THE SPACE OF LOOPS ON A LIE GROUP

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INTRODUCTION

In this paper we describe the Hopf algebra $H_*(\Omega' K)$,† where K is a connected compact Lie group and $\Omega' K$ is the e-component of the loop space on K. As an application, we compute the stable homotopy groups $\pi_k(U)$ (by a method other than that envisaged in [5]) and the group $\pi_{2n}\{U(n)\}$. These results, presented in Section 8, have recently been used by Kervaire [11], and independently by Milnor [6], to prove that the n-sphere is parallelizable only if n = 1, 3 or 7.

We shall need the following known information about $\Omega' K$:

(1.1) *the space* $\Omega' K$ *is a homotopy-commutative Hopf space;*

(1.2) *both* $H_*(\Omega' K; Q)$ *and* $H^*(\Omega' K; Q)$ *are primitively generated polynomial rings.*

((1.1) follows from the fact that K is itself a Hopf space; (1.2) follows from the Serre C-theory and the fact that over Q, the group K looks like a product of odd spheres [16]). From the application of Morse theory [4], [8], it is further known that

(1.3) *the* Z-*module* $H_q(\Omega' K)$ *is free* $(q = 0, 1, 2, \cdots)$ *and vanishes for odd* q;

(1.4) *one has an (explicit) additive basis of singular cycles for the classes of* $H_*(\Omega' K)$.

Unfortunately, this basis is not closed under Pontryagin multiplication, and therefore it is not directly applicable to our problem. Nevertheless, the construction of (1.4) is the main tool in the proof of Theorem 1. This theorem, in turn, is the main step towards our description.

Our first new result is that the Pontryagin ring $H_*(\Omega' K)$ is always finitely generated. The explicit generators are described in Theorem 1.

A homomorphism $s: R \to K$ of the real numbers R into K, whose kernel contains the group of integers in R, will be called a *circle on* K. With such a circle we associate two spaces: K_s, the centralizer of the image of s, and K^s, the space K/K_s of left cosets. The formula

$$x \to x\, s(t)\, x^{-1} s(t)^{-1} \qquad (x \in K;\ 0 \leq t \leq 1)$$

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†In general we use the singular theory, and we use the lower star for homology, the upper star for cohomology. If no coefficients are indicated, the integers Z are to be understood. The rational numbers are denoted by Q, the reals by R.

defines a map of K into $\Omega'K$, which clearly is constant along the left cosets of K_s. This transformation therefore induces a map

$$g^s: K^s \to \Omega'K, \tag{1.5}$$

and the question arises under what circumstances $g^s_* H_*(K^s)$ generates $H_*(\Omega'K)$ multiplicatively.

To formulate an answer, we need the following definition. Let s be a circle on K, and T a maximal torus of K containing s. Let $\mathfrak{W}(K; T)$ be the subgroup of the automorphisms of T which are induced by the inner automorphisms of K, and let $\Sigma(K, T)$ denote the set of roots of K, interpreted as elements of $H^1(T)$ (that is, as homomorphisms of $H_1(T)$ into Z). We also write s for the homology class in $H_1(T)$ determined by the restriction of s to $0 \leq t < 1$, and we denote by Λ_s the module generated in $H_1(T)$ by s and all its transforms under $\mathfrak{W}(K, T)$.

Definition 1.1. A circle $s \subset T \subset K$ is called a *generating circle for* K if each root $\theta \in \Sigma(K, T)$ takes on the value 1 on some element of Λ_s.

In other words, s is a generating circle if for each $\theta \in \Sigma(K, T)$ there exists an $x_\theta \in \Lambda_s$ such that $\theta(x_\theta) = 1$. For convenience, we shall refer to K^s as a *generating variety* of K, and to g^s as a *generating map*, whenever s is a generating circle.

THEOREM 1. *If* s *is a generating circle for* K, *then the image of*

$$g^s_*: H_*(K^s) \to H_*(\Omega'K)$$

generates the Pontryagin ring $H_*(\Omega'K)$.

A companion theorem which assures us that Theorem 1 is not trivial reads as follows:

THEOREM 2. *If* K *is a compact Lie group with trivial center, then* K *has a generating circle.*

In general, a semisimple compact group need not have a generating circle, as is easily verified. However, the restriction concerning the center is not serious, because the identity components of the loop space of K and of the loop space of the adjoint group of K coincide. Further, the adjoint group Ad K of K (that is, the linear group given by the adjoint representation of K on its Lie algebra) has trivial center. Clearly, Theorems 1 and 2 imply

COROLLARY. *The Pontryagin ring* $H_*(\Omega'K)$ *is finitely generated.*

The homogeneous spaces K^s play a central part in many topological questions about Lie groups, and they have many pleasing properties. In particular we recall that

(1.6) *if* s *is any circle on* K, *then* $H_*(K^s)$ *is a free module and* $H_q(K^s) = 0$ *for odd* q [4];

(1.7) *there is an algorithm for constructing the ring* $H^*(K^s)$ *in terms of the Cartan integers of* K [7], [8].

Together with Theorems 1 and 2, these facts imply that the diagonal map $\Delta_*: H_*(\Omega) \to H_*(\Omega) \otimes H_*(\Omega)$ can be computed in terms of the generators (that is, in terms of the elements of $H_*(K^s)$, for a suitable s). Indeed, this diagonal map is determined by the diagonal map $K^s \to K^s \times K^s$, and is thus described by $H^*(K^s)$. It therefore remains to describe a procedure for determining the relations between the generators. In practice, the relations are rather complicated; however, theoretically the problem can be solved completely. First, it is an easy consequence of (1.1), (1.2), (1.3), Theorem 1, and (1.6) that the following is true.

THEOREM 3. *Let* s *be a generating circle for* K, *and let* $g^s: K^s \to \Omega' K$ *be the generating map. Let* $\mathfrak{P}^* \subset H^*(\Omega' K)$ *be the subspace of primitive elements in* $H^*(\Omega' K)$, *and set* $\mathfrak{P}^*_s \subset H^*(K^s)$ *equal to* $g^{s*}\mathfrak{P}^*$. *Then*

(a) $\mathfrak{P}^*_s$ *is a direct summand of* $H^*(K^s)$;

(b) *the subspace* $\mathfrak{P}^*_s$ *and the ring* $H^*(K^s)$ *completely determine the Hopf-algebras* $H^*(\Omega' K)$ *and* $H_*(\Omega' K)$.

The mechanism by which this description proceeds is given in Section 6.

In view of Theorem 3, the problem we set out to solve is reduced to finding $\mathfrak{P}^*_s \otimes Q$ in $H^*(K^s; Q)$. This last gap is filled in Theorem 4 of Section 7, by means of the beautiful and complete description of $H^*(K^s; Q)$ due to A. Borel and J. Leray.

I am very much indebted to A. Borel, who read and criticized the first version of this manuscript. In particular, he pointed out to me that with the strong definition of a generating circle which I had used, the orthogonal groups SO(4n) have no generating circles. With the present, much weaker and more appropriate definition, Theorem 2 is, I hope, correct.

2. PROOF OF THEOREM 1 (PRELIMINARIES)

We shall use the symbol K exclusively for semisimple, compact, connected Lie groups. The space of loops on K, that is, ΩK, is based on the identity $e \in K$. We use the variable endpoint definition for ΩK and the symbol $\vee$ for Pontryagin multiplication. This multiplication is associative, and it is homotopy-commutative because K is an H-space, as remarked earlier. The parametrization of paths in ΩK is usually left to the reader.

If s is a circle on K, the restriction of s to [0, 1] defines a point in ΩK which is also denoted by s, and we write $\Omega_s K$ for the component of ΩK containing s.

The *inverse* of a circle s is written s^{-1}, and it is defined by $s^{-1}(t) = s(-t) = s(t)^{-1}$. Two circles s and s^1 are called *conjugate* if there exists an $x \in K$ such that $x s(t) x^{-1} = s^1(t)$. Throughout, g^s denotes the map $K^s \to \Omega' K$ defined in the introduction, and we write A_s for the image of $g^s_*: H_*(K^s) \to H_*(\Omega' K)$. The following is an immediate consequence of the connectedness of K.

PROPOSITION 2.1. *If* s *and* s^1 *are conjugate circles, then* $A_s = A_{s^1}$.

For the circles s and s^{-1} only a weaker proposition holds:

PROPOSITION 2.2. *The rings generated by* A_s *and* $A_{s^{-1}}$ *in* $H_*(\Omega' K)$ *are identical.*

Proof. It is convenient to introduce auxiliary maps $f^s: K^s \to \Omega_s K$ defined by

$$f^s(q)(t) = k\, s(t)\, k^{-1} \qquad (k \cdot K_s = q;\ 0 \le t \le 1). \tag{2.1}$$

Note that g^s is homotopic to f^s followed by Pontryagin multiplication with the point s^{-1} in $\Omega_{s^{-1}} K$; that is, $g^s = s^{-1} \vee f^s$.

