SOME INTEGRAL FORMULAS AND THEIR APPLICATIONS

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0. INTRODUCTION

In a previous paper [9] (see also Yano and Bochner [10]), we have proved the integral formula

$$(0.1) \qquad \int_{V_n} [K_{ji} v^j v^i + (\nabla^j v^i)(\nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)]\, d\sigma = 0 .$$

The formula is valid for any vector field v^h in an n-dimensional compact orientable Riemannian space V_n, where ∇_j is the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with the fundamental tensor g_{ji} of V_n, where $\nabla^j = g^{ji}\nabla_i$, where K_{ji} is the Ricci tensor $K_{aji}{}^a$, where $K_{kji}{}^h$ is the curvature tensor, and where $d\sigma$ is the volume element of the space.

Equation (0.1) can be written in the following three forms:

$$(0.2) \qquad \int_{V_n} [K_{ji} v^j v^i + (\nabla^j v^i)(\nabla_j v_i) - \frac{1}{2}(\nabla^j v^i - \nabla^i v^j)(\nabla_j v_i - \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)]\, d\sigma = 0,$$

$$(0.3) \qquad \int_{V_n} [K_{ji} v^j v^i - (\nabla^j v^i)(\nabla_j v_i) + \frac{1}{2}(\nabla^j v^i + \nabla^i v^j)(\nabla_j v_i + \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)]\, d\sigma = 0,$$

$$(0.4) \qquad \int_{V_n} [K_{ji} v^j v^i - (\nabla^j v^i)(\nabla_j v_i) - \frac{n-2}{n}(\nabla_j v^j)(\nabla_i v^i)$$
$$+ \frac{1}{2}(\nabla^j v^i + \nabla^i v^j - \frac{2}{n} g^{ji} \nabla_b v^b)(\nabla_j v_i + \nabla_i v_j - \frac{2}{n} g_{ji} \nabla_a v^a)]\, d\sigma = 0.$$

From these equations, we can easily obtain

THEOREM A (Myers [5], Bochner [1]; see also Yano and Bochner [10]). *If, in a space* V_n, *the form* $K_{ji}v^j v^i$ *is positive definite, then there does not exist a harmonic vector other than the zero vector.*

THEOREM B (Bochner [1]; see also Yano and Bochner [10]). *If, in a space* V_n, *the form* $K_{ji}v^j v^i$ *is negative definite, then there does not exist a Killing vector other than the zero vector.*

THEOREM C. *If, in a space* V_n, *the form* $K_{ji} v^j v^i$ *is negative definite, then there does not exist a conformal Killing vector other than the zero vector.*

On the other hand, applying Green's formula

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$$\int_{V_n} g^{ji}\nabla_j\nabla_i f\,d\sigma = 0 \tag{0.5}$$

to $f = \frac{1}{2}v^h v_h$, we find that

$$\int_{V_n} [(g^{ji}\nabla_j\nabla_i v^h)v_h + (\nabla^j v^i)(\nabla_j v_i)]\,d\sigma = 0. \tag{0.6}$$

Forming the difference (0.6) - (0.2) and the sum (0.6) + (0.3), we obtain respectively

$$\int_{V_n}\Big[(g^{ji}\nabla_j\nabla_i v^h - K_i^{\cdot h}v^i)v_h + \frac{1}{2}(\nabla^j v^i - \nabla^i v^j)(\nabla_j v_i - \nabla_i v_j) + (\nabla_j v^j)(\nabla_i v^i)\Big]d\sigma = 0, \tag{0.7}$$

$$\int_{V_n}\Big[(g^{ji}\nabla_j\nabla_i v^h + K_i^{\cdot h}v^i)v_h + \frac{1}{2}(\nabla^j v^i + \nabla^i v^j)(\nabla_j v_i + \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)\Big]d\sigma = 0. \tag{0.8}$$

These two equations yield, respectively, the following two theorems.

