \inf_{C\in \mathscr{S}_{x_0}}\left(\int_C (g_{ij}((x)_0)\, dx^i dx^j)^{1/2}\right), \tag{1.8}$$

*where the metric tensor is the tensor whose components are constant and equal to the values of* $g_{ij}$ *at the initial point.*

We remark here incidentally that, if one replaces the independent variables (x) through a unimodular linear transformation followed by a translation, so that $(x)_0$ becomes the origin O and $g_{ij}(O) = \delta_{ij}$ (Kronecker delta) in the new coordinate system, then the affine distance of $(x)_0$ from the boundary is the same as the Euclidean distance of the origin from the nearest singularity, with respect to the new coordinate system. Thus $\delta(x)_0 = \infty$ if and only if u is a single-valued function, defined over the whole numerical n-space. In addition, we point out that since u satisfies (1.1), an upper bound for the second partial derivatives of u determines a positive lower bound for the eigenvalues of $g_{ij}$; hence, if one wants to estimate a lower bound for $\delta(x)$ for any given $(x) \in D$, one needs only the Euclidean distance of $(x)$ from the boundary and an upper bound for the second derivatives of $u$ at $(x)$ itself, while, if one wants to estimate a lower bound for $\gamma(x)$, one needs in addition a uniform upper bound for the second derivatives in some neighborhood of $(x)$. Having thus introduced the principal notations and explanations, we can now state the two main theorems with their respective corollaries.

THEOREM 1. *Let* $u = f(x^1, \cdots, x^n)$ *be a convex solution of* (1.1) *in an open domain* $D$, *and suppose that* $u$ *is of class* $\mathscr{C}^5$. *Consider the Riemannian structure in* $D$ *defined by* (1.4) *and* (1.2), *and the corresponding norm* $|\psi|$ *for the third derivatives of* $u$, *defined by* (1.3) *and* (1.6), *at each point* $(x) \in D$. *Then the inequality*

$$|\psi|(x) \leq \frac{c_n}{\gamma(x)} \qquad (c_n < n\sqrt{2}), \tag{1.9}$$

*holds, where* $\gamma(x)$ *is the geodesic distance of* $(x)$ *from the boundary of* $D$.

COROLLARY. *Let* $u = f(x^1, \cdots, x^n)$ *be a convex solution of* (1.1) *in an open domain* $D$, *and suppose that* $u$ *is of class* $\mathscr{C}^5$. *If the Riemannian structure defined in* $D$ *by* (1.4) *and* (1.2) *is that of a complete Riemannian manifold, then* $u$ *is a quadratic polynomial in* $x^1, \cdots, x^n$, *defined over the whole numerical space.*

The corollary above, restated in the language of affine differential geometry, asserts the following (apparently new) uniqueness theorem.

COROLLARY. (restatement of the previous one). *Let* $V_n$ *be an* $n$-*dimensional, improper affine hypersphere, immersed and of class* $\mathscr{C}^5$ *in the affine* $(n+1)$-*space* $E_{n+1}$. *If the Schwarz-Pick metric induced in* $V_n$ *is definite, and if* $V_n$ *is complete with respect to that metric, then* $V_n$ *is a convex paraboloid in* $E_{n+1}$.

The other main theorem of this paper has been stated [6, 9] only in the case of two independent variables; here we present its generalization to up to four variables.

THEOREM 2. *Let* $u = f(x^1, \cdots, x^n)$ $(n \leq 5)$ *be a convex solution of* (1.1), *defined and of class* $\mathscr{C}^5$ *in an open domain* $D$ *containing the open Euclidean ball with radius* $r$ *and center at the origin* $O$. *Let* $g_{ij}(x) = \partial^2 u / \partial x^i \partial x^j$, *and suppose that* $g_{ij}(O) = \delta_{ij}$. *Then*

$$\sum_{i,j,k=1}^{n} \left( \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}(O) \right)^2 \leq \frac{4 M_n^2}{r^2}, \tag{1.10}$$

*where* $M_n$ *is a universal constant.*

In other words, under the restriction $n \leq 5$ and the assumptions in Theorem 1, the inequality

$$|\psi|(x) \leq \frac{M_n}{\delta(x)} \tag{1.11}$$

is valid at each point $(x) \in D$, where $|\psi|(x)$ and $\delta(x)$ are defined by (1.3), (1.6), and by (1.8) respectively. (For upper bounds on the $M_n$, see (4.14) and (4.15) at the end of the paper.)

COROLLARY. *Let* $u = f(x^1, \cdots, x^n)$ $(n \leq 5)$ *be a convex solution of* (1.1), *defined and of class* $\mathscr{C}^5$ *in the whole numerical* $n$-*space. Then* $u$ *is a quadratic polynomial.*

It is natural to conjecture that the assertion of Theorem 2 should remain valid also for $n \geq 6$: this guess is almost forced by the fortuitous way in which the condition $n \leq 5$ occurs in the proof of the theorem. It also seems very likely that the assertions of both Theorems 1 and 2 remain valid without any assumptions on the existence of fifth derivatives of u; for the sake of simplicity we have not tried to relax the assumptions in that direction.

## 2. PRELIMINARY RESULTS

Here and for the rest of the paper, we shall consider $u = f(x^1, \cdots, x^n)$ to be a convex solution of (1.1), admitting continuous partial derivatives at least up to order 5, in an open domain D draped over a region in the numerical n-space $R^n$ (*domaine étalé*). We introduce in D the Riemannian metric (1.4) defined by (1.2) and the tensor $A_{ijk}$ defined by (1.3). Both definitions require the fixed coordinate system $(x) = (x^1, \cdots, x^n)$ or one related to it by a unimodular affine transformation, since in other coordinate systems equation (1.1) and the definitions of the two fundamental tensors have a more complicated expression. If we confine our computations to the coordinate systems affinely related to (x), we verify immediately the equation

(2.1) $$g^{hk}A_{ijh} = A_{ij}{}^{k} = -\Gamma^{k}{}_{ij}\,,$$

where $\Gamma^k{}_{ij}$ denotes the Christoffel symbol of the second kind. Since $g = \det(g_{ij}) = 1$ identically, we verify the fact already mentioned in (1.5),

(2.2) $$\frac{\partial \log \sqrt{g}}{\partial x^j} = \Gamma^{i}{}_{ij} = -A_{ij}{}^{i} = g^{ik}A_{ijk} = 0\,,$$

and, by differentiating the last two members covariantly,

(2.3) $$g^{ik}A_{ijk,\ell} = A_{ij}{}^{i}{}_{,\ell} = 0\,,$$

where the index of covariant differentiation is written to the right, following the comma.