Consider now the sequence of maps

$$K^s \xrightarrow{\Delta} K^s \times K^{s^{-1}} \to \Omega_s K \times \Omega_{s^{-1}} K \xrightarrow{\vee} \Omega' K. \tag{2.2}$$

Here the first map Δ is the diagonal map in the following sense. Clearly $K_s = K_{s^{-1}}$; hence $kK_s \to kK_s \times kK_{s^{-1}}$ can be interpreted as the diagonal map. The second map in (2.2) is $f^s \times f^{s^{-1}}$, while the last one is the Pontryagin multiplication. The composition of these three maps is homotopically trivial, by inspection. In homology this is expressed by the identity

$$f^s_* y \vee s^{-1} + \sum f^s_* x_\alpha \vee f^{s^{-1}} x_\beta + s \vee f^{s^{-1}}_* y = 0, \tag{2.3}$$

which holds whenever the diagonal expansion of $y \in H_*(K^s)$ in $H_*(K^s \times K^s)$ takes the form

$$\Delta_* y = y \otimes 1 + \sum x_\alpha \otimes x_\beta + 1 \otimes y \qquad (\dim x_\alpha, x_\beta < \dim y). \tag{2.4}$$

Because $H_*(K^s)$ is free, $H_*(K^s \times K^s) = H_*(K^s) \otimes H_*(K^s)$, so that every element $y \in H_*(K^s)$ with $\dim y > 0$ has such an expansion.

Because $\vee$ is commutative on the homology level, (2.3) implies that in the lowest nontrivial (positive) dimension, $s^{-1} \vee f^s_*$ and $s \vee f^{s^{-1}}_*$ differ by sign. Hence the same is true of g^s_* and $g^{s^{-1}}_*$. Now, using induction on the dimensions in (2.3), we see that A_s is contained in the ring generated by $A_{s^{-1}}$, and vice versa. This proves the proposition.

A product of the form $\Pi = K^{s_1} \times K^{s_2} \times \cdots \times K^{s_n}$ will be called an s-product, if

$$s_1 \vee s_2 \vee \cdots \vee s_n = e \qquad \text{(trivial loop)}, \tag{2.5}$$

$$\text{each } s_i \text{ is conjugate to } s \text{ or to } s^{-1}. \tag{2.6}$$

With every such s-product we associate a map $f^\pi \colon \Pi \to \Omega' K$ defined by the multiplication

$$f^\pi = f^{s_1} \vee f^{s_2} \vee \cdots \vee f^{s_n}$$

(by condition (2.5), f^π has its values in $\Omega' K$).

PROPOSITION 2.3. *The subring generated by* A_s *in* $H_*(\Omega' K)$ *contains the image of* f^π_* *for every* s-*product* Π.

Proof. Again because $H_*(K^s)$ is free, it is sufficient to consider f^π_* on elements of the form $u = y^1_1 \otimes y^1_2 \otimes \cdots \otimes y^1_n$ $[y^1_i \in H_*(K^{s_i})]$. Then, if $f^\pi_* y^1_i = y_i$,

$$f^\pi_* u = y_1 \vee y_2 \vee \cdots \vee y_n.$$

By (2.5), this element is also given by $(s_1^{-1} \vee y_1) \vee (s_2^{-1} \vee y_2) \cdots (s_n^{-1} \vee y_n)$. Thus f^π_* is in the ring generated by the A_{s_i}. By (2.6) and Propositions 2.1 and 2.2, this ring is already generated by A_s.

We come now to the main part of the proof, which is carried out in the next two sections. It rests squarely on the results of [8], and can be sketched as follows:

Corresponding to a generating circle s, we construct a family of singular cycles (Γ, f_Γ) on $\Omega' K$. By appealing to [8], it is verified that this family contains an additive base for $H_*(\Omega' K)$. Next we construct for each (Γ, f_Γ) an s-product Π, and a map $\phi: \Gamma \to \Pi$ such that

$$f_\Gamma = f^\pi \circ \phi. \tag{2.7}$$

Once this program is carried out, the proof of Theorem 1 is complete. One merely combines (2.7) with Proposition 2.3.

3. THE CYCLES OF $\Omega' K$

From now on we choose a fixed maximal torus $T \subset K$, and we denote its universal covering group by $\mathfrak{T}$. The covering map shall be η.

The adjoint representation of K on its own Lie algebra $\mathfrak{k}$, when restricted to T, decomposes $\mathfrak{k}$ into a direct sum

$$\mathfrak{k} = \mathfrak{t} + \mathfrak{e}_1 + \cdots + \mathfrak{e}_m,$$

where $\mathfrak{t}$ is the tangent space to T, and the $\mathfrak{e}_i$ are 2-planes on which T is represented nontrivially by rotations. The kernel of this representation of T on $\mathfrak{e}_i$ is denoted by U_i, and *any component* of $\eta^{-1}(U_i)$ $(i = 1, \cdots, m)$ is called a *singular plane* (of K) in $\mathfrak{T}$.

Let $P = \{p_1, p_2, \cdots, p_k\}$ be a finite ordered collection of singular planes. Our purpose is to assign a homology class $P_* \in H_*(\Omega' K)$ to P. Let $\overline{p}_i$ be the image of p_i under the projection $\mathfrak{T} \to T$, and let $K(p_i)$ be the centralizer of $\overline{p}_i$. Just as in [8], we construct the manifold $W(P) = \Pi_1^k K(p_i)$ and its twisted quotient

$$\Gamma(P) = K(p_1) \times_T K(p_2) \times_T \cdots \times_T K(p_k)/T.$$

By definition, $\Gamma(P)$ is the quotient of $W(P)$ under the right action of T^k on $W(P)$ defined by

$$(x_1, x_2, \cdots, x_k) \circ (t_1, \cdots, t_k) = (x_1 t_1, t_1^{-1} x_2 t_2, \cdots, t_{k-1}^{-1} x_k t_k) \tag{3.1}$$
$$(x_i \in K(p_i);\ t_i \in T;\ i = 1, \cdots, k).$$

From [8] we recall that $\Gamma(P)$ *is always connected and orientable.*

We define next a family of maps $f_P^c: \Gamma(P) \to \Omega' K$ (all of which will be homotopic). The image of a fundamental cycle of $\Gamma(P)$ (chosen arbitrarily but once for all, for each $\Gamma(P)$) will be the class P_*.

The maps f_P^c are defined by means of a *chain subject to* P. By definition, such a chain is to be an ordered sequence of polygons $c = \{c_0, c_1, \cdots, c_k\}$ in $\mathfrak{T}$ satisfying to the following boundary conditions:

$$(C)\begin{cases}(C_1) & \text{the polygon } c_0 \text{ starts at the identity } e \text{ of } \mathfrak{F};\\ (C_2) & \text{the endpoint of } c_{i-1} \text{ coincides with the initial point of } c_i \text{ and lies on the plane } p_i;\\ \cdots\cdots & \\ (C_{k+1}) & \text{the endpoint of } c_k \text{ is } e.\end{cases}$$

Let $c = \{c_i\}$ be a chain subject to P, and denote the projection of c_i into T by $\overline{c}_i$. Let $\tilde{f}_P^c: W(P) \to \Omega' K$ be given by

$$(x_1, x_2, \cdots, x_k) \to \overline{c}_0 + \mathrm{Ad}\, x_1 \cdot \overline{c}_1 + \mathrm{Ad}\,(x_1 x_2) \cdot \overline{c}_2 + \cdots + \mathrm{Ad}\,(x_1 \cdots x_k) \cdot \overline{c}_k .$$

The meaning of the + is surely clear: we let the parameter run through $\overline{c}_0$, then through $\mathrm{Ad}\,(x_1) \cdot \overline{c}_1$, and so forth. Because of the boundary conditions on c, no breaks occur.

It is easily verified that this map is constant along the orbits of the action (3.1) and so defines the promised map $f_P^c: \Gamma(P) \to \Omega' K$. All the maps f_P^c are homotopic, because two chains, both subject to P, can be moved into one another without violating the boundary conditions imposed by P.

We must now strongly appeal to [8] for the proof of the following fact.

PROPOSITION 3.1. *Let* P *range over the finite ordered sequences of singular planes in* $\mathfrak{F}$. *Then the corresponding homology classes* P_* *generate* $H_*(\Omega' K)$ *additively.*

Indeed we have essentially copied the constructions of Chapter I in [8]. Only the emphasis has changed. There we started from a straight line segment c in $\mathfrak{F}$ (in general position) which terminates at e. The order in which c meets the singular planes then define a sequence $P = \{p_i\}$. Finally, we used the obvious chain for this P furnished by c itself, in the construction of f_P^c.

To prevent misunderstanding, let me emphasize that the totality of cycles P_* described here is by no means independent. It merely generates $H_*(\Omega' K)$. In [8] we selected the subcollection which formed a basis; here, there is no need for that.

4. PROOF OF THEOREM 1 (COMPLETED)

Let s be a circle on K. A parametrized curve $u: [n, n+1] \to \mathfrak{F}$ $(n \in Z)$ is called an *elementary* s-*segment* if the projection of u on T agrees with the map $s^1 | [n, n+1]$, where s^1 is a circle on K, conjugate either to s or to s^{-1}. A chain c subject to P is called an s-*chain* if each of its polygons c_i has only elementary s-segments for its straight edges.

PROPOSITION 4.1. *Suppose that* c *is an* s-*chain subject to* $P = \{p_1, \cdots, p_k\}$. *Then there exists an* s-*product* Π, *and a map* $\phi: \Gamma(P) \to \Pi$ *such that* $f_P^c = f^\pi \circ \phi$.