THEOREM D (de Rham and Kodaira [6]; see also Yano and Bochner [10]). *A necessary and sufficient condition for* v_j *in a space* V_n *to be harmonic is that*

$$g^{ji}\nabla_j\nabla_i v^h - K_i^{\cdot h}v^i = 0. \tag{0.9}$$

THEOREM E. *A necessary and sufficient condition for* v^h *in a space* V_n *to be a Killing vector is that*

$$g^{ji}\nabla_j\nabla_i v^h + K_i^{\cdot h}v^i = 0 \quad \textit{and} \quad \nabla_i v^i = 0. \tag{0.10}$$

A necessary and sufficient condition for v^h to define an infinitesimal affine collineation is that

$$\underset{v}{\mathfrak{L}}\left\{{}_j{}^h{}_i\right\} = \nabla_j\nabla_i v^h + K_{kji}^{\cdot\cdot\cdot h}v^k = 0, \tag{0.11}$$

where $\underset{v}{\mathfrak{L}}$ denotes the Lie derivation with respect to v^h. Theorem E now yields the following result.

THEOREM F. *An infinitesimal affine collineation in a space* V_n *is a motion.*

The main purpose of the present paper is to derive some other integral formulas in Riemannian and pseudo-Kählerian spaces, and to state some applications of these formulas. A part of the paper was announced at the Summer Institute for Differential Geometry in the Large, held at Seattle in 1956.

1. CONFORMAL KILLING VECTORS

Forming the sum (0.6) + (0.4) and taking account of

$$\int_{V_n} [(v_h \nabla^h \nabla_i v^i + (\nabla_j v^j)(\nabla_i v^i)] \, d\sigma = 0,$$

we obtain

$$\int_{V_n} \Big[\Big(g^{ji} \nabla_j \nabla_i v^h + K_i^{\cdot h} v^i + \frac{n-2}{n} \nabla^h \nabla_i v^i \Big) v_h \tag{1.1}$$
$$+ \frac{1}{2}\Big(\nabla^j v^i + \nabla^i v^j - \frac{2}{n} g^{ji} \nabla_b v^b\Big)\Big(\nabla_j v_i + \nabla_i v_j - \frac{2}{n} g_{ji} \nabla_a v^a\Big)\Big] \, d\sigma = 0,$$

from which we have

THEOREM 1.1 (Lichnerowicz [3], [4]; Sato [8]). *A necessary and sufficient condition for* v^h *in a* V_n *to be a conformal Killing vector is that*

$$g^{ji} \nabla_j \nabla_i v^h + K_i^{\cdot h} v^i + \frac{n-2}{n} \nabla^h \nabla_i v^i = 0. \tag{1.2}$$

For an infinitesimal conformal motion, we have

$$\underset{v}{\mathfrak{L}}\, g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\phi g_{ji}, \tag{1.3}$$

and consequently

$$\underset{v}{\mathfrak{L}} \left\{ {}_j{}^h{}_i \right\} = \nabla_j \nabla_i v^h + K_{kji}^{\cdot\cdot\cdot h} v^k = \phi_j A_i^h + \phi_i A_j^h - g_{ji} \phi^h, \tag{1.4}$$

where

$$\phi_j = \nabla_j \phi.$$

We call a *conformal collineation* an infinitesimal transformation $\xi^h \to \xi^h + v^h dt$ which satisfies (1.4). From Theorem 1.1 we have

THEOREM 1.2. *An infinitesimal conformal collineation is a conformal motion.*

Suppose that a vector v^h defines an infinitesimal conformal motion; then (1.3) and (1.4) hold, and consequently

$$\underset{v}{\mathfrak{L}}\, K_{kji}^{\cdot\cdot\cdot h} = -A_k^h \nabla_j \phi_i + A_j^h \nabla_k \phi_i - \nabla_k \phi^h g_{ji} + \nabla_j \phi^h g_{ki}; \tag{1.5}$$

it follows that

$$\nabla_j \nabla_i \phi = \frac{1}{n-2} \underset{v}{\mathfrak{L}}\, L_{ji}, \tag{1.6}$$

where

$$L_{ji} = -K_{ji} + \frac{K}{2(n-1)} g_{ji} \qquad \text{and} \qquad K = g^{ji} K_{ji}.$$