We compute now the covariant derivative $A_{ijk,\ell}$ of $A_{ijk}$ explicitely:

$$A_{ijk,\ell} = -\frac{1}{2}\frac{\partial^4 u}{\partial x^i\,\partial x^j\,\partial x^k\,\partial x^\ell} + g^{hm}(A_{hjk}\,A_{mi\ell} + A_{ihk}\,A_{mj\ell} + A_{ijh}\,A_{mk\ell})\,,$$

and we verify that it is symmetric in all pairs of indices; for instance

(2.4) $$A_{ijk,\ell} = A_{ij\ell,k}\,.$$

Equations (2.3) and (2.4) imply that the "divergence" of $A_{ijk}$ vanishes:

(2.5) $$g^{i\ell}A_{ijk,\ell} = 0\,.$$

Continuing these formal calculations, one verifies that the Riemann curvature tensor $R_{ijk\ell}$ has the following simple expression in terms of the $A_{ijk}$:

(2.6) $$R_{ijk\ell} = g^{hm}(A_{hi\ell}A_{mjk} - A_{hik}A_{mj\ell}$$

Using (2.2), we shall give here similarly the formulas for the Ricci tensor

$$R_{ik} = g^{j\ell} R_{ijk\ell} = g^{j\ell} g^{hm} A_{h\ell i} A_{mjk}, \tag{2.7}$$

and for the scalar curvature

$$R = g^{ik} R_{ik} = A^{ijk} A_{ijk} = |\psi|^2, \tag{2.8}$$

where $|\psi|$ is the symbol introduced in (1.6).

It is clear from the last equations that the scalar curvature is nonnegative, and that it vanishes if and only if $A_{ijk} = 0$, that is, if the full Riemann tensor $R_{ijk\ell}$ vanishes. It is also easy to see that the Ricci curvature is nonnegative, a fact of fundamental importance in the sequel: indeed, if $\xi^i$ is any contravariant vector, we have

$$R_{ik}\xi^i\xi^k = g^{hm} g^{\ell j} (A_{h\ell i}\xi^i)(A_{mjk}\xi^k) \geq 0, \tag{2.9}$$

equality holding only if $A_{ijk}\xi^k = 0$. From this fact we obtain also the inequality

$$R_{ik}\,\xi^i\,\xi^k \leq R\, g_{ik}\xi^i\xi^k, \tag{2.10}$$

as one may verify by choosing an orthonormal basis for the vectors at any point.

[All the equations derived here, (2.1) through (2.8), are special cases applied to improper affine hyperspheres of standard relations between the Schwarz-Pick metric and the Darboux-Fubini-Pick cubic form $\psi$. These two differential forms determine any differentiable hypersurface in $E_{n+1}$ uniquely, in analogy with the first and second fundamental forms in classical differential geometry, while the existence is subject to the integrability conditions, which in the case studied here specialize to (1.5), (2.4) and (2.6). In other words, all the results to follow can also be formulated in the language of Riemannian geometry, under the assumption of a positive definite Riemannian metric (1.4) together with a symmetric tensor $A_{ijk}$ satisfying (1.5), (2.4) and (2.6).]

We shall prove now an algebraic inequality about the curvature tensors, valid in every Riemannian manifold with positive definite metric.

LEMMA 1. *In any* n*-dimensional Riemannian manifold with a positive definite metric tensor* $g_{ij}$ *of class* $C^2$*, the following inequalities pertaining to the Riemann tensor* $R_{ijk\ell}$*, the Ricci tensor* $R_{ik}$*, and the scalar curvature* R *are valid at each point:*

$$\frac{1}{n} R^2 \leq R_{ij} R^{ij}, \tag{2.11}$$

$$\frac{2}{n-1} R_{ij} R^{ij} \leq R_{ijk\ell} R^{ijk\ell}. \tag{2.12}$$

*Proof.* Let

$$R'_{ij} = R_{ij} - \frac{1}{n} R\, g_{ij};$$

then $g^{ij} R'_{ij} = 0$, so that the decomposition of the Ricci tensor given by

$$R_{ij} = R'_{ij} + \frac{1}{n} R\, g_{ij}$$

is orthogonal with respect to the metric tensor. We have then immediately

$$(2.13)\qquad 0 \leq R'_{ij}R'^{ij} = R_{ij}R^{ij} - \left(\frac{1}{n}R g_{ij}\right)\left(\frac{1}{n}R g^{ij}\right) = R_{ij}R^{ij} - \frac{1}{n}R^2,$$

and this proves (2.11). We remark that equality holds in (2.11), at any point, if and only if $R'_{ij}$ vanishes at that point; it holds identically, if and only if either $n = 2$ or the metric is that of an Einstein space.

Before proving (2.12) we observe that for $n = 2$ both terms of that inequality are identical, so that we can limit our consideration to $n \geq 3$. Consider the conformal curvature tensor of H. Weyl,

$$C_{ijk\ell} = R_{ijk\ell} - \frac{1}{n-2}(g_{ik}R_{j\ell} + g_{j\ell}R_{ik} - g_{i\ell}R_{jk} - g_{jk}R_{i\ell})$$

$$+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}),$$

and let $R'_{ijk\ell}$ denote the difference $R_{ijk\ell} - C_{ijk\ell}$. Since the contraction of any two of the four indices of $C_{ijk\ell}$ with the contravariant component of the metric tensor gives the zero tensor, the two tensors $R'_{ijk\ell}$ and $C_{ijk\ell}$ are mutually orthogonal. Computing the inner product of $R'_{ijk\ell}$ with itself, we obtain

$$R'_{ijk\ell}R'^{ijk\ell} = \frac{4}{n-2}R_{ij}R^{ij} - \frac{2}{(n-1)(n-2)}R^2,$$

so that from the orthogonal decomposition $R_{ijk\ell} = C_{ijk\ell} + R'_{ijk\ell}$ we obtain

$$(2.14)\qquad R_{ijk\ell}R^{ijk\ell} \geq R'_{ijk\ell}R'^{ijk\ell} = \frac{4}{n-2}R_{ij}R^{ij} - \frac{2R^2}{(n-1)(n-2)},$$

equality holding if and only if $C_{ijk\ell} = 0$. The latter condition is verified identically if and only if either $n = 3$, or the space is conformally flat. Combining (2.14) linearly with (2.13) so as to eliminate the term containing $R^2$, we obtain the desired inequality (2.12). We observe that equality holds in (2.12), at any given point, if and only if it holds in both (2.13) and (2.14). Therefore, in that case, the metric is one of constant curvature at that point,

$$R_{ijk\ell} = \frac{R}{(n-1)(n-2)}(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}),$$

a condition which, if verified identically for $n \geq 3$, implies by Schur's theorem that the Riemannian manifold has constant scalar curvature $R$ whose value determines the manifold uniquely within a local isometry. This completes the proof of the lemma.

Before we state the next result, we must generalize the concept of an elliptic, linear differential inequality (of the type, for example that defines subharmonic functions with respect to the Laplace-Beltrami operator), so as to make it applicable to continuous, nondifferentiable functions. Such a generalization was introduced in [2], where it was shown that every function which is subharmonic in the generalized sense satisfies the strong maximum principle in the sense of E. Hopf.