Proof. Let $\{c_i\}$ $(i = 1, \cdots, k)$ be the polygons of c, and let $c_i = a_i^1 + a_i^2 + \cdots + a_i^{n_i}$ be the ordered decomposition of c_i into elementary s-segments. We assume that $a_0^1: [0, 1] \to \mathfrak{F}$, so that a_0^2 is defined for $1 \le t \le 2$, and so forth. Let s_i^j be the circle determined by a_i^j, and write K^{ij} for $K^{s_i^j}$. The s-product Π is now defined by

$$\Pi = K^{01}\times\cdots\times K^{0n_0}\times K^{11}\times\cdots\cdots\times K^{kn_k}.$$

That Π is indeed an s-product is seen as follows: Because the total chain

$$c_0 + c_1 + \cdots + c_k$$

represents a closed polygon in $\mathfrak{T}$, the Pontryagin product of all the s^i_j represents the trivial loop. Hence condition (2.5) is satisfied. By the definition of an elementary s-chain, all the s^i_j are conjugate to s or to s^{-1}. Hence condition (2.6) is also fulfilled.

Next let $W = K\times\cdots\times K\times K\times\cdots\cdots\times K$ (one copy over each K^{ij} of Π), and let $W \to \Pi$ be the product of the natural projections $K \to K^{ij}$. The promised map $\phi: \Gamma(P) \to \Pi$ shall be described by a map $\tilde{\phi}$: $W(P) \to W$ which takes the fibers of $W(P)$ (over $\Gamma(P)$) into the fibers of W (over Π). We define $\tilde{\phi}(x_1, \cdots, x_p)$ to be the point $\{y_i\}$ in W, where

$$y_i = e \text{ for } 1 \leq i \leq n_0,$$

$$y_i = x_1 \text{ for } n_0 < i \leq n_0 + n_1,$$

$$y_i = x_1 x_2 \text{ for } n_0 + n_1 < i \leq n_0 + n_1 + n_2,$$

$$\cdots,$$

$$y_i = x_1 x_2 \cdots x_k \text{ for } m - n_k < i \leq m \qquad (m = n_0 + n_1 + \cdots + n_k).$$

In words, $g = \tilde{\phi}(x_1, \cdots, x_p)$ is described as follows. The first n_0 coordinates of g are e, the next n_1 coordinates are x_1, the next n_2 coordinates are $x_1 x_2$, and so forth.

The verification that $\tilde{\phi}$ is fiber preserving, and that the induced ϕ factors f^c_P by f^π, is left to the reader.

In view of Propositions 3.1 and 4.1, the program outlined in Section 2 is nearly completed. We still have to show that we can find elementary s-chains subject to P for every P, under the suitable conditions. This gap is filled by the following proposition, which therefore also completes the proof of Theorem 1.

PROPOSITION 4.2. *Let* s *be a generating circle for* K. *Then every sequence of singular planes* P *admits an* s-*chain subject to* P.

Let s be a generating circle for K on T, and consider the module Λ_s generated in $H_1(T)$ by s and its transforms under $\mathcal{W}(K, T)$. Let Λ be the kernel of the covering map $\eta: \mathcal{F} \to T$. Under the identification $H_1(T) \approx \Lambda$, Λ_s goes over into a sublattice of Λ.

(4.1) *Every singular plane of* K *on* $\mathfrak{T}$ *contains a point of the lattice* Λ_s.

Proof. The roots of K are by definition the pre-images of generators of $H^1(T/U_i)$ under the projection of T on T/U_i ($i = 1, \cdots, m$). (Here the U_i are the kernels of Ad$|$ T on e_i, as in Section 3).

The definition of a generating circle can therefore also be given as follows (see Definition 1.1): the circle s is generating for K if the natural homomorphism $H_1(T) \to H_1(T/U_i)$ maps Λ_s onto $H_1(T/U_i)$ for each $i = 1, \cdots, m$. On the other hand, the singular planes of K on $\mathfrak{T}$ are precisely the components of the pre-images of U_i under η. Hence (4.1) is true.

Clearly every point of Λ_s can be joined to the origin by a sequence of elementary s-segments. Therefore, by (4.1), every sequence P of singular planes also admits an s-chain subject to P. This proves Proposition 4.2.

5. GENERATING CIRCLES

A circle s on K is called *minimal* if the dimension of the center of K_s is precisely 1. These circles correspond to the edges of the "Cartan simplex," and they are of interest in this context because, by means of Theorem 1, they lead to the most economical description of $H_*(\Omega K)$.

THEOREM 2'. *Every compact, connected Lie group* K *with trivial center has a minimal generating circle.*

To prove this theorem, we recall some facts concerning the set of roots of K. The reader is referred to [9] and [14] for details.

Because the class of groups under consideration splits canonically into simple groups of the same class, we may assume K to be simple, in this discussion.

The Weyl group $\mathscr{W} = \mathscr{W}(K, T)$ acts on the roots $\Sigma = \Sigma(K, T)$ as a permutation group. If this action is transitive, we call K *simply laced.* The only other possibility is that Σ is the union of two disjoint orbits under $\mathscr{W}$. In that case, K is called *doubly laced.* A nondegenerate inner product is defined on $H_1(T)$ by setting

$$(x, y) = \sum \theta(x) \cdot \theta(y) ,$$

where θ ranges over Σ. The induced inner product on $H^1(T)$ is the Killing form, and the orbits of $\mathscr{W}$ on Σ are precisely the roots of equal length. There are therefore at most two lengths. We refer to roots with maximal length as *long roots.*

One has the notion of a *fundamental system of roots in* Σ: such a system $\mathscr{F}$ is characterized by the property that every root is a linear combination of elements in $\mathscr{F}$ with entirely nonpositive or entirely nonnegative coefficients.

LEMMA. *Let* $\mathscr{F}$ *be a fundamental system of roots for the doubly laced group* K. *If* $a \in \mathscr{F}$ *is a long root, then there exists a shorter root* $b \in \Sigma$ *such that the* a-*coefficient of* b *in the base* $\mathscr{F}$ *is precisely* 1.

This lemma is a consequence of the following proposition, which emerges from the classification theory by direct verification.

PROPOSITION 5.1. *Let* K *be doubly laced. Then there exists a linear ordering* $a_1, a_2, \cdots, a_\ell$ *of the elements of* $\mathscr{F}$ *such that* (1) a_i *is perpendicular to all roots of* $\mathscr{F}$ *which are not adjacent to it, while adjacent roots are not perpendicular;* (2) *the length of* a_i *is a nondecreasing, nonconstant function of the index* i. (See for instance [9, p. 18].)

The lemma now follows by this argument: Let $\mathscr{F}$ be ordered according to Proposition 5.1, and suppose that in this ordering $a = a_k$. By (1) in the proposition, there exists an index $j < k$ such that a_j is the last root of $\mathscr{F}$ shorter than a. Let $R_i^* \in \mathscr{W}$ denote the reflections in the root planes a_i of $\mathscr{F}$, and set $b = R_k^* \cdots R_{j+1}^* \circ a_j$. Then b is a short root, because it is conjugate to a_j. Moreover,

$$R_{j+1}^* \circ a_j = a_j - \frac{2(a_j, a_{j+1})}{(a_{j+1}, a_{j+1})} \cdot a_{j+1} .$$

Because the length of a_{j+1} is not less than the length of a_j, the Cartan integer $2(a_j, a_{j+1})/(a_{j+1}, a_{j+1})$ is -1. Thus $R^*_{j+1} \circ a_j = a_j + a_{j+1}$. By (1) and (2) in the proposition, one can now use induction to show that a is involved with coefficient 1 in b.

Proof of Theorem 2'. The circles on T are completely determined by their homology classes. Furthermore, every class of $H_1(T)$ can be realized by a circle. We shall therefore not labor with this distinction, hereafter. Let $\mathscr{F}$ be a fixed fundamental system of roots in Σ. Because K has trivial center, Σ constitutes a base for $H^1(T)$. The dual base of circles in $H_1(T)$ is denoted by $\mathscr{F}_*$. As is well known, a circle is minimal if and only if it is conjugate under $\mathscr{W}$ to an element of $\mathscr{F}_*$. We shall call circles of $\mathscr{F}_*$ *short* if they are duals of long roots in $\mathscr{F}$. Thus, if K is simply laced, all circles of $\mathscr{F}_*$ are short. In any case, *short minimal* circles always exist, by Proposition 5.1 (2). Suppose now that $s \in \mathscr{F}_*$ is dual to the long root a. We shall show that s is a generating circle. Assume first that K is simply laced. Then $a(s) = 1$, by definition. For each $w \in \mathscr{W}$, it follows that $w^* \circ a(w_* \circ s) = 1$. Since $\mathscr{W}$ is transitive on Σ, and $w_* s$ is clearly in Λ_s, we see that each root takes the value 1 on Λ_s, in other words, that s is a generating circle. If K is doubly laced, let b be a short root in Σ with $b(s) = 1$, guaranteed by the lemma. Since each root is now conjugate to a or to b under $\mathscr{W}$, s is again a generating circle.

We have therefore proved Theorem 2' in the following stronger form:

Every short minimal circle is a generating circle. In particular, if K *is simply laced, all minimal circles are generating circles.*

We close this section with a list of the generating varieties for the classical groups corresponding to short minimal circles.

The unitary groups. These groups are simply laced. Thus all minimal circles are generating circles. For $1 \leq n \leq m$, let s_{nm} be the circle on Ad SU(n + m), given by $t \to \mathrm{Ad}\, s'_{nm}(t)$, where $s'_{nm}(t)$ is the diagonal matrix with first n entries $e^{\alpha t}$ $[\alpha = 2\pi i m/(n+m)]$, and subsequent m entries $e^{-\beta t}$ $[\beta = 2\pi i n/(n+m)]$. These circles are all minimal circles, and the generating variety corresponding to s_{nm} is the Grassmannian $G_{nm} = U(n+m)/U(n) \times U(m)$. The most economical generating variety for Ad SU(n + 1) is therefore the complex projective n-space $P_n(C)$.