From (1.6), we obtain

$$g^{ji} \nabla_j \nabla_i \phi = -\frac{1}{2(n-1)} (\underset{v}{\mathfrak{L}} K + 2K\phi). \tag{1.7}$$

Thus, if K is a constant, we have

$$g^{ji} \nabla_j \nabla_i \phi = -\frac{K}{n-1} \phi. \tag{1.8}$$

Now, applying Green's formula (0.5) to $f^2/2$, we obtain

$$\int_{V_n} [f g^{ji} \nabla_j \nabla_i f + g^{ji} (\nabla_j f)(\nabla_i f)] d\sigma = 0. \tag{1.9}$$

Thus, if the function f satisfies an equation of the form

$$\Delta f = g^{ji} \nabla_j \nabla_i f = \lambda f \tag{1.10}$$

with λ = constant, and if $\lambda > 0$, then $f = 0$; and if $\lambda = 0$, then f = constant.

From (1.8) it now follows that if $K < 0$, then $\phi = 0$ and the conformal motion is a motion. If $K = 0$, then ϕ = constant, and the conformal motion is homothetic. But a homothetic transformation, being an affine motion, is a motion, by Theorem F. Thus we have

THEOREM 1.3. *An infinitesimal conformal motion in a* V_n *of constant nonpositive* K *is a motion.* (Essentially the same result has been obtained by T. Sumitomo and M. Kurita.)

COROLLARY. *If a* V_n *with* K = constant *admits an infinitesimal nonhomothetic conformal motion, then* $K > 0$.

When V_n is an Einstein space with $K > 0$, we deduce from (1.6) that

$$\nabla_j \nabla_i \phi = \lambda \phi g_{ji} \qquad \left(\lambda = -\frac{K}{n(n-1)} < 0\right), \tag{1.11}$$

from which

$$\nabla_j \phi_i + \nabla_i \phi_j = 2\lambda \phi g_{ji}. \tag{1.12}$$

From (1.3) and (1.12), we find that

$$\nabla_j w_i + \nabla_i w_j = 0, \tag{1.13}$$

where

$$w_i = v_i - \frac{1}{\lambda}\phi_i. \tag{1.14}$$

Thus we have

THEOREM 1.4. *If an Einstein space* V_n *with* $K > 0$ *admits an infinitesimal nonhomothetic conformal motion defined by* v^h, *then* v^h *can be decomposed into*

$$v^h = w^h + \frac{1}{\lambda}\phi^h \qquad \left(\lambda = -\frac{K}{n(n-1)} < 0\right), \tag{1.15}$$

where w^h *is a Killing vector and where* $\phi_i = \nabla_i\phi$ *is a conformal Killing vector.* (A. Lichnerowicz [3], [4] obtained this result by using the de Rham decomposition of a vector in a compact orientable space. But the proof above shows that this theorem is also true locally.)

Suppose that there exist two infinitesimal nonhomothetic conformal motions v^h and v^{*h}; then

$$v^h = w^h + \frac{1}{\lambda}\phi^h \quad \text{and} \quad v^{*h} = w^{*h} + \frac{1}{\lambda}\phi^{*h}.$$

It is easily verified that

$$(\underset{w}{\mathfrak{L}}\,\underset{w*}{\mathfrak{L}})g_{ji} = (\underset{w}{\mathfrak{L}}\,\underset{w*}{\mathfrak{L}} - \underset{w*}{\mathfrak{L}}\,\underset{w}{\mathfrak{L}})g_{ji} = \underset{[ww*]}{\mathfrak{L}}g_{ji} = 0,$$

$$\underset{w}{\mathfrak{L}}\phi_i = \nabla_i(\underset{w}{\mathfrak{L}}\phi) \qquad (\nabla_j(\underset{w}{\mathfrak{L}}\phi_i) = \lambda(\underset{w}{\mathfrak{L}}\phi)g_{ji}),$$

$$\underset{[\phi\phi*]}{\mathfrak{L}}g_{ji} = 0,$$

$$[\phi\phi*]^h = \underset{\phi}{\mathfrak{L}}\phi^{*h} = \lambda(\phi^h\phi^* - \phi^{*h}\phi).$$

This implies

THEOREM 1.5 (Lichnerowicz [3], [4]). *If* V_n *is an Einstein space with* $K > 0$, *then*

$$L = L_1 + L_2, \quad \text{with } [L_1 L_1] \subset L_1,\ [L_1 L_2] \subset L_2 \text{ and } [L_2 L_2] \subset L_1, \tag{1.16}$$

where L *is the Lie algebra of the Lie group of conformal motions,* L_1 *is the subalgebra defined by motions, and* L_2 *is the vector space of the gradient of* ϕ *which appears in* (1.3).