DEFINITION 2. *Let* L *denote a linear, elliptic, second-order differential operator with* n *independent variables, of the form*

$$L = a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(x)\frac{\partial}{\partial x^i},$$

*where the only assumptions on the coefficients* $a^{ij}(x)$ *and* $b^i(x)$ *are that in some neighborhood of each* $(x)$, $a^{ij}(x)$ *and* $b^i(x)$ *are bounded, and that the* $a^{ij}(x)$ *form a symmetric, uniformly positive definite matrix. If* $v = v(x)$ *is any upper semicontinuous, real-valued function of* $(x)$ *in an open domain* U, *then for any given point* $(x)_0 \in U$ *and any real number* a, *we say that*

$$L[v]((x)_0) \geq a$$

*if and only if, for each* $\varepsilon > 0$, *there exists a neighborhood* $V_\varepsilon \subset U$ *of* $(x)_0$ *and a function* $v_\varepsilon(x)$ *of class* $\mathscr{C}^2$ *in* $V_\varepsilon$ *which satisfy the following two conditions:*

a) $v(x) - v_\varepsilon(x)$ *achieves its minimum value over* $V_\varepsilon$ *at* $(x)_0$;

b) *at* $(x)_0$, *the function* $v_\varepsilon(x)$ *satisfies the inequality*

$$L[v_\varepsilon]((x)_0) = a^{ij}((x)_0)\frac{\partial^2 v_\varepsilon}{\partial x^i \partial x^j}((x)_0) + b^i((x)_0)\frac{\partial x_\varepsilon}{\partial x^i}((x)_0) > a - \varepsilon .$$

*Similarly, if* $w = w(x)$ *is a real-valued function defined in a subset* $K \subset U$, *we say that* $L[v] \geq w$ *everywhere in* K, *if for each point* $(x) \in K$, *have*

$$L[v](x) \geq w(x)$$

*in the sense just specified. If* $v(x)$ *is lower semicontinuous in* U, *and if* $w(x)$ *is defined in* $K \subset U$, *we say that* $L[v] \leq w$ *everywhere in* K *if and only if* $L[-v] \geq -w$ *in* K *in the same sense as above.*

We can state now the first result.

PROPOSITION 1. *Let* u *be a convex solution of* (1.1), *of class* $\mathscr{C}^5$ *in an open domain; let* $g_{ij}$ *be the metric tensor defined in that domain by the second partial derivatives of* u; *and let* $\Delta$ *denote the Laplace-Beltrami operator with respect to* $g_{ij}$. *Then the scalar curvature* R *is of class* $\mathscr{C}^2$ *and is nonnegative, and its square root satisfies (in the sense of Definition 2) the partial differential inequality*

$$\Delta \sqrt{R} \geq \frac{n+1}{n(n-1)} R^{3/2}. \tag{2.15}$$

*Proof.* It is clear from (2.8) that the scalar curvature R is nonnegative; since it can be expressed algebraically in terms of $g^{ij}$ and $A_{ijk}$, in a way which involves partial derivatives of u up to the third order, it follows that R is of class $\mathscr{C}^2$, and hence $\sqrt{R}$ is of class $\mathscr{C}^2$ everywhere, except where $R = 0$; at the latter set of points $\sqrt{R}$ is merely continuous. At the points where $R = 0$, $\sqrt{R}$ achieves its minimum value, so that, if we apply Definition 2 (rather trivially) with $v_\varepsilon$ replaced by 0 and L by $\Delta$, the verification of (2.15) is immediate. Thus we can limit our consideration in the sequel to the points where $R > 0$, that is, where $\sqrt{R}$ has the required derivatives for formal calculation.

Consider now the Laplace-Beltrami operator $\Delta$ applied to the tensor $A_{ijk}$ defined by (1.3), and the resulting tensor

$$(\Delta A)_{ijk} = g^{\ell m} A_{ijk,\ell m}.$$

Since the first covariant derivative $A_{ijk,\ell}$ of $A_{ijk}$ is symmetric in each pair of indices by (2.4), we can rewrite the Beltramian $(\Delta A)_{ijk}$ and then apply successively the Ricci identity and equations (2.5) and (2.6), with the following result:

$$\begin{aligned}(\Delta A)_{ijk} &= g^{\ell m} A_{\ell jk,im} = g^{\ell m} A_{\ell jk,mi} + g^{\ell m}(A_{\ell jk,im} - A_{\ell jk,mi})\\ &= (g^{\ell m} A_{\ell jk,m})_{,i} + g^{\ell m}(A_{hjk} R^h{}_{\ell im} + A_{\ell hk} R^h{}_{jim} + A_{\ell jh} R^h{}_{kim})\\ &= A_{hjk} R^h{}_i + A^m{}_{hk}(A^h{}_{am} A^a{}_{ji} - A^h{}_{ai} A^a{}_{jm}) + A^m{}_{jh}(A^h{}_{am} A^a{}_{ki} - A^h{}_{ai} A^a{}_{km})\\ &= A^{h\ell m}(A_{h\ell i} A_{mjk} + A_{h\ell j} A_{mki} + A_{h\ell k} A_{mij}) - 2A^a{}_{ib} A^b{}_{jc} A^c{}_{ka}.\end{aligned}$$

This last expression is useful in computing $\Delta R$ in terms of $A_{ijk}$ and its derivatives from (2.8); the outcome is the following:

$$\begin{aligned}\tfrac{1}{2}\Delta R &= \tfrac{1}{2} g^{\ell m} R_{,\ell m} = A^{ijk}(\Delta A)_{ijk} + A^{ijk,\ell} A_{ijk,\ell}\\ &= 3A^{ijk} A^{h\ell m} A_{h\ell i} A_{mjk} - 2A^{ijk} A^a{}_{ib} A^b{}_{jc} A^c{}_{ka} + A^{ijk,\ell} A_{ijk,\ell} \qquad (2.16)\\ &= R_{ij} R^{ij} + R_{ijk\ell} R^{ijk\ell} + A^{ijk,\ell} A_{ijk,\ell};\end{aligned}$$

the last step can be verified by direct substitution of the curvature tensors by equations (2.6) and (2.7). Applying Lemma 1, we obtain the inequality

$$\tfrac{1}{2}\Delta R \geq \frac{n+1}{n(n-1)} R^2 + A_{ijk,\ell} A^{ijk,\ell}. \qquad (2.17)$$

Consider now the first-order Beltrami parameter applied to R: from (2.8), we have

$$g^{ij} R_{,i} R_{,j} = 4A^{abc,i} A_{abc} A_{jk\ell,i} A^{jk\ell}. \qquad (2.18)$$

We now introduce the seven-indexed tensor

$$B_{abcijk\ell} = \tfrac{1}{2}(A_{abc,i} A_{jk\ell} - A_{abc} A_{jk\ell,i}),$$

and we point out that its inner product with itself can be expressed in terms of the quantities occurring in (2.17) and (2.18), as follows:

$$\begin{aligned}B_{abcijk\ell} B^{abcijk\ell} &= \tfrac{1}{2} A^{abc,i} A^{jk\ell}(A_{abc,i} A_{jk\ell} - A_{abc} A_{jk\ell,i})\\ &= \tfrac{1}{2} R A_{abc,i} A^{abc,i} - \tfrac{1}{8} g^{ij} R_{,i} R_{,j} \geq 0.\end{aligned}$$

Combining this last inequality with (2.16) and (2.17), we obtain the following identity and inequality, at all points where $R > 0$;

$$\Delta \sqrt{R} = \frac{\Delta R}{2\sqrt{R}} - \frac{g^{ij}R_{,i}R_{,j}}{4R^{3/2}} = \frac{R_{ij}R^{ij} + R_{ijk\ell}R^{ijk\ell}}{\sqrt{R}} + \frac{2B_{abcijk\ell}B^{abcijk\ell}}{R^{3/2}}$$

$$\geq \frac{n+1}{n(n-1)} R^{3/2};$$

this verifies our assertion (2.15).