The orthogonal groups. The even orthogonal groups are simply laced, while the odd ones are doubly laced. For $n \geq 3$ and $m \geq 1$, let s_{nm} be the circle on Ad SO(2m + n) given by $t \to \mathrm{Ad}\, s'_{nm}(t)$, where $s'_{nm}(t)$ is the "diagonal" matrix consisting of m two-by-two diagonal boxes

$$\begin{pmatrix} \cos 2\pi i t & -\sin 2\pi i t \\ \sin 2\pi i t & \cos 2\pi i t \end{pmatrix}$$

followed by n entries equal to 1. These circles are seen to be short minimal circles. The corresponding generating varieties are the spaces $SO(2m+n)/U(m) \times SO(n)$. For the even orthogonal groups SO(2m), we have in addition the minimal circle s_{0m} which assigns to t the element $\mathrm{Ad}\, s'_{0m}(t)$, where now $s'_{0m}(t)$ consists of m two-by-two diagonal boxes

$$\begin{pmatrix} \cos \pi i t & -\sin \pi i t \\ \sin \pi i t & \cos \pi i t \end{pmatrix}.$$

The generating variety for this circle is $SO(2m)/U(m)$. The most economical generating variety for Ad $SO(n)$ $(n \geq 5)$, is therefore the real Grassmannian of oriented 2-planes in n-space.

The symplectic groups. The only short minimal circle in this case is given by

$$s(t) = \text{Ad}\,\{e^{\pi i t}\cdot \text{Identity}\}\,.$$

The corresponding generating variety is $Sp(n)/U(n)$.

6. SOME GENERALITIES

Let X be an arcwise connected space with base-point e. We write X^n for $X \times X \times \cdots \times X$ (n factors), and ε for the imbedding $X^{n-1} \to X^n$ which uses e in the last factor. The permutation group on n letters is denoted by $W(n)$. It acts on X^n by permuting the factors; one therefore has a representation of $W(n)$ on $H_*(X^n)$ and $H^*(X^n)$.

Let $\mathscr{S}^n H^*(X)$ be the subring of $H^*(X^n)$, kept pointwise fixed under $W(n)$.

Let $S^n H_*(X)$ be the quotient of $H_*(X^n)$ by the subspace generated by elements of the form $\sigma_*(u) - u$ $(\sigma \in W(n),\ u \in H_*(X_n))$. The maps ε then define a directed system of modules

$$H_*(X) \to S^2H_*(X) \to S^3H_*(X) \cdots \tag{6.1}$$

and an inverse system of rings

$$H^*(X) \leftarrow \mathscr{S}^2H^*(X) \leftarrow \mathscr{S}^3H^*(X) \cdots . \tag{6.2}$$

We define the *infinite symmetric product* of $H_*(X)$, written $SH_*(X)$, as the direct limit of (6.1).

The *infinite symmetric power* of $H^*(X)$, written $\mathscr{S}H^*(X)$, is by definition the inverse limit of (6.2).

These two objects arise naturally if one studies the maps of X into a homotopy-commutative (or homology-commutative) Hopf space. Indeed, if $f\colon X \to \Omega$ is such a map, then the multiplication on Ω induces a map $f^n\colon X^n \to \Omega$ for each positive integer n, and because Ω is commutative on the homology level, f^n_* factors through the projection $H_*(X^n) \to S^nH_*(X)$, while $(f^n)^*$ takes values in $\mathscr{S}^nH^*(X)$. In the limit, one therefore obtains homomorphisms f_{**} and f^{**} which make the following diagrams commutative:

$$\begin{array}{ccc} SH_*(X) & \xrightarrow{f_{**}} & H_*(\Omega)\,, \\ \varepsilon_*\uparrow & \nearrow_{f_*} & \\ H_*(X) & & \end{array} \tag{6.3}$$

$$\begin{array}{ccc} \mathscr{S}H^*(X) & \xleftarrow{f^{**}} & H^*(\Omega)\,. \\ \downarrow\varepsilon^* & \swarrow_{f^*} & \\ H^*(X) & & \end{array} \tag{6.4}$$

Here ε_* and ε^* are the natural injection and projection of (6.1) and (6.2) respectively. This situation is considerably simplified under the following assumption on $H_*(X)$:

(F) $H_*(X)$ is free and of finite type, and $H_q(X) = 0$ for odd q.

Since we are only interested in this case, at present, we briefly summarize the main consequences of the assumption (F). The proofs are left to the reader.

Consequences of (F)

(6.5) As a W(n) module, $H_*(X^n)$ is canonically isomorphic to

$$\{H_*(X)\}^n = H_*(X) \otimes \cdots \otimes H_*(X)$$

(n factors), where W(n) acts by permutations on this tensor product. Similarly, the algebra $H^*(X^n)$ is equal to $\{H^*(X^*)\}^n$. Hence $H_*(X) \to SH_*(X)$ is a functor in the realm of modules of type (F), and $H^*(X) \to \mathscr{S}H^*(X)$ is a functor in the realm of algebras of type (F).

In view of this fact, we shall write H_* for $H_*(X)$, and H^* for $H^*(X)$.

(6.6) The module isomorphism $H^* = \operatorname{Hom}(H_*, Z)$ induces a module isomorphism $\mathscr{S}H^* = \operatorname{Hom}(SH_*, Z)$.

(6.7) The diagonal map $\Delta: X \to X \times X$ induces a diagonal map $\Delta_*: SH_* \to SH_* \otimes SH_*$ which defines the multiplication in $\mathscr{S}H^*$.

(6.8) The canonical identifications $H_*^n \otimes H_*^m = H_*^{n+m}$ induce in the limit a multiplication

$$h_*: SH_* \otimes SH_* \to SH_*$$

which dually defines a Hopf homomorphism

$$h^*: \mathscr{S}H^* \to \mathscr{S}H^* \otimes \mathscr{S}H^*.$$

In short, both SH_* and $\mathscr{S}H^*$ are Hopf algebras, and they are natural duals. Further, if Ω is a Hopf space of the type we have been considering, and $f: X \to \Omega$ is a map, then f_{**} and f^{**} preserve the Hopf structures.

Let $i_n: H^* \to \mathscr{S}^n H^*$ be the linear map which takes 1 into 0, and, in case $\dim x > 0$, assigns to x the element $\sum \sigma^*(x \otimes 1 \otimes \cdots \otimes 1)$ $(\sigma \in W(n))$. In the limit, $\{i_n\}$ defines a linear map $i : H^* \to \mathscr{S}H^*$.

(6.9) The image of H^* under i is precisely the subspace of primitive elements in $\mathscr{S}H^*$.

(Here $x \in \mathscr{S}H^*$ is called primitive, if $h^*x = x \otimes 1 + 1 \otimes x$). We are now ready to state our main conclusion:

PROPOSITION 6.1. *Let* $f: X \to \Omega$ *be a map of an arcwise connected space* X *into the homology-commutative Hopf space* Ω. *Assume further that* $H_*(X)$ *and* $H_*(\Omega)$ *are subject to* (F), *and that* $f_*H_*(X)$ *generates the Pontryagin ring of* $H_*(\Omega)$.

Let $\mathfrak{P}^* \subset H^*(\Omega)$ *be the subspace of primitive elements in* $H^*(\Omega)$. *Then*

(a) *Under* f^{**}, $H^*(\Omega)$ *is identified with the rational closure of the subring generated by* 1 *and by* $i \circ f^* \mathfrak{P}^*$ *in* $\mathscr{P} H^*(X)$.

(b) *Under* f_{**}, $H_*(\Omega)$ *is identified with the quotient* $SH_*(X)/J$, *where* J *is the ideal of elements anihilated by* $f^{**}H^*(\Omega)$.

(Here the rational closure of a subring $A \subset \mathscr{P} H^*(X)$ is the smallest subring of $\mathscr{P} H^*(X)$ which contains A and is additively a direct summand of $\mathscr{P} H^*(X)$.)

Proposition 6.2 is proved as follows: To say that $f_* H_*(X)$ generates $H_*(\Omega)$ means precisely that $f_{**}: SH_*(X) \to H_*(\Omega)$ is onto. Because $H_*(\Omega)$ and $H_*(X)$ satisfy (F), the dual to this property is that f^{**} imbeds $H^*(\Omega)$ in $\mathscr{P} H^*(X)$ as a *direct summand.* Finally, over the rational numbers, $H^*(\Omega)$ is generated by its primitive elements [10] and by 1. Hence $f^{**}\mathfrak{P}^*$, together with 1, generates $f^{**}H^*(\Omega)$ rationally. Since f^{**} preserves the Hopf structure, $f^{**}\mathfrak{P}^* = i \circ f^* \mathfrak{P}^*$.

COROLLARY. *Under the hypotheses of Proposition* 6.1, $H^*(\Omega)$ *and* $H_*(\Omega)$ *(as rings) are algebraically determined by the ring* $H^*(X)$ *and the subspace* $f^* \mathfrak{P}^*$ *of* $H^*(X)$.

This corollary proves Theorem 3 of the Introduction.