2. PSEUDO-ANALYTIC VECTORS IN PSEUDO-KÄHLERIAN SPACES

We now consider a pseudo-Kählerian space K_{2n}, that is, a space V_{2n} which carries a tensor $F_i^{\cdot h}$ satisfying the conditions

$$F_j^{\cdot i} F_i^{\cdot h} = -A_j^h \qquad (F_j^{\cdot m} F_i^{\cdot l} g_{ml} = g_{ji}, \; F_{ji} = -F_{ij}), \tag{2.1}$$

$$\nabla_j F_i^{\cdot h} = 0. \tag{2.2}$$

First we recall some important formulas in the theory of pseudo-Kählerian spaces:

$$K_{kji}^{\cdot\cdot\cdot a} F_a^{\cdot h} - K_{kja}^{\cdot\cdot\cdot h} F_i^{\cdot a} = 0, \tag{2.3}$$

$$K_i^{\cdot a} F_a^{\cdot h} = -\frac{1}{2} K_{kji}^{\cdot\cdot\cdot h} F^{kj}, \tag{2.4}$$

$$K_i^{\cdot a} F_a^{\cdot h} - F_i^{\cdot a} K_a^{\cdot h} = 0, \tag{2.5}$$

$$K = -\frac{1}{2} K_{kjih} F^{kj} F^{ih}, \tag{2.6}$$

$$\nabla_{[j} H_{ih]} = 0 \quad \text{and} \quad \nabla_j H^j_{\cdot i} = F_i^{\cdot a} \nabla_a K, \tag{2.7}$$

where

$$H_{ih} = K_{kjih} F^{kj}. \tag{2.8}$$

From (2.6) and (2.7), we obtain

THEOREM 2.1. *If, in a* K_{2n}, K *is a constant, then the tensor* H_{ih} *is harmonic.*

THEOREM 2.2. *If, in a* K_{2n}, $K = 0$, *then* H_{ih} *is harmonic and effective.*

THEOREM 2.3. *If, in a* K_{2n}, K = constant *and* $B_2 = 1$, *then the* K_{2n} *is an Einstein space.*

If a vector field v_h satisfies the condition

$$F_j^{\cdot a} \nabla_i v_a - F_i^{\cdot a} \nabla_a v_j = 0, \tag{2.9}$$

we call it a *covariant pseudo-analytic vector field;* and if a vector field v^h satisfies the condition

$$\mathfrak{L} F_i^{\cdot h} = F_a^{\cdot h} \nabla_i v^a - F_i^{\cdot a} \nabla_a v^h = 0, \tag{2.10}$$

we call it a *contravariant pseudo-analytic vector field* (Sasaki and Yano [7]). It is easily seen that if v^h is covariant (contravariant) pseudo-analytic, then so is $F_i^{\cdot h} v^i$; and that if u^h and v^h are contravariant pseudo-analytic, then $[u, v]^h$, $[Fu, v]^h$, $[u, Fv]^h$, $[Fu, Fv]^h$ are also contravariant pseudo-analytic.