We observe that, in order to have equality in (2.15) at a point where $R > 0$, it is necessary first of all that all sectional curvatures at that point be equal to each other (this is seen from the remarks following the proof of Lemma 1). In addition, it is necessary that the tensor $B_{abcijk\ell}$ vanish. The first condition is impossible in any neighborhood, for $n \geq 3$; for it would imply by Schur's theorem that R be a constant, contrary to the proposition just proved. The second condition would lead to a contradiction, if verified even at a single point where $A_{ijk} \neq 0$, since one can show by algebraic manipulation that the vanishing of $B_{abcijk\ell}$ at such a point would contradict (2.5). For this reason it seems that one could improve somewhat the numerical estimates that will be obtained in the sequel; however, at the present stage it does not seem useful to seek the best possible estimates.

We quote now as Proposition 2 a well-known theorem, whose proof was first given by H. Hopf and W. Rinow [5], and later simplified by G. de Rham [11]. The present statement of the theorem differs from the one in the references in that it applies to all Riemannian manifolds with positive definite metric, instead of only to complete ones; the proof, however, is identical, so that it can be omitted; the applications in the present article will occur mainly in noncomplete manifolds. The definition of $\gamma(p)$ in Proposition 2 agrees with our definition of the geodesic distance of (x) from the boundary (see Definition 1, (1.7)).

PROPOSITION 2 (Hopf, Rinow, de Rham). *Let* X *be a Riemannian manifold with positive definite metric of class* $\mathscr{C}^1$, *and denote by* d(p, q) *the Riemannian distance between any two points* $p, q \in X$ *(that is, the lower bound of the Riemannian length of all arcs joining* p *and* q). *Let* X'*.denote the complement of* X *in its completion with respect to the distance function* d(p, q), *and* $\gamma(p)$ *the distance (extended from* X *to its completion) of any point* $p \in X$ *from* X' *(if* X *is already complete, we set* $\gamma(p) = \infty$). *Then the following statements are true.*

(i) *The closed metric ball* $\overline{\Sigma}(p, r)$ *for any* $p \in X$ *and real number* $r > 0$, *defined as the set of points* $q \in X$ *such that* $d(p, q) \leq r$, *is compact if and only if* $r < \gamma(p)$.

(ii) *For each* $p \in X$, *every geodesic arc with* p *as initial point can be extended to any Riemannian length less than* $\gamma(p)$; *however, if* $\gamma(p) < \infty$, *there exists at least one geodesic arc, originating at* p, *whose maximal extension in* X *is of length* $\gamma(p)$, *and with no end point in* X.

(iii) *For any two points* $p, q \in X$ *such that* $d(p, q) \leq \max(\gamma(p), \gamma(q))$, *there exists at least one arc* C, *necessarily a geodesic, which joins* p *and* q *and whose length achieves the minimum value* d(p, q).

(iv) *For any two points* $p, q \in X$, *the inequality*

$$\gamma(q) - d(p, q) \leq \gamma(p) \leq \gamma(q) + d(p, q)$$

*is valid; in particular* $\gamma(q) = \infty$ *if and only if* $\gamma(p) = \infty$.

Our next proposition deals with a property which the Riemannian distance function d(p, q) has for those pairs of points p and q that can be joined by a geodesic of length d(p, q), as for instance under the conditions of statement (iii) in Proposition 2. In order to avoid confusion, we adopt a different name and notation for the restricted distance to which this property gives rise. The proposition itself, related to a lemma by S. B. Myers [8], is proved in detail in [2]. Here we shall elaborate only the formal aspects of the proof, since part of it is needed in the proof of Theorem 2.

DEFINITION 3. *Let* X *be a Riemannian manifold with a positive definite metric tensor of class* $\mathscr{C}^1$. *For any two points* $p, q \in X$ *we define the* geodesic distance r(p, q) *to be the Riemannian distance* d(p, q), *provided that* p *and* q *can be joined by at least one geodesic in* X *of length* d(p, q); *if* p *and* q *can not be joined by an arc in* X *of length* d(p, q), *then the geodesic distance shall remain undefined.*

PROPOSITION 3. *Let* X *be an* n-*dimensional Riemannian manifold with positive definite metric of class* $\mathscr{C}^3$ *and with nonnegative Ricci curvature; that is, let*

$$R_{ij}\xi^i\xi^j \geq 0 \quad \textit{for all } \xi^i \ (i = 1, \cdots, n); \tag{2.19}$$

*let* $p_0$ *be an arbitrary point in* X, *fixed for the sequel; let*

$$r = r(p) = r(p_0, p) \qquad (p \in Y = Y_{p_0})$$

*denote the geodesic distance between* $p_0$ *and* p, *where* p *ranges over the domain* Y (*depending on* $p_0$) *for which that geodesic distance is defined; and let* $\phi(t)$ *be a twice differentiable function of a real variable* t *for* $t \geq 0$, *such that* $\phi'(t) \geq 0$ *for all* t *and* $\phi'(0) = 0$. *Then the function* $v = v(p) = \phi(r(p))$, *continuous for* $p \in Y$, *satisfies the inequality*

$$v(p) \leq \phi''(r(p)) + \frac{n-1}{r(p)}\phi'(r(p)) \tag{2.20}$$

*in the sense of Definition* 2, *where* $\triangle$ *is the Laplace-Beltrami operator; for* $p = p_0$ *the conclusion is valid, if the right-hand member of* (2.24) *is replaced by its limit as* p *approaches* $p_0$.

*Proof.* Under the differentiability assumptions on the metric, there exists an open, everywhere dense subdomain Y' of Y, in which r(p) is twice differentiable. In fact, if $p_1 \in Y$, let $C_1$ be a geodesic of length $d(p_0, p_1)$, joining $p_0$ and $p_1$; then for each point p in $C_1$, other than the end points, the segment C of $C_1$ between $p_0$ and p is the only arc of length $d(p_0, p)$ joining its end points (because of the unique continuation property of the geodesics). Furthermore, since p can not be a conjugate point of $p_0$ along C, there is a connected neighborhood $U \subset Y$ of p, such that for each point $p' \in U$ there is a unique geodesic C' of length $d(p_0, p')$ joining $p_0$ and p', and this geodesic can be constructed by the continuity method, that is, by joining p and p' by a differentiable arc in U, and by obtaining the variational formula by Jacobi's equation for the geodesic joining $p_0$ with a moving point on the arc. Thus one shows that the function r is differentiable in U, and that it satisfies the equation

$$g^{ij}r_{,i}r_{,j} = 1. \tag{2.21}$$

Since $\phi(t)$ is twice differentiable, it follows immediately from (2.21) that

$$\triangle v = \triangle\phi(r) = \phi''(r) + \phi'(r)\triangle r \tag{2.22}$$

in the open, dense subset $Y'$ of $Y$ where $r$ satisfies (2.21). Thus the proof can be completed by showing that $\Delta r \leq (n-1)/r$, since $\phi'(t) \geq 0$.