Remark. The Steenrod operations in $H^*(\Omega)$ can also be deduced from their effect on $H^*(X)$, as is easily verified. An example may be instructive. If S_{2n+1} is the (2n + 1)-sphere, then ΩS_{2n+1}, the loop space on S_{2n+1}, is a *homology*-commutative Hopf-space, so that our remarks are still applicable. (This is not true for the even spheres.) Moreover, there exists a map f of S_{2n} into ΩS_{2n+1}, such that $f_* H_*(S_{2n})$ generates $H_*(\Omega S_{2n+1})$. Hence $f^{**}: H^*(\Omega S_{2n+1}) = \mathscr{P} H^*(S_{2n})$. (There is no choice: $f^* \mathfrak{P}^*$ must equal $H^{2n}(S_{2n})$.) Since all nontrivial Steenrod operations vanish on S_{2n}, the same is true in $H^*(\Omega S_{2n+1})$. To verify that $\mathscr{P} H^*(S_{2n})$ is indeed the divided polynomial ring that $H^*(\Omega S_{2n+1})$ is known to be, one argues as follows. In general, under the condition (F), the map $\varepsilon^*: \mathscr{P}^n H^*(X) \to \mathscr{P}^{n-1} H^*(X)$ is easily seen to be bijective in dimensions at most n - 1. Hence, in this range of dimensions, $\mathscr{P} H^*(X)$ can be replaced by $\mathscr{P}^n H^*(X)$.

In the special case under consideration, one sees analogously that

$$\mathscr{P}^k H^*(S_{2n}) = \mathscr{P} H^*(S_{2n}),$$

in dimensions at most 2nk. Hence, if x generates $H^{2n}(S_{2n})$, the obvious formula

$$(x \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes x)^k = k!\, x \otimes x \otimes \cdots \otimes x$$

in $\{H^*(S_{2n})\}^k$ implies that the kth power of the generator in dimension 2n of $\mathscr{P} H^*(S_{2n})$ is divisible by precisely k!.

7. SUSPENSION OF THE GENERATING MAP

If L is an acyclic space fibered by F over X, then the contraction of the singular complex of F to a point in L defines a suspension homomorphism $L_*: H_*(F) \to H_*(X)$ in homology. Its dual $L^*: H^*(X) \to H^*(F)$ decreases dimensions by one.

If X is an odd sphere S_{2n+1} $(n \geq 0)$, L is the space of paths starting at $e \in S_{2n+1}$, and $\pi: L \to X$ assigns to each path its endpoint, then we obtain the Serre fiber space with fiber Ω, the space of loops over S_{2n+1}. In this case, the image of L^* coincides precisely with the primitive elements of $H^*(\Omega)$ [15].

Over Q, a 1-connected compact Lie group K looks just like a product of odd spheres of dimension at least 3. Hence, in the Serre fiber-space over K, with ΩK as fiber, the suspension again maps $H^*(K: Q)$ onto the primitive elements of $H^*(\Omega K; Q)$.

Consider next the universal bundle for $K: K \to M \to B_K$, where M is an acyclic space on which K acts principally on the right, and $B_K = M/K$ is the universal base-space of K. According to A. Borel [1], the suspension in this case maps $H^*(B_K; Q)$ onto a set of primitive generators for $H^*(K; Q)$. Since the suspension takes decomposable elements into 0, the following proposition becomes evident.

PROPOSITION 7.1. *If* K *is a* 1*-connected compact Lie group, and*

$$\mathfrak{P}^*_Q \subset H^*(\Omega K; Q)$$

is the subspace of primitive elements in $H^*(\Omega K; Q)$, *then* $\mathfrak{P}^*_Q$ *is precisely the image of* $L^* \circ M^*$, *where* M^* *denotes the suspension from* B_K *to* K, *and* L^* *denotes the suspension from* K *to* ΩK.

Suppose now that K is 1-connected, and that s is an arbitrary circle on K. We let $f: K^s \to \Omega K$ be defined by

$$f(q)(t) = ks(t)k^{-1} \qquad (q = kK_s,\ 0 \leq t \leq 1), \tag{7.1}$$

and we propose to compute $f^* \circ L^* \circ M^*$.

In general, if $h: P \to F$ maps a polyhedron into the fiber of an acyclic fiber space $F \to L \xrightarrow{\pi} X$, the composition $h^* \circ L^*$ can be computed as follows: Let $i: F \to L$ be the injection of F into a particular fiber, and let w be a point of i(F). Set $H = i \circ f$. Because L is acyclic, $H: P \to L$ is homotopic to the constant map of P into w. Let $H_u: P \to L$ describe this homotopy $(0 \leq u \leq 1,\ H_0 = H,\ H_1 = \text{constant map})$. Let $Eh: P \times S^1 \to X$ be given by the formula

$$Eh(p, t) = \pi\{H_t(p)\} \qquad (p \in P,\ 0 \leq t \leq 1).$$

(Here we have identified S^1 with the space obtained from the unit interval by identifying 0 with 1. Because $w \in i(F)$, $Eh(p, 0) = Eh(p, 1)$ for all $p \in P$, so that Eh is well-defined.)

The map Eh is called a suspension for h. Next, let $x \in H^1(S^1)$ be an appropriate generator. In the canonical decomposition $H^*(P \times S^1) = H^*(P) \otimes H^*(S^1)$, each element $u \in H^*(P \times S^1)$ can be written uniquely in the form $u_1 \otimes 1 + u_2 \otimes x$ $(u_i \in H^*(P);\ i = 1, 2)$. We call the element u_2 the x-component of u. With this understood, it is clear from [15] that $h^* \circ L^* v$ is the x-component of $(Eh)^* v$, for each $v \in H^*(X)$.

We apply this procedure twice to the map f [see (7.1)]. A first suspension of f (in $\Omega K \to L \to K$) is easily determined: $Ef: K^s \times S^1 \to K$ is given by

$$Ef(q, t) = ks(t)k^{-1} \qquad (q = kK_s,\ 0 \leq t \leq 1). \tag{7.2}$$

Next, let i be the injection of K into M. If we write the operation of K on M as $m \cdot k$ $(m \in M,\ k \in K)$, then i is given by $i(k) = w \cdot k$, where w is a point of M. Let $F = i \circ f$, and let F_u be a homotopy of F into the constant map $K \times S^1 \to w$. Then $E^2 f: K^s \times S^1 \times S^1 \to B_K$ is given by

(7.3) $$E^2f(q;\, t_1,\, t_2) = \pi\{F_{t_2}(q,\, t_1)\} \qquad (q \in K^s,\; 0 \leq t_i \leq 1;\; i = 1,\, 2),$$

where π denotes the projection $M \to B_K$.

From the foregoing it is now clear that if x and x' are appropriate generators for $H^1(S^1)$, then

(7.4) $$f^* \circ L^* \circ M^* \quad \textit{is the } x \otimes x' \textit{ component of } (E^2 f)^*.$$

To compute $(E^2f)^*$, we need an auxiliary map λ, which is defined below.

Let T be a maximal torus of K containing s. We write K^T for K/T. There is a natural map $\tau\colon K^T \to K^s$. Consider next the Borel fiber spaces

(7.5) $$T \to M \xrightarrow{\pi'} B_T$$

and

(7.6) $$K^T \to B_T \xrightarrow{\rho} B_K.$$

The first is obtained by decomposing M according to its T-orbits. Thus $M/T = B_T$ is a universal base space for T. The second fibering is obtained from the natural map $\rho\colon M/T \to M/K$.

The map λ will take $K^T \times S^1 \times S^1$ into B_T according to the law

(7.7) $$\lambda(q,\, t_1,\, t_2) = \pi'\{F_{t_2}[\tau(q),\, t_1] \circ k\} \qquad (q = kT,\; 0 \leq t_i \leq 1;\; i = 1,\, 2).$$

This map is well-defined: Because π' is insensitive to right translations by T, λ depends only on q, and not on $k \in K$. We see that $\lambda(q, 0, t_2) = \lambda(q, 1, t_2)$ because $F_{t_2}(\tau(q), 0) = F_{t_2}(\tau(q), 1)$ for all $0 \leq t_2 \leq 1$. Finally,

$$\lambda(q,\, t_1,\, 0) = \pi'\{w \circ k\, s(1)\, k^{-1} \circ k\} = \pi'(w \circ k),$$

while $\lambda(q, t_1, 1)$ is given by $\pi'(w \circ k)$. Hence $\lambda(q, t_1, 0) = \lambda(q, t_1, 1)$.

Let $\alpha\colon K^T \to K^T \times S^1 \times S^1$ take $q \in K^T$ into $(q, 0, 0)$, and let

$$\beta\colon S^1 \times S^1 \to K^T \times (S^1 \times S^1)$$

be the injection into the second factor which uses the identity coset $q_0 = T$ on the first factor.

LEMMA. *The map* $\lambda\colon K^T \times S^1 \times S^1 \to B_T$, *just defined, has the following properties:*

(7.8) $$\lambda \circ \alpha = j;$$

(7.9) $\lambda \circ \beta$ *is a suspension in* (7.5) *for the map* $S^1 \to T$ *given by* $t \to s(t)$ $(0 \leq t \leq 1)$;

(7.10) *the following diagram (where* $\tau_1 = \tau \times \mathit{Identity}$*) is commutative:*

$$\begin{array}{ccc} K^T \times S^1 \times S^1 & \xrightarrow{\lambda} & B_T . \\ \downarrow \tau_1 & & \downarrow \rho \\ K^S \times S^1 \times S^1 & \xrightarrow{E^2 f} & B_K \end{array}$$

Proof. We have already seen that $\lambda(q, t_1, 0) = \pi (w \circ k)$. Hence $q \to \lambda(q, 0, 0)$ injects K^T into B_T as a fiber over B_K. This proves (7.8). Next,

$$\lambda(q_0, t_1, t_2) = \pi' \{F_{t_2}(q_0, t_1)\} .$$

By the definition of F_t, $\lambda(q_0, t_1, 0) = \pi'\{w \circ s(t_1)\}$ $(0 \leq t_1 \leq 1)$, while

$$\lambda(q_0, t_1, 1) = \pi'\{w\} .$$

Hence $\lambda \circ \beta$ is a suspension of the map $t \to s(t)$ $(0 \leq t \leq 1)$. This proves (7.9). Finally, because $\pi = \rho \circ \pi'$, and π is not affected by right translations of M by K, the diagram (7.10) is commutative.