Now, forming the square of the tensor appearing in the left-hand member of (2.9), we find that

$$(F^{jb} \nabla^i v_b - F^{ib} \nabla_b v^j)(F_j^{\cdot a} \nabla_i v_a - F_i^{\cdot a} \nabla_a v_j) = 2[g^{ji}(\nabla_j v_h)(\nabla_i v^h) - F^{jb} F^{ia} (\nabla_i v_b)(\nabla_a v_j)]. \tag{2.11}$$

On the other hand we have, from Green's formula,

$$0 = \int_{K_{2n}} g^{ji} \nabla_j(v_h \nabla_i v^h)\, d\sigma = \int_{K_{2n}} [g^{ji}(\nabla_j v_h)(\nabla_i v^h) + v_h g^{ji} \nabla_j \nabla_i v^h] d\sigma$$

and

$$0 = \int_{K_{2n}} \nabla_i [F^{jb} F^{ia} v_b (\nabla_a v_j)]\, d\sigma$$

$$= \int_{K_{2n}} [F^{jb} F^{ia} (\nabla_i v_b)(\nabla_a v_j) + F^{jb} F^{ia} v_b (\nabla_i \nabla_a v_j)]\, d\sigma$$

$$= \int_{K_{2n}} [F^{jb} F^{ia} (\nabla_i v_b)(\nabla_a v_j) + K_{ji} v^j v^i]\, d\sigma,$$

by virtue of (2.4) and (2.5); consequently equation (2.11) gives

THEOREM 2.4. *In a* K_{2n}, *we have*

$$(2.12) \quad \int_{K_{2n}} \left[(g^{ji} \nabla_j \nabla_i v^h - K_i^{\cdot h} v^i) v_h + \frac{1}{2} (F^{jb} \nabla^i v_b - F^{ib} \nabla_b v^j)(F_j^{\cdot a} \nabla_i v_a - F_i^{\cdot a} \nabla_a v_j) \right] d\sigma = 0.$$

From this follows

THEOREM 2.5. *A necessary and sufficient condition for a vector field* v_i *in a* K_{2n} *to be covariant pseudo-analytic is that*

$$(2.13) \quad g^{ji} \nabla_j \nabla_i v^h - K_i^{\cdot h} v^i = 0.$$

Combining Theorem D and Theorem 2.5, we obtain the famous theorem: *A necessary and sufficient condition for a vector field* v^h *in a* K_{2n} *to be harmonic is that* v^h *be covariant pseudo-analytic.*

Next, forming the square of the tensor appearing in the middle member of (2.10), we find that

$$(2.14) \quad \begin{aligned} &(F^{jb} \nabla_b v^i - F_b^{\cdot i} \nabla^j v^b)(F_j^{\cdot a} \nabla_a v_i - F^{a}_{\cdot i} \nabla_j v_a) \\ &= 2[g^{ji}(\nabla_j v_a)(\nabla_i v^a) + F^{jb} F^{ia} (\nabla_b v_i)(\nabla_j v_a)], \end{aligned}$$

from which we obtain

THEOREM 2.6. *In a* K_{2n}, *we have*

$$\int_{K_{2n}} \Big[(g^{ji} \nabla_j \nabla_i v^h + K_i^{\cdot h} v^i) v_h \tag{2.15}$$
$$+ \frac{1}{2}(F^{jb} \nabla_b v^i - F_b^{\cdot i} \nabla^j v^b)(F_j^{\cdot a} \nabla_a v_i - F^a_{\cdot i} \nabla_j v_a) \Big] d\sigma = 0.$$

From this, we have

THEOREM 2.7. *A necessary and sufficient condition for a vector field* v^h *in a* K_{2n} *to be contravariant pseudo-analytic is that*

$$g^{ji} \nabla_j \nabla_i v^h + K_i^{\cdot h} v^i = 0. \tag{2.16}$$

Combining equations (0.6) and (2.16), we can easily prove a theorem of Bochner [1]: *If, in a* K_{2n}, $K_{ji} v^j v^i$ *is positive definite, then there does not exist a contravariant pseudo-analytic vector field other than the zero vector.* Also, combining (2.13) and (2.16), we can prove a further theorem of Bochner [2]: *In a* K_{2n}, *if* u^h *is covariant pseudo-analytic and* v^h *is contravariant pseudo-analytic, then* $u^h v_h$ *is a constant.*