By formal computations one verifies that in $Y'$ the geodesic hyperspheres defined by the equation $r =$ (positive constant) are locally like twice differentiable hypersurfaces. From (2.21) the vector $\xi^i = g^{ij} r_{,j}$ is the (outward) unit normal vector to each such hypersphere, and the restriction of the covariant tensor $-r_{,ij}$ to any such hypersphere defines its second fundamental form relative to the orientation of the normal vector $\xi^i$. Consequently, if we denote by $\kappa_\nu$ $(\nu = 1, \cdots, n-1)$ the $n-1$ principal normal curvatures of the geodesic hypersphere passing through each point $p \in Y'$ and by $H$ the mean curvature

$$H = \frac{1}{n-1}\left(\sum_{\nu=1}^{n-1} \kappa_\nu\right),$$

we obtain the identity in $Y'$

$$\Delta r = -(n-1)H. \tag{2.23}$$

Let $p$ be a point in $Y'$, $C$ the shortest geodesic joining $p_0$ to $p$, parametrized by the geodesic distance $r$ from $p_0$. Denote by $\eta_\nu{}^i(p)$ $(\nu = 1, \cdots, n-1)$ a choice of unit vectors at $p$, each of them tangent to the geodesic hypersphere in the (mutually orthogonal) directions corresponding to the principal curvatures $\kappa_\nu$ respectively. We define the $n-1$ vectors $\eta_\nu{}^i$ elsewhere in $C$ by parallel translation from $p$ along $C$, and we let $\kappa_\nu{}'$ denote the normal relative curvature of any geodesic hypersphere in the direction $\eta_\nu{}^i$. Then the Jacobi equation along $C$ yields immediately the following Riccati equation:

$$\frac{d\kappa_\nu{}'}{dr} = (\kappa_\nu{}')^2 + R_{ijk\ell}\, \xi^i \eta_\nu{}^j \xi^k \eta_\nu{}^\ell \quad (\nu = 1, \cdots, n-1).$$

Taking the average over $\nu$ in the above equation and recalling the elementary identity $\Sigma_{\nu=1}^{n-1} \kappa_\nu{}' = (n-1)H$, we obtain the inequality

$$\frac{dH}{dr} = \frac{1}{n-1}\left(R_{ij}\, \xi^i \xi^j + \sum_{\nu=1}^{n-1} (\kappa'_\nu)^2\right) \geq H^2 + \frac{1}{n-1} R_{ij}\, \xi^i \xi^j. \tag{2.24}$$

From the assumption (2.19) that the Ricci curvature is nonnegative and from the fact that $H$ is asymptotic to $-1/r$ as $r$ approaches zero, we obtain the inequality $H \geq -1/r$, which, combined with (2.23) and (2.22), proves (2.20) everywhere in $Y'$.

The extension of the proof to include the rest of $Y$ involves an application of Definition 2, and, since the details are not needed in the sequel, we merely refer for them to [2]. The statement concerning the relation (2.20) in the limit as $p$ approaches $p_0$ is a trivial exercise. This concludes the proof of Proposition 3.

## 3. PROOF OF THEOREM 1

We can rephrase the statement of Theorem 1 to read that there exists a positive constant $c_n$, depending only on the number $n$ of variables, such that, if at any point $(x)_0 \in D$ we have

$$|\psi|((x)_0) > a > 0 \tag{3.1}$$

for any positive constant a, then $\gamma((x)_0) \leq c_n/a$ (without loss of generality we shall henceforth take $(x)_0$ to be the origin O); indeed we shall deduce a contradiction by assuming that

$$\gamma(O) > \frac{c_n}{a}. \tag{3.2}$$

The estimate $c_n < n\sqrt{2}$ is obtained from the numerical solution of a certain ordinary differential equation and by majorizing the result. Its chief interest lies in the fact that it probably describes the order of magnitude of $c_n$ as n becomes large.

From (2.8) and (2.15) (Proposition 1) we obtain the inequality

$$\Delta|\psi| \geq \frac{n+1}{n(n-1)}|\psi|^3. \tag{3.3}$$

This relation is the starting point for the whole proof: the rest of the argument follows from results by E. K. Haviland [4], and from a method due to R. Osserman [10], generalized by the author [2] to spaces with nonnegative Ricci curvature. The essential point consists of constructing an auxiliary function $v = v(x)$ in an open subdomain $\Sigma \subset D$ with the following four properties:

(a) *The domain* $\Sigma$ *contains the origin* O *in its interior, and the closure of* $\Sigma$ *relative to* D *is compact in* D.

(b) *The function* v *satisfies the inequality*

$$\Delta v \leq \frac{n+1}{n(n-1)} v^3 \tag{3.4}$$

*everywhere in* $\Sigma$, *in the sense of Definition* 2.

(c) *The function* v *satisfies*

$$v(O) = a, \tag{3.5}$$

*where* a *is the constant occurring in* (3.1) *and* (3.2); *at other points in* $\Sigma$,

$$v(x) > 0. \tag{3.6}$$

(d) *As* (x) *approaches the boundary of* $\Sigma$, v(x) *becomes uniformly infinite.*

We shall prove first that, if there exists a function v with the four properties listed above, then we have a contradiction with (3.1). From properties (a) and (d) and the fact that the function $|\psi| - v$ becomes $-\infty$ uniformly on the boundary of the relatively compact domain $\Sigma$, it follows that the function attains its maximum value M at some interior point $(x_1)$ of $\Sigma$. Since, by (3.1) and (3.5), $|\psi| - v$ is positive at the origin, it follows *a fortiori* that $M > 0$. But from (3.3) and (3.4) we would have the inequality

$$\Delta(|\psi| - v) \geq \frac{n+1}{n(n-1)}(|\psi|^3 - v^3) \geq \frac{3(n+1)}{n(n-1)}(|\psi| - v)v^2$$

(in the sense of Definition 2) wherever $|\psi| \geq v$, since both $|\psi|$ and v are nonnegative. In particular, we would have, in a neighborhood of the maximum value point, the inequality

$$\Delta(|\psi| - v) > 0,$$

contrary to the maximum principle [2] applied to solutions, in the sense of Definition 2, of the above inequality. This contradiction would then prove the theorem.

It is natural, in seeking to construct a function $v(x)$ with the four required properties, to make it a function whose values depend only on the geodesic distance $r(O, (x))$ from the origin (see Definition 3). We abbreviate $r(O, (x))$ by $r(x)$, and we let $v(x) = \phi(r(x))$, where $\phi(t)$ is a function of a real variable $t \geq 0$ to be determined. To ensure that $v(x)$ satisfy (3.4), we apply Proposition 3, using the fact that by equation (2.9) the Ricci curvature is nonnegative. Accordingly, it is sufficient that $\phi(t)$ satisfy the ordinary differential equation

$$\phi''(t) + \frac{n-1}{t}\phi'(t) = \frac{n+1}{n(n-1)}(\phi(t))^3 \qquad (\phi'(0) = 0), \tag{3.7}$$

provided that at the same time we have

$$\phi'(t) \geq 0 \qquad \text{for } t > 0. \tag{3.8}$$

The condition (3.5) that $v(O) = a$ is translated into the initial condition

$$\phi(0) = a. \tag{3.9}$$

If we can prove that the solution of (3.7) with the initial condition (3.9) automatically satisfies (3.8), and that it diverges to $+\infty$ as $t$ approaches a positive number $t_1$ (depending on a), then, if $\gamma(0) > t_1$, the domain of definition $\Sigma$ of the function $v(x) = \phi(r(x))$ has a compact closure in D, by the Hopf-Rinow-de Rham theorem (Proposition 2), and $v$ has all the properties needed for the proof of the theorem. Therefore we examine the properties of equation (3.7). This equation, in a more general form, has been considered by R. Osserman [10] in connection with the inequality $\Delta u \geq f(u)$ in Euclidean space. It is worth remarking that, for $n = 3$, equation (3.7) is a modified form of Emden's equation, obtained from the spherically symmetric, polytropic expansion of gases [7, pp. 559-561].