The homomorphism τ^* is injective (in the fibering K^T over K^s, base and fiber have only even-dimensional homology), hence τ_1^* is also injective. It will therefore be sufficient to compute $\tau_1^* \circ (E f)^*$. By (7.10), this homomorphism is equal to $\lambda^* \circ \rho^*$. By (7.8) and (7.9), λ^* is easily written down for a 2-dimensional element of $H^*(B_T)$:

(7.11) $$\lambda^* v = j^* v \otimes 1 \otimes 1 + \langle M^1_* s, v\rangle \otimes x \otimes x' .$$

Here s is the homology class of the circle s in $H_1(T)$, M^1_* denotes suspension in the fibering (7.5), v is in $H^2(B_T)$, $\langle , \rangle$ stands for the inner product, and x, x' are again appropriate generators of $H^1(S^1)$.

The great virtue of the fibering (7.5) is its simplicity: the suspension M^1_* is a bijection of $H_1(T)$ onto $H_2(B_T)$, and B_T is the product of Eilenberg-Mac Lane spaces K(Z, 2). Hence M^1_* defines an identification of $H^*(B_T)$ with the symmetric algebra over $H^1(T)$; that is, if $x_1, \cdots, x_\ell$ are a base for $H^1(T)$, then M^1_* identifies $H^*(B_T)$ with $Z[x_1, \cdots, x_\ell]$). In particular, $H^*(B_T)$ is generated by 2-dimensional elements, so that (7.11) determines λ^* on all of $H^*(B_T)$. Explicitly: let $\theta_s' : H^*(B_T) \to H^*(B_T)$ be the derivation which extends the homomorphism $v \to \langle M^1_* s, v\rangle$ $[v \in H^2(B_T)]$. Then, because $1 \otimes x \otimes x'$ has square zero, λ^* is given by

(7.12) $$\lambda^* v = j^* v \otimes 1 \otimes 1 + j^* \theta_s' v \otimes x \otimes x' \qquad (v \in H^*(B_T)) .$$

We have proved the following proposition.

PROPOSITION 7.2. *In the notation already introduced,*

(7.13) $$\tau^* \circ f^* \circ L^* \circ M^* = \pm\, j^* \circ \theta_s \circ \rho^*$$

(I would rather not commit myself on the sign; but the + sign seems likely.)

To proceed further, we need to review the fundamental results of A. Borel [1] and J. Leray [12] concerning ρ^* and j^*:

(7.14) *under* ρ^*, $H^*(B_K; Q)$ *is mapped isomorphically onto the invariants of* $\mathcal{W}(K, T)$ *in* $H^*(B_T; Q)$;

(7.15) *the kernel of* j^* *is the ideal generated by the invariants of* $\mathcal{W}(K, T)$ *in* $H^*(B_T; Q)$ *of positive degree, and* j^* *is onto;*

(7.16) *the* j^**-image of the invariants under* $\mathcal{W}(K_s, T)$ *in* $H^*(B_T; Q)$ *is the image of* $\tau^*: H^*(K^s; Q) \to H^*(K^T; Q)$.

Borel has further shown that

(7.17) *if* K *has no torsion, the propositions* (7.14) *to* (7.16) *hold over the integers* Z *as well.*

Here the operation of $\mathcal{W}(K, T)$ and $\mathcal{W}(K_s, T)$ on $H^*(B_T)$ is the one obtained from the natural operation of these groups on $H^1(T; Q)$ (and therefore on the symmetric algebra over $H^1(T; Q)$) through the suspension M_*. Hence (7.14) to (7.16) imply the following description of $H^*(K^s; Q)$ which, in the last analysis, depends entirely on the Cartan integers of K and on the position of s in T.

Let A_K *be the symmetric algebra over* $H'(T; Q)$, *and let* I_K *and* I_{K_s} *be the invariants of* A_K *under* $\mathcal{W}(K, T)$ *and* $\mathcal{W}(K_s, T)$, *respectively. Let* J_K *be the ideal (in* A_K*) generated by the elements of positive degree in* I_K. *Then* $H^*(K^s; Q)$ *can be identified with* I_{K_s}/J_K. *We call this the Borel description of* $H^*(K^s; Q)$.

If $s \in H_1(T)$, then the homomorphism $s: H^1(T) \to Q$ extends uniquely to a derivation $\theta_s: A_K \to A_K$. The composition $I_K \xrightarrow{\theta_s} I_{K_s} \to I_{K_s}/J_K$ is denoted by θ_s^*. We use this notation in Theorem 4 below, which in view of Propositions 7.1 and 7.2, and the formula (7.14) to (7.17) is now evident.

THEOREM 4. *Let* s *be a circle on the* 1*-connected compact Lie group* K, *and let* $f: K^s \to K$ *be given by*

$$f(q)(t) = ks(t)k^{-1} \qquad (q = kK_s;\ 0 \leq t \leq 1). \tag{7.18}$$

Let $\mathfrak{P}^*$ *denote the primitive elements in* $H^*(\Omega K)$. *Then, in the Borel description of* $H^*(K^s; Q)$, *the subspace* $f^*P^* \otimes_Z Q$ *is the image of* I_K *under* $\theta_s^*: I_K \to I_{K_s}/J_K$. *If* K *has no torsion, then this formula is valid over the integers.*

Our program is now nearly completed. What is still needed is a remark concerning the case when K is not 1-connected.

Let then K be only 0-connected and semisimple, and let s be a circle in the maximal torus T of K. Also, let K' be the universal covering group of K, and denote the pre-images of s and T in K' by s' and T'. Let $g^s: K^s \to \Omega' K$ be given by (1.5) in the Introduction. Let $f: K'^s \to \Omega K'$ be given by the analogue of (7.18). As is well known, $K'^s = K^s$. Further, $\Omega K'$ and $\Omega' K$ are of the same homotopy type. It therefore makes sense to compare f^* with the homomorphism induced by g^s, which we write g_s^*.

PROPOSITION 7.1. *Let* $\mathfrak{P}^*$ *denote the primitive elements of* $H^*(\Omega' K)$. *Then, in the situation just described,*

$$f^*\mathfrak{P}^* = [s'/s] \cdot g_s^* \mathfrak{P}^*,$$

where $[s'/s]$ *denotes the degree of the covering map* $s' \to s$.

Proof. Let $X = \Omega'K \times \Omega'K \times \cdots \times \Omega'K$ ([s'/s] factors), and let $\Delta\colon \Omega'K \to X$ be the diagonal map $q \to (q, q, \cdots, q)$. Also, let $u\colon X \to \Omega'K$ be the map given by the Pontryagin product. It is easily seen that $u \circ \Delta \circ g^s \approx f$. For primitive elements, this decomposition of f immediately yields the formula of the proposition.

8. THE STABLE HOMOTOPY OF THE UNITARY GROUP

From the fibering $U(n) \to U(n+1) \to S_{2n+1}$ it follows easily that $\pi_k\{U(n)\}$ is independent of n, for $n > 2k$. These stable values of $\pi_k\{U(n)\}$ will be denoted by $\pi_k(U)$. As our first application of the preceding theory, we prove the following theorem.

THEOREM 5. *The stable homotopy of the unitary groups is of period 2:*

$$\pi_k(U) = \pi_{k+2}(U) \qquad (k = 0, 1, \cdots). \tag{8.1}$$

The first nonstable group is given by

$$\pi_{2n}\{U(n)\} = Z/n!Z \qquad (n \geq 1). \tag{8.2}$$

The formula (8.1) was already announced in [5]. With the aid of that relation, the Borel-Hirzebruch divisibility theorems [2] imply (8.2), except for the prime 2. The proof of (8.2) was sketched in [6].

The proof of (8.1) to be presented is rather round-about. However, the more direct proof which I had in mind in [5], and which will be described at another time, involves more of the Morse theory. In view of the good use Kervaire and Milnor have made of (8.2), it therefore seems worthwhile to include a proof entirely in the context of this paper. The first step is to construct a model for $H^*\{\Omega SU(n+1)\}$.

PROPOSITION 8.1. *Let* Ω *denote the loopspace* $\Omega SU(n+1)$ $(n \geq 1)$, *and let* $P_n(C)$ *denote the complex projective* n*-space. Then*

$$H^*(\Omega) \approx \mathscr{S} H^*(P_n(C)). \tag{8.3}$$

The Hopf algebra $H_*(\Omega)$ *is a polynomial ring* $Z[\sigma_1, \sigma_2, \cdots, \sigma_n]$ *with generators* σ_i (*dim* $\sigma_i = 2i$; $i = 1, 2, \cdots, n$) *and with diagonal map*

$$\Delta_* \sigma_i = \sum \sigma_\alpha \times \sigma_\beta \qquad (\alpha + \beta = i,\ \sigma_0 = 1). \tag{8.4}$$

Proof. Let $g\colon P_n(C) \to \Omega$ be the generating map corresponding to the circle s_{n1} of Section 5. We need to show that the induced homomorphism

$$g^{**}\colon H^*(\Omega) \to \mathscr{S} H^*\{P_n(C)\}$$

is onto. (It is injective, because g is a generating map.) By Proposition 6.1, it is sufficient to show that under g^* the primitive subspace of $H^*(\Omega)$, say $\mathfrak{P}^*$, maps onto $H^+\{P_n(C)\} = \sum_{i>0} H^i\{P_n(C)\}$. Over the rationals, $SU(n+1)$ looks like the product $S^3 \times S^5 \times \cdots \times S^{2n+1}$. Hence $H^*(\Omega)$ has primitive elements in all even dimensions greater than 0 and less than or equal to 2n. The ring $H^+\{P_n(C)\}$ has precisely a free module of rank 1 in each of these dimensions. Hence, since $g^* \mathfrak{P}^*$ is in any case a direct summand of $H^*\{P_n(C)\}$, it follows that $g^* \mathfrak{P}^* = H^+\{P_n(C)\}$. This proves (8.3).