Let a vector field v^h be given in a V_n, and consider a geodesic $\xi^h(s)$ in V_n. The condition that the infinitesimal transformation $\xi^h \to \xi^h + v^h dt$ transform the geodesic $\xi^h(s)$ into a geodesic and preserve affine character of the arc length is given by

$$(\nabla_j \nabla_i v^h + K_{kji}^{\cdot\cdot\cdot h} v^k) \frac{d\xi^j}{ds} \frac{d\xi^i}{ds} = 0. \tag{2.17}$$

If we take a point ξ^h and a unit vector h^h at ξ^h, the geodesic which passes through ξ^h and is tangent to h^h is uniquely determined, and we can consider the vector

$$u^h = (\nabla_j \nabla_i v^h + K_{kji}^{\cdot\cdot\cdot h} v^k) h^j h^i \tag{2.18}$$

appearing in the left-hand member of (2.17). We shall call (2.18) the geodesic deviation vector of the unit vector h^h at the point ξ^h with respect to v^h.

Now consider n mutually orthogonal unit vectors $h^h_{(a)}$ $(a = 1, 2, \cdots, n)$ and the geodesic deviation vectors $u^h_{(a)}$ of $h^h_{(a)}$ with respect to v^h. For the mean of $u^h_{(a)}$ we have

$$\frac{1}{n} \sum_a u^h_{(a)} = \frac{1}{n}(g^{ji} \nabla_j \nabla_i v^h + K_i^{\cdot h} v^i),$$

which shows that the mean is independent of the choice of $h^h_{(a)}$. We shall call $\frac{1}{n} \sum_a u^h_{(a)}$ the *mean geodesic deviation vector with respect to* v^h. From Theorem 2.7, we obtain

THEOREM 2.8. *A necessary and sufficient condition for a vector field* v^h *in a* K_{2n} *to be contravariant pseudo-analytic is that the mean geodesic deviation vector with respect to* v^h *vanish.*

In a previous paper [9; Theorem 2], we have proved that the Lie derivative of a harmonic tensor in a V_n with respect to a motion vanishes. Thus, since F_{ji} is a harmonic tensor, we have $\underset{v}{\mathfrak{L}} F_{ji} = 0$, where v^h is a Killing vector, and consequently we have

THEOREM 2.9. *A one-parameter group of motions in a* K_{2n} *preserves the pseudo-complex structure of the space.*

Conversely, if a K_{2n} admits an infinitesimal transformation $\xi^h \to \xi^h + v^h dt$ which preserves the pseudo-complex structure of the space and also the volume element, then we have (2.16), and $\nabla_i v^i = 0$. From Theorem E, we now obtain

THEOREM 2.10. *If an infinitesimal transformation* $\xi^h \to \xi^h + v^h dt$ *preserves the pseudo-complex structure of the space* K_{2n} (*that is, if the vector* v^h *is contravariant pseudo-analytic*) *and if it also preserves the volume element* (*that is, if the vector* v^h *satisfies* $\nabla_i v^i = 0$), *then the transformation is a motion.*

3. KILLING VECTORS IN KÄHLER-EINSTEIN SPACES

We now consider an equation of the form

$$\Delta f = g^{ji}\nabla_j \nabla_i f = \lambda f \qquad (\lambda = \text{constant},\ \lambda < 0) \tag{3.1}$$

in a K_{2n}, from which

$$\Delta f_h = g^{ji}\nabla_j \nabla_i f_h - K_h^{\cdot a} f_a = \lambda f_h \qquad (f_h = \nabla_h f), \tag{3.2}$$

whence, by virtue of (2.5),

$$\Delta v_h = g^{ji}\nabla_j \nabla_i v_h - K^a_{\cdot h} v_a = \lambda v_h, \tag{3.3}$$

where

$$v_h = F^a_{\cdot h} f_a. \tag{3.4}$$

Substituting (3.3) into (0.8) and taking account of the condition $\nabla_i v^i = 0$, we find

$$\int_{K_{2n}} \left[(2K_{ji} + \lambda g_{ji}) v^j v^i + \frac{1}{2}(\nabla^j v^i + \nabla^i v^j)(\nabla_j v_i + \nabla_i v_j) \right] d\sigma = 0. \tag{3.5}$$