It is convenient to normalize the differential equation (3.7) and the condition (3.9) by replacing $t$ and $\phi(t)$ respectively by $s$ and $f(x) = y$, where

$$t = \frac{s}{a}\sqrt{\frac{n(n-1)}{n+1}}, \qquad y = f(x) = \frac{1}{a}\phi(t). \tag{3.10}$$

Then $y$ satisfies the differential equation

$$y'' + \frac{n-1}{s}y' = y^3 \qquad \left(y' = \frac{dy}{ds},\ y'' = \frac{d^2y}{ds^2}\right), \tag{3.11}$$

with initial conditions

$$y = 1, \quad y' = 0 \qquad \text{for } s = 0. \tag{3.12}$$

We shall now discuss this differential equation.

LEMMA 2. *The differential equation* (3.11) *with initial conditions* (3.12) *admits a unique, analytic solution in the neighborhood of* $s = 0$. *This solution is an even function; it is monotone increasing for all* $s > 0$ *in the domain of regularity, and becomes infinite as* $s$ *approaches a certain number* $k_n$ *which satisfies the conditions*

$$(3.13)\quad \begin{cases} \sqrt{n} \le k_n \le \sqrt{2n} & (n \ge 4), \\ \sqrt{2n} \le k_n \le \sqrt{n}\displaystyle\int_0^{\pi/2}\left(1 - \frac{1}{2}\sin^2\theta\right)^{-1/2} d\theta \cong 1.8541\sqrt{n} & (n = 2, 3). \end{cases}$$

One can show immediately that there is a recursion formula for the expansion of the solution of (3.11), (3.12) as a formal power series in $s$; also, that all coefficients of even powers of $s$ are positive, and that those of odd powers vanish. Comparing the expansion for $y$ with the formal power series expansion for the solution of $\frac{n}{s}\frac{dz}{ds} = z^3$ with the same initial conditions, one sees that the latter converges to the actual solution $z = \left(1 - \frac{s^2}{n}\right)^{-1/2}$, and that its coefficients dominate those of the former. Thus the formal power series expansion of $y$ in terms of $s$ actually converges to the unique, analytic solution of (3.11) with (3.12). Since all of the coefficients of the power series are nonnegative, the series converges for all positive values of $s$ up to the first real singularity; therefore, for all positive values of $s$ on the maximal real interval of regularity of $y$, the solution $y$ as well as all of its derivatives are positive. Comparing now the power series expansion for $y$ with that of the solution $w$ of the equation $nw'' = w^3$ with the same initial conditions, we find that the coefficients of the expansion of $y$ dominate the corresponding coefficients for $w$. On the other hand, $w$ can be evaluated by quadratures, and it can be shown to be the elliptic function defined by inverting the elliptic integral

$$s = \sqrt{n}\int_0^{\operatorname{arcsec} w} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}.$$

Since $w$ becomes infinite as $s$ approaches the value corresponding to the complete elliptic integral in (3.13), $y$ has a singularity as $s$ approaches a value $k_n$ dominated by that integral. Therefore $y$ is regular in a bounded, open interval $|s| < k_n$, where

$$(3.14)\quad \sqrt{n} \le k_n \le 1.8541 \cdots \sqrt{n}.$$

We can sharpen the estimate for $k_n$ as follows: for $n = 4$, we can express $y$ in closed form by

$$(3.15)\quad y = 8/(8 - s^2) \qquad (n = 4),$$

so that $k_4 = \sqrt{8}$; this is the only known case where the solution appears to be elementary. In the remaining cases, since all derivatives of $y$ (and in particular the third) are positive for $s > 0$, and since $y' = 0$ for $s = 0$, we use the inequality, obtained from the law of the mean,

$$(3.16)\quad y'' - \frac{1}{s}y' > 0 \qquad (s > 0).$$

Multiplying the left member of this inequality by $(n - 4)/4$ and adding it to (3.11), one obtains the differential inequality

(3.17) $$\frac{n}{4}y'' + \frac{3n}{4s}y' \begin{cases} > y^3 & (n > 4), \\ < y^3 & (n < 4) \end{cases}$$

for all $s > 0$. If we replace the inequality signs by equality, we obtain a differential equation, whose solution with the same initial values as in (3.12) is

(3.18) $$h(s) = 2n/(2n - s^2).$$

One can now apply the principle of the maximum (respectively, minimum) to the function $y - h(s)$ from the inequality (3.17) for $n > 4$ (respectively, $n < 4$); according to this, $y - h(s)$ can not attain its maximum (minimum) value in the interior of the segment $0 < s < \min(\sqrt{2n}, k_n)$. On the other hand, by comparing the coefficients of the power series expansions of $y$ and of $h(s)$, one sees that, in a neighborhood of $s = 0$, $y > h(s)$ for $n > 4$, and $y < h(s)$ for $s < 4$. Therefore the same inequalities hold throughout the whole interval in which both $y$ and $h(s)$ are regular. This yields immediately the inequalities

$$k_n \leq \sqrt{2n} \qquad (n \geq 4),$$

$$k_n \geq \sqrt{2n} \qquad (n < 4),$$

which together with (3.14) imply (3.13).

It remains to be proved that $y$ becomes infinite as $s$ approaches $k_n$. To this end we subtract (3.16) from (3.11), with the result of the following inequality (always for $s > 0$)

(3.19) $$\frac{n}{s}y' < y^3;$$

for if $y$ did not become infinite, then, being monotone increasing, it would converge to a finite number; (3.19), $y'$ would also be bounded and, being monotone, it would converge to a finite number. This would imply that for $s = k_n$ the limiting values of $y$ and $y'$ could be taken as initial values for a further regular continuation, which is impossible. This completes the proof of the lemma.

We return now to the function $\phi(t)$ defined in terms of $y = f(s)$ by (3.10). This function, which is the solution of (3.7) and (3.9), has now been shown to be monotone increasing for $t > 0$ (see (3.8)), and to diverge to infinity as $t$ approaches $c_n/a$, where

$$c_n = k_n \sqrt{n(n-1)/(n+1)}.$$

From the estimates (3.13) on $k_n$ it follows that $c_n \leq n\sqrt{2}$, as was claimed in the statement of Theorem 1 in the Introduction. Thus the auxiliary function $v(x) = \phi(r(x))$ becomes infinite uniformly on the relative boundary in D of the subdomain $\Sigma \subset D$ which consists of those points $(x) \in D$ whose geodesic distance $r(x)$ from the origin is less than $c_n/a$. By Proposition 2 (Hopf, Rinow, de Rham), $\Sigma$ is relatively compact in D, provided that $\gamma(0) > c_n/a$; hence, if this last condition is satisfied, then v meets the four requirements (a), (b), (c), (d), including (3.4), (3.5), (3.6); these show that (3.3), (3.1), and (3.2) are incompatible. This completes the proof of Theorem 1. The corollary is clearly a special case.