Now let $u \in H^2\{P_n(C)\}$ be a generator. Then $H^*\{P_n(C)\} = Z[u]/\{u^{n+1}\}$, where $\{u^{n+1}\}$ denotes the ideal generated by u^{n+1}. Hence, if σ_i is the g_*-image of the dual to u^i, then, together with 1, the σ_i $(i = 1, \cdots, n)$ generate $H_*(\Omega)$, and (8.4) must be the diagonal map. Finally, $g_{**}: SH_*\{P_n(C)\} \to H_*(\Omega)$ is injective, because g^{**} is onto. There are therefore no relations between the σ_i, other than commutativity; that is, $H_*(\Omega) = Z[\sigma_1, \cdots, \sigma_n]$.

COROLLARY. *Let* $S_1^n(u)$ *denote the graded ring of symmetric functions in* n *variables* $u_1, \cdots, u_n$ $(\dim u_i = 2)$. *Then, for dimensions at most* 2n, *there is a ring isomorphism* ϕ *of* $H^*(\Omega)$ *onto* $S_1^n(u)$.

From the relation $H^*\{P_n(C)\} = Z[u]/\{u^{n+1}\}$, one concludes without difficulty that $\mathscr{S}^n H^*\{P_n(C)\} = S_1^n(u)$ for dimensions at most 2n. However in this same range of dimensions the projection $\mathscr{S} H^*(P_n(C)) \to \mathscr{S}^n H^*\{P_n(C)\}$ is bijective. Hence (8.1) implies the corollary. Further:

(8.5) *Under* ϕ, *the primitive subspace* $\mathfrak{P}^*$ *goes onto the module generated additively by the power sums* $\Sigma_k(u) = u_1^k + u_2^k + \cdots + u_n^k$ $(k = 1, 2, \cdots, n)$.

Proof. The primitive subspace of $\mathfrak{P}H^*\{P_n(C)\}$ is the image of the map $i: H^*\{P_n(C)\} \to \mathscr{S}H^*\{P_n(C)\}$. Since $i(u^k)$ is clearly represented by $\Sigma_k(u)$ in $S_1^n(u)$, (8.5) is evident.

Remark. The universal base-space of $U(n+1)$, denoted by $B_{U(n+1)}$, is known to have a cohomology ring which, in dimensions at most 2n, is identical with the one just found for $\Omega SU(n+1)$. This fact strongly suggests the periodicity of $\pi(U)$.

The next proposition is the essential step in the proof of (8.1).

PROPOSITION 8.2. *Let* $f: G_{mm} \to \Omega SU(2m)$ *be the generating map of the Grassmannian* $U(2m)/U(m) \times U(m)$ *into* $\Omega SU(2m)$ *induced by the generating circle* s_{mm} *of Section 5. Then* $f^*: H^*(\Omega SU(2m)) \to H^*(G_{mm})$ *is onto.*

Proof. We again let $\mathfrak{P}^*$ denote the primitive elements in $H^*(\Omega)$, where now $\Omega = \Omega SU(2m)$. We first determine the image of $\mathfrak{P}^*$ under f^* in the Borel description of $H^*(G_{mm})$. *Because* $U(2m)$ *has no torsion, this description is valid over the integers.*

Let $\tilde{s}: R \to SU(2m)$ be the circle which assigns to t the diagonal matrix with first m entries $e^{2\pi i t}$, and last m entries $e^{-2\pi i t}$. Then $\{SU(2m)\}^{\tilde{s}} = G_{mm}$, and we let $\tilde{f}: G_{mm} \to \Omega SU(2m)$ be the map [see (7.1)] determined by this circle. Because under Ad, $\tilde{s}$ maps onto the generating circle s_{mm} with degree 2, Proposition 7.1 implies

$$\tilde{f}^*\mathfrak{P}^* = 2f^*\mathfrak{P}^*. \tag{8.6}$$

The homomorphism $\tilde{f}^*$ is computed with the aid of Theorem 4. For this purpose, let T be the group of diagonal matrixes in $SU(2m)$. Let $x_i \in H^1(T)$ be the class which corresponds to the character

$$t \to i\text{th coordinate of } t,$$

under the canonical isomorphism $\mathrm{Hom}(T, S^1) \approx H^1(T^1)$. The $x_1, \cdots, x_{2m}$ span $H^1(T)$, but are not independent, since $x_1 + \cdots + x_{2m} = 0$. The roots of $SU(2m)$ are given by $x_i - x_j$, and the Weyl group is just the full permutation group on the x_i.

It follows that $H^*\{B_{SU(2m)}\}$ is given by

(8.7) $$H^*\{B_{SU(2m)}\} = S_1^{2m}(x)/\{x_1 + \cdots + x_{2m}\},$$

where $S_1^{2m}(x)$ denotes the symmetric polynomials in the variables $x_1, \cdots, x_{2m}$, and where $\{x_1, \cdots, x_{2m}\}$ is the ideal generated by $x_1 + \cdots + x_{2m}$. Let $I_1^{2m}(x)$ be the ideal generated in $Z[x_1, \cdots, x_{2m}]$ by the elements of positive degree in $S_1^{2m}(x)$. Then the Borel description of $H^*(G_{mm})$ in these same variables is

(8.8) $$H^*(G_{mm}) = S_1^m(x) \otimes S_{m+1}^{2m}(x)/I_1^{2m}(x).$$

This is apparent, because the subgroup of the Weyl group which keeps s_{mm} fixed is precisely the subgroup of permutations which preserves the first, and last, m variables.

Let $\rho: S_1^m(x) \to H^*(G_{mm})$ be defined by $\rho = \text{Identity} \otimes 1$. It is then well known that

(8.8) *the homomorphism* ρ *is onto, and it is bijective in dimensions up to* 2m.

LEMMA 8.1. *The subspace* $\bar{f}^*\mathfrak{P}^*$ *is spanned by the elements* $\{2\rho \circ \Sigma_k(x)\}$ $(k = 1, \cdots, 2m - 1)$ *where* $\Sigma_k(x)$ *is the power sum* $x_1^k + \cdots + x_m^k$ *in* $S_1^m(x)$.

Proof. By Theorem 4, $\bar{f}^*\mathfrak{P}^* = \theta^* H^*\{B_{SU(2m)}\}$, where θ^* is the derivation corresponding to $\bar{s}_{mm}$. In terms of the $\{x^i\}$, the derivation θ^* is clearly represented by $\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_m} - \frac{\partial}{\partial x_{m+1}} - \cdots - \frac{\partial}{\partial x_{2m}}$. Hence

$$\begin{aligned}\theta^* \circ (x_1^{k+1} + \cdots + x_{2m}^{k+1}) &= (k+1)[\{x_1^k + \cdots + x_m^k\} \otimes 1 - 1 \otimes \{x_{m+1}^k + \cdots + x_{2m}^k\}] \\ &= 2(k+1)\rho \cdot \Sigma_k(x)\end{aligned}$$

in $H^*(G_{mm})$. The $(k + 1)$st symmetric function c_{k+1}, in $S_1^{2m}(x)$ is congruent to $(x_1^k + \cdots + x_{2m}^k)/(k + 1)$ modulo decomposable elements of $S_1^{2m}(x)$. Because θ^* anihilates decomposable elements, we therefore have $\theta^* \circ c_{k+1} = 2\rho \circ \Sigma_k(x)$. Finally $S_1^{2m}(x)$ is generated by the c_i. This proves the lemma.

LEMMA 8.2. *In dimensions up to* 2m, *the homomorphism*

$$\chi = \rho^{-1} \circ f^* \circ \phi^{-1} : S_1^{2m-1}(u) \to S_1^m(x)$$

is well-defined. For $k = 1, \cdots, m$, χ *maps the power sum* $\Sigma_k(u)$ *onto* ± 1 *times the power sum* $\Sigma_k(x)$.

Proof. The homomorphism χ is well-defined in this range of dimensions, because of the corollary to Proposition 8.1 and (8.8). The rest follows from (8.5), (8.6) and Lemma 8.1.

LEMMA 8.3. *In dimensions up to* 2m, *the homomorphism* χ *takes* $S_1^{2m-1}(u)$ *onto* $S_1^m(x)$.

Proof. Let $c_i(u)$ denote the ith elementary symmetric function in $S_1^{2m-1}(u)$. Then $S_1^{2m-1}(u) \approx Z[c_1(u), \cdots, c_{2m-1}(u)]$, and similarly $S_1^m(x) \approx Z[c_1(x), \cdots, c_m(x)]$.