From this integral formula, we have

THEOREM 3.1. *If, in a* K_{2n}, *the form* $(2K_{ji} + \lambda g_{ji}) v^j v^i$ *is positive definite, then the equation* $\Delta f = \lambda f$ *has no solution other than zero.*

THEOREM 3.2. *If, in a Kähler-Einstein space* K_{2n} *with* $K > 0$ $(K_{ji} = (K/2n) g_{ji})$, $K/n + \lambda > 0$, *then the equation* $\Delta f = \lambda f$ *has no solution other than zero.*

Consequently if the equation $\Delta f = \lambda f$ admits a solution other than zero, then

$$\frac{K}{n} + \lambda \leq 0, \qquad \text{that is,} \qquad \lambda \leq -\frac{K}{n}. \tag{3.6}$$

THEOREM 3.3. *If, in a Kähler-Einstein space* K_{2n} *with* $K > 0$, *the equation* $\Delta f = -(K/n) f$ *admits a solution other than zero, then* $v_i = F^a_{\cdot i} f_a$ *is a Killing vector.*

Now suppose that a general K_{2n} admits a Killing vector v^h, then we have

$$2\nabla_{[j}(F^{a}_{\cdot i]}v_a) = 0 \quad \text{and} \quad \nabla_j(F^{ji}v_i) = F^{ji}\nabla_j v_i, \tag{3.7}$$

by virtue of the condition $\underset{v}{\mathfrak{L}} F_{ji} = 0$

THEOREM 3.4. *In an irreducible* K_{2n}, $F^{ji}\nabla_j v_i \neq 0$ *for a Killing vector* v^h.

For the proof, we note that if $F^{ji}\nabla_j v_i = 0$, then $F^{a}_{\cdot i}v_a$ is harmonic, and consequently v^h is also harmonic. Thus v^h, being at the same time a Killing vector and a harmonic vector, is a parallel vector field, a fact which contradicts the irreducibility.

Consider now an irreducible Kähler-Einstein space K_{2n} with $K > 0$, and suppose that K_{2n} admits a Killing vector v^h, then

$$f = \frac{n}{K}F^{ji}\nabla_j v_i \neq 0. \tag{3.8}$$

On the other hand, using $\nabla_j \nabla_i v_h + K_{kjih}v^k = 0$, we find that

$$f_j = \nabla_j f = \nabla_j\left(\frac{n}{K}F^{ih}\nabla_i v_h\right) = F^{a}_{\cdot j}v_a,$$

and consequently

$$f_i = F^{a}_{\cdot i}v_a \quad \text{and} \quad v_i = -F^{a}_{\cdot i}f_a, \tag{3.9}$$

from which

$$g^{ji}\nabla_j\nabla_i f = -\frac{K}{n}f. \tag{3.10}$$

Thus we have

THEOREM 3.5. *If an irreducible Kähler-Einstein space* K_{2n} *with* $K > 0$ *admits a Killing vector* v^h *other than zero, then the equation* (3.10) *admits a solution* f *other than zero, and conversely.*

Suppose that an irreducible Kähler-Einstein space K_{2n} with $K > 0$ admits two Killing vectors v^h and w^h, corresponding to solutions f and g of (3.10), respectively. Then

$$F^{ji}\nabla_j[v, w]_i = F^{ji}\nabla_j \underset{v}{\mathfrak{L}} w_i = \underset{v}{\mathfrak{L}}(F^{ji}\nabla_j w_i) = \frac{K}{n}\underset{v}{\mathfrak{L}} g = \frac{K}{n}v^i\nabla_i g$$

$$= -\frac{K}{n}F^{ai}f_a\nabla_i g$$

$$= -\frac{K}{n}F^{ji}(\nabla_j f)(\nabla_i g).$$

Thus, if we define $[f, g]$ by

$$[f, g] = -F^{ji}(\nabla_j f)(\nabla_i g),$$

we have

THEOREM 3.6 (Lichnerowicz [3], [4]). *If an irreducible Kähler-Einstein space* K_{2n} *with* $K > 0$ *admits two Killing vectors* v^h *and* w^h *to which correspond* f *and* g *respectively, then* $[v, w]^h$ *and* [f, g] *correspond to each other.*

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