## 4. PROOF OF THEOREM 2

We shall rephrase the statement of Theorem 2 in a way analogous to the reformation of Theorem 1 at the beginning of Section 3. Assuming, (as we do in the Introduction) that the function u is a convex solution of (1.1) in a domain D containing the origin O in its interior; that

$$\frac{\partial^2 u}{\partial x^i \partial x^j}(O) = \delta_{ij}, \tag{4.1}$$

and that $|\psi|(O) \geq a > 0$; we shall seek to construct a curve in D, from the origin to the boundary of D, whose *Euclidean* length in terms of the rectangular coordinates (x) can be bounded in terms of the positive constant a. It follows from Theorem 1 and Proposition 2 that under these assumptions there exists a geodesic C with respect to the Riemannian metric (1.2) (namely a shortest possible curve with O as initial point and no end point in D) whose Riemannian length $\gamma$ is finite and satisfies

$$\gamma = \gamma(O) \leq \frac{c_n}{a} \qquad (c_n < n\sqrt{2}). \tag{4.2}$$

It is a remarkable and fortunate fact that, for $n \leq 5$, this same geodesic C with respect to the Riemannian metric has bounded Euclidean length, as we shall prove. As in the case of Theorem 1, the ultimate step depends on a lemma in ordinary differential equations. For the proof of this lemma, the author is indebted to N. Levinson.

Let the shortest geodesic C with initial point O and no end point in D be parametrized by its Riemannian length s from the origin; let its total length be $\gamma$, where $\gamma$ satisfies (4.2); and denote by $\sigma = \sigma(s)$ the corresponding Euclidean length. Then the following inequality holds.

$$0 < \frac{d\sigma}{ds} = \sqrt{\delta_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}} < \sqrt{g^{ij}\delta_{ij}} = \rho(s) = \rho.$$

Differentiating the distortion parameter $\rho$ with respect to s, we obtain

$$\frac{d}{ds}(g^{ij}\delta_{ij}) = 2\rho\frac{d\rho}{ds} = 2\delta_{ij}\, A^{ij}{}_k\frac{dx^k}{ds} \tag{4.3}$$

directly from (2.1). We regard this last expression as the Riemannian inner product between the tensors $\delta_{ij}$ and $A^{ij}{}_k\, dx^k/ds$, and prepare to apply the Cauchy-Schwarz inequality, in majorizing that expression by the product of the Riemannian absolute values. Actually a slightly better inequality is obtained by exploiting the fact that the $\delta_{ij}$ form a positive definite matrix, while the $A^{ij}{}_k\, dx^k/ds$ form a matrix with trace zero with respect to the metric $g_{ij}$. Let

$$a_{ij} = \delta_{ij} - \frac{\rho^2}{n} g_{ij}, \qquad a^{ij} = g^{ik} g^{j\ell} a_{k\ell};$$

then the decomposition $\delta_{ij} = a_{ij} + (\rho^2/n)g_{ij}$ is orthogonal with respect to the metric $g_{ij}$ at each point, and the second term is orthogonal to $A^{ij}{}_k\, dx^k/ds$. Thus we have

$$\left|\delta_{ij} A^{ij}{}_k\frac{dx^k}{ds}\right| = \left|a_{ij} A^{ij}{}_k\frac{dx^k}{ds}\right| \leq (a_{ij}a^{ij})^{1/2}\left(A^{ij}{}_k\frac{dx^k}{ds}A_{ij\ell}\frac{dx^\ell}{ds}\right)^{1/2}.$$

The norm of the $a_{ij}$ is majorized as follows:

$$a_{ij}a^{ij} = g^{ik}g^{j\ell}\delta_{ij}\delta_{k\ell} - \frac{\rho^4}{n} < \rho^4\left(1 - \frac{1}{n}\right), \tag{4.4}$$

while the squared absolute value of $A^{ij}{}_k dx^k/ds$ can be computed from (2.9):

$$g_{i\ell}g_{jm}A^{ij}{}_h\frac{dx^h}{ds}A^{\ell m}{}_k\frac{dx^k}{ds} = R_{hk}\frac{dx^h}{ds}\frac{dx^k}{ds}. \tag{4.5}$$

In what follows, we shall denote by K(s) the function

$$K(s) = \sqrt{R_{hk}\frac{dx^h}{ds}\frac{dx^k}{ds}}. \tag{4.6}$$

Combining (4.3) through (4.6), and applying the Cauchy-Schwarz inequality, we obtain

$$\rho\left|\frac{d\rho}{ds}\right| \leq \left(\frac{n-1}{n}\right)^{1/2}\rho^2 K(s),$$

from which we deduce the integral inequality

$$\sigma(s) < \sqrt{n}\int_0^s \exp\left[\int_0^{s_1}\left(\frac{n-1}{n}\right)^{1/2}K(s_2)\,ds_2\right]ds_1 \qquad (0 < s < \gamma), \tag{4.7}$$

because of the initial conditions $\sigma(O) = 0$, $\rho(O) = \sqrt{n}$, the latter being a consequence of (4.1). Our purpose now is to prove, by means of the available information on K(s), that the integral (4.7) converges as s approaches $\gamma$.

From the definition (4.6) of K(s) and from (2.10), (2.8), and Theorem 1, and using the obvious identity $\gamma(x(s)) = \gamma(O) - s = \gamma - s$, we obtain the inequality

$$0 \leq K(s) \leq \frac{c_n}{\gamma - s}. \tag{4.8}$$

If one could prove the stronger inequality $K(s) \leq \frac{1-\varepsilon}{\gamma - s}\left(\frac{n}{n-1}\right)^{1/2}$ for some $\varepsilon > 0$, then the convergence of (4.7) as $s \to \gamma$ would be immediate. It seems doubtful, however, that this hypothesis be true, especially for high values of n (as a matter of fact, we did not succeed in proving it even for n = 2). Therefore some other property of K(s), restraining its rate of growth, has to be used: the crucial inequality in this case is the Riccati inequality (2.24), where the "unknown" function H(s) is the relative mean curvature of the geodesic hypersphere with center O that passes through the point (x(s)). Thus in the present notation we have the following majorant for K(s):

$$0 \leq K(s) \leq \sqrt{(n-1)\left(\frac{dH(s)}{ds} - H^2(s)\right)}, \tag{4.9}$$

where H(s) is a differentiable function of s in the open interval $0 < s < \gamma$, satisfying

$$H'(s) = \frac{dH}{ds} \geq H^2(s). \tag{4.10}$$

We shall now quote Levinson's lemma.