The well-known Newton formulas express $\Sigma_i(x)$ in terms of the elementary symmetric functions

$$\Sigma_n(x) = \pm n c_n(x) + r_n(x) \qquad (n = 1, \cdots, m),$$

where $r_n(x)$ is a polynomial in $c_1(x), \cdots, c_{n-1}(x)$. Similarly,

$$\Sigma_i(u) = \pm ic_i(u) + r_i(u) \qquad (i = 1, \cdots, 2m - 1).$$

These two relations, together with the condition that $\chi \circ \Sigma_k(u) = \pm \Sigma_k(x)$, imply that $\chi \circ c_k(u) = \pm c_k(x) + Q_k(x)$, where $Q(x)$ is a polynomial in $c_{k-1}(x), \cdots, c_1(x)$. Clearly, $\chi \circ c_1(u) = \pm c_1(x)$. Hence, by induction, $c_i(x)$ is in the image of χ for all i $(1 \leq i \leq m)$.

Proposition 8.2 follows as a corollary. Indeed, we have shown that $f^*H^*(\Omega)$ contains $\rho \circ c_i(x)$ $(i = 1, \cdots, 2m)$. Since the $c_i(x)$ generate $S_1^m(x)$, and ρ is onto, f^* must be onto.

PROPOSITION 8.3. *The map* $f\colon G_{mm} \to \Omega SU(2m)$ *induces an isomorphism in homology, up to dimension* $(2m + 1)$.

Proof. The Poincaré series for $\Omega SU(2m)$ is clearly

$$(1 - t^2)^{-1}(1 - t^4)^{-1} \cdots (1 - t^{4m-2})^{-1}.$$

By the Hirsch formula, [1], the Poincaré series for G_{mm} is

$$(1 - t^2)(1 - t^4) \cdots (1 - t^{4m}) \cdot \{(1 - t^2)^{-2} \cdots (1 - t^{2m})^{-2}\}.$$

Hence the spaces in question have equal Betti numbers, up to dimension $2m + 1$. Since they are both free of torsion, the proposition follows from Proposition 8.2.

We are now in a position to prove the formula (8.1). The two spaces G_{mm} and $\Omega\, SU(2m)$ are simply connected. Hence f induces a homotopy equivalence up to dimension $2m$. Thus $f_*\colon \pi_k(G_{mm}) \to \pi_k(\Omega)$ is bijective for $k \leq 2m$. Combining this with the well-known facts that

$$\pi_k(G_{mm}) = \pi_{n-1}\{U(m)\} \quad (1 \leq k \leq 2m) \qquad \text{and} \qquad \pi_k(\Omega) = \pi_{k+1}\{SU(2m)\} \quad (1 \leq k),$$

we obtain the isomorphism $\pi_k\{U(m)\} = \pi_{k+2}\{SU(2m)\}$, which is valid for $k < 2m$. Taking m large, we obtain (8.1).

To prove the second part of Theorem 5, we first determine the spherical cycles in $\Omega SU(n + 1)$. The following proposition will be needed.

PROPOSITION 8.3. *The subspace* $\mathfrak{P}_*$ *of primitive elements in* $H_*\{\Omega SU(n + 1)\}$ *is spanned by elements* $\{p_i\}$ $(i = 1, \cdots, n)$ *which are inductively determined according to the formula*

$$p_k - p_{k-1} \cdot \sigma_1 + p_{k-2} \cdot \sigma_2 - \cdots \pm k\sigma_k = 0 \qquad (k = 1, 2, \cdots, n). \tag{8.10}$$

(*Here the* σ_i *are the generators employed in Proposition* 8.1.)

Note that the relation between the p_i and σ_i is precisely the Newton relation between the "power sums" and the elementary symmetric functions alluded to earlier. This suggests the following proof. Let $S_n^{(m)}$ be the subring of $Z[x_1, \cdots, x_m]$ $(m \geq n)$ generated by the first n symmetric functions $c_i^{(m)} = \sum x_1 \cdots x_i$ $(i = 1, \cdots, n)$. Then $S_n^{(m)} = Z[c_1^{(m)}, \cdots, c_n^{(m)}]$, as is well known, so that the assignment $\sigma_i \to c_i^{(m)}$ identifies each $S_n^{(m)}$ $(m \geq n)$ with $Z[\sigma_1, \cdots, \sigma_n]$. Now let

$$\phi\colon Z[x_1, \cdots, x_{2n}] \to Z[x_1, \cdots, x_n] \otimes Z[x_1, \cdots, x_n]$$

be given by $\phi(x_i) = x_i \times 1$ and $\phi(x_{n+i}) = 1 \otimes x_i$, for $1 \leq i \leq n$. Then ϕ induces a homomorphism of $S_n^{(2n)}$ into $S_n^{(n)} \otimes S_n^{(n)}$ which, in view of our identification of $S_n^{(2n)}$ and $S_n^{(n)}$ with $Z[\sigma_1, \cdots, \sigma_n]$, is equivalent to (8.2). But now it is clear that each power sum $\sum x_j^i$ corresponds to a primitive element which is not divisible. This proves the proposition.

The following fact will also be needed in the sequel (we have set $\Omega_n = \Omega SU(n+1)$):

PROPOSITION 8.4. *Let* i: $\Omega_n \to \Omega_{n+1}$ *be induced by the standard inclusion* $SU(n) \subset SU(n+1)$ $(n \geq 2)$. *Then, in the basis of* (8.4), *the image of*

$$i_*: H_*(\Omega_n) \to H_*(\Omega_{n+1})$$

is the subring $Z[\sigma_1, \cdots, \sigma_{n-1}]$ *of* $Z[\sigma_1, \cdots, \sigma_n]$.

Proof. The map i can be considered as the fiber inclusion in the fibering of Ω_{n+1} over ΩS^{2n-1}. Because the degrees in both fiber and base space are even, i_* is injective. But then the image of i_* must be $Z[\sigma_1, \cdots, \sigma_{n-1}]$.

THEOREM 6. *Let* $\Omega = \Omega SU(n+1)$, *and let* $\{p_k\}$ *be the generators of* $\mathfrak{P}_*$ *as given by* (8.10). *Then the image of* $\pi_{2k}(\Omega)$ *in* $H_*(\Omega)$ *is generated by* $(k-1)!\, p_k$ $(k = 1, \cdots, n)$.

Because $\pi_{2k}(\Omega)$ is in the stable range, and by virtue of Proposition 8.4, we can choose n arbitrarily large to prove the theorem for a fixed k. In the sequel, we choose m sufficiently large relative to k, and set $n + 1 = 2m$. As before, $f: G_{mm} \to \Omega$ denotes the generating map, and we write B for $B_{U(2m)}$. The inclusion $U(m) \subset U(2m)$ defines a map $i: G_{mm} \to B$ which is a homotopy equivalence for dimensions up to 2m. The inclusion $SU(2m) \subset U(2m)$ induces a map $j: B_{SU(2m)} \to B$. Now let $S: \pi_r(\Omega) \to \pi_{k+2}\{B_{SU(2m)}\}$ be the double suspension in homotopy. Because we are suspending through acyclic fiber spaces, S is bijective. Let $\lambda = f_* \circ i_*^{-1} \circ j_* \circ S$. Then $\lambda: \pi_k(\Omega) \to \pi_{k+2}(\Omega)$ is defined for dimensions smaller than 2m, and it is bijective for $1 \leq r \leq 2m$. Thus

$$\pi_{2k}(\Omega) = \lambda^{k-1} \circ \pi_2(\Omega). \tag{8.11}$$

We now let $j_* \circ S_*$ be the double suspension in homology from $H_*(\Omega)$ to $H_*(B)$, and we set $\lambda_* = f_* \circ i_*^{-1} \circ j_* \circ S_*$, in the same range of dimensions.

It is clear that if $\mathfrak{h}$ denotes the Hurewicz homomorphism $\mathfrak{h}: \pi_k(X) \to H_k(X)$, then we have the commutativity relation

$$\lambda_* \circ \mathfrak{h} = \mathfrak{h} \circ \lambda. \tag{8.12}$$

Hence (8.11) implies that

$$\mathfrak{h}\, \pi_{2k}(\Omega) = \lambda_*^{k-1} \circ H_2(\Omega). \tag{8.13}$$

The homomorphism λ_* has the following three properties, in the dimensions under consideration:

(a) λ_* *anihilates decomposable elements;*

(b) λ_* *preserves the primitive subspace* $\mathfrak{P}_*$;

(c) *the image of* λ_* *is not divisible in dimensions greater than* 0.

The first property holds because λ_* involves a suspension (see [13]). Next, it is clear that λ_* preserves spherical classes, by (8.12). From general theory, such classes are always in $\mathfrak{P}_*$. Further, because SU(2m) looks like a product of odd spheres over the rationals, $\mathfrak{P}_*$ must in fact be generated rationally by the spherical classes. This proves (b). Finally, in the proof of Lemma 8.2, we have computed the double suspension composed with f* in cohomology, and found that its image is not divisible. Thus (c) is established. An immediate consequence of (a) and (8.10) is that

$$\lambda_* p_r = \pm r \lambda_* \sigma_r . \tag{8.14}$$

Now, by (b) and (c), it follows that $\lambda_* \sigma_r = \pm p_{r+1}$. Hence $\lambda_*^{k-1} \circ p_1 = \pm(k-1)!\, p_k$. This proves the theorem.

The formula (8.2) is an easy consequence of Theorem 6: Let $\Omega_n \xrightarrow{i} \Omega_{n+1} \xrightarrow{j} \Omega S_{2n+1}$ be the fibering induced by the fibering $SU(n+1)/SU(n) = S_{2n+1}$. (Again, we have set $\Omega_