LEMMA 3. *Let* $H = H(s)$ *be a continuously differentiable, real-valued function of* $s$ $(0 < s < \gamma < \infty)$, *satisfying* (4.10). *Then, for every* $\varepsilon > 0$ *and every constant* $c' < 2$,

$$\int_\varepsilon^\gamma \exp\left[c' \int_\varepsilon^s \sqrt{H'(s_1) - H^2(s_1)}\, ds_1\right] ds < \infty. \tag{4.11}$$

*Proof.* We perform the substitution

$$s = \gamma(1 - e^{-t}), \tag{4.12}$$

and introduce the following two functions of $t$:

$$w = w(t) = \gamma e^{-t} H(s) - \frac{1}{2}$$

and

$$z = z(t) = \gamma e^{-t} \sqrt{\frac{dH(s)}{ds} - H^2(s)}\,.$$

These substitutions lead to the following restatement of the lemma: if $z = z(t)$ is a continuous, nonnegative function of $t$ for all $t > 0$, such that the Riccati equation

$$\frac{dw}{dt} = w^2 + z^2 - \frac{1}{4} \tag{4.13}$$

has a continuous solution $w = w(t)$ on the whole half-line $t > 0$, then for every $\varepsilon' > 0$ and every constant $c' < 2$ the following statement is true:

$$\gamma \int_{\varepsilon'}^\infty \exp\left[c' \int_{\varepsilon'}^t z(t_1)\, dt_1\right] e^{-t} dt < \infty.$$

In order to see this, we consider $w = w(t)$ as a solution of the differential inequality

$$\frac{dw}{dt} \geq w^2 - \frac{1}{4}$$

obtained from (4.13), and we compare it with the solutions of the corresponding equation (obtained by replacing the inequality by equality) that are regular on the whole positive half-line of $t$. We obtain immediately the bounds

$$-\frac{1}{2} \coth \frac{t}{2} \leq w(t) \leq \frac{1}{2}.$$

From these we obtain the following inequality, for every $t \geq \varepsilon'$:

$$\int_{\varepsilon'}^t z(t_1)\, dt_1 = \int_{\varepsilon'}^t \left(w'(t_1) - w^2(t_1) + \frac{1}{4}\right)^{1/2} dt_1 \leq \int_{\varepsilon'}^t \left(w'(t_1) + \frac{1}{4}\right)^{1/2} dt_1$$

$$\leq \sqrt{t - \varepsilon'} \left[ \int_{\varepsilon'}^{t} \left( w'(t_1) + \frac{1}{4} \right) dt_1 \right]^{1/2} = \left[ (t - \varepsilon') \left( w(t) - w(\varepsilon') + \frac{t - \varepsilon'}{4} \right) \right]^{1/2}$$

$$\leq \frac{1}{2} \left[ (t - \varepsilon')^2 + 2(t - \varepsilon') \left( 1 + \coth \frac{\varepsilon'}{2} \right) \right]^{1/2} < \frac{t - \varepsilon'}{2} + \frac{e^{\varepsilon'}}{e^{\varepsilon'} - 1} .$$

Consequently the left-hand member of (4.11) can be majorized by direct computation as follows: for any constant $c' < 2$,

$$\int_{\varepsilon}^{\gamma} \exp \left[ c' \int_{\varepsilon}^{s} \sqrt{H'(s_1) - H^2(s_1)}\, ds_1 \right] ds$$

$$= \gamma \int_{\varepsilon'}^{\infty} \exp \left[ c' \int_{\varepsilon'}^{t} z(t_1)\, dt_1 \right] e^{-t} dt < \frac{2(\gamma - \varepsilon)}{(2 - c')} \exp(c'\gamma/\varepsilon) ,$$

where $\varepsilon' = \log \left( \frac{\gamma}{\gamma - \varepsilon} \right)$, according to the reverse of the substitution (4.12). This completes the proof of the Lemma.

We remark that the condition $c' < 2$ under which the assertion of Lemma 3 is valid is the best possible one; in fact, if we take, for example, $\gamma = 1$ and $H(s) = (2 - 2s)^{-1}$, then the integral (4.11) diverges for every $c' \geq 2$.

We now conclude the proof of Theorem 2. The function $K(s)$ occurring in the integral in (4.7) has two majorants, given by (4.8) and (4.9) respectively: using the former in the interval from 0 to $\varepsilon$, for any $\varepsilon$ $(0 < \varepsilon < \gamma)$, and the latter for $s > \varepsilon$, and setting $c' = (n - 1)/\sqrt{n}$, we can evaluate the limit of the integral in (4.7) as $s$ approaches $\gamma$ in terms of $\varepsilon' = \log \frac{\gamma}{\gamma - \varepsilon}$. If $c_n > 1$ and $c' < 2$, that is, under the condition $n \leq 5$, the result of the computation from (4.9) is

$$\lim_{s \to \gamma} \sigma(s) = \sigma(\gamma) < \frac{\gamma \sqrt{n}}{c_n - 1} \left( e^{(c_n - 1)\varepsilon'} - 1 \right) + \frac{2\gamma \sqrt{n}}{2 - c'} \exp \left[ (c_n - 1)\varepsilon' + \frac{c' e^{\varepsilon'}}{e^{\varepsilon'} - 1} \right] .$$

In order to simplify the last expression, we use the inequality

$$\frac{e^{\varepsilon'}}{e^{\varepsilon'} - 1} < \frac{1}{\varepsilon'} + 1$$

and set $\varepsilon' = \sqrt{c'/(c_n - 1)}$; we write $\sqrt{c'(c_n - 1)} = C_n$ and use the assertion of Theorem 1 that $\gamma \leq c_n/a$; this gives

$$\delta(0) \leq \sigma(\gamma) \leq \frac{M_n}{a} ,$$

where, for $n = 2, 3, 4, 5$,

$$M_n < \frac{c_n \sqrt{n}}{c_n - 1} \left( e^{C_n} - 1 \right) + \frac{2\, c_n \sqrt{n}}{2 - c'} e^{(2C_n + c')} .$$

Using the upper bound from Theorem 1, $c_n < n\sqrt{2}$, and the definitions of the remaining constants, we have calculated approximately the following numerical estimates for $M_n$:

(4.14) $$M_2 < 127; \quad M_3 < 2660; \quad M_4 < 40100; \quad M_5 < 653000.$$

This concludes the proof of Theorem 2 and, as a special case, of its Corollary.

The extreme wastefulness of our method of estimating the constants $M_n$, resulting in (4.14), is evidenced by comparing our upper bound for $M_2$, the only $M_n$ whose existence has been reported so far in the literature, with the bound calculated by J. Nitsche [9], by means of complex variable techniques. By direct computation in terms of the local complex parameter

$$\sigma = x^1 + \frac{\partial u}{\partial x^1} + \sqrt{-1}\left(x^2 + \frac{\partial u}{\partial x^2}\right)$$

and of the holomorphic generating function

$$f(\sigma) = x^1 - \frac{\partial u}{\partial x^1} + \sqrt{-1}\left(\frac{\partial u}{\partial x^2} - x^2\right),$$

where $\sigma$ and $f(\sigma)$ are Nitsche's notations, one obtains the identity

$$|\psi| = \frac{4|f''(\sigma)|}{(1 - |f'(\sigma)|^2)^{3/2}}.$$

From this and from Nitsche's arguments one obtains the following, considerable improvement on the upper bound for the constant $M_n$ appearing in (1.10) and (1.11) for $n = 2$:

(4.15) $$M_2 \leq 4.$$

It is not surprising that the results obtained by complex variable techniques for $n = 2$ should yield much sharper estimates; on the other hand, the methods used in the present paper are applicable to any number of variables; the weakness of our numerical estimates for $n = 2$ is a strong indication of the room for possible improvement for $n \geq 3$, including the very likely possibility, mentioned in the Introduction, that the assertion of Theorem 2 may be valid also for $n \geq 6$.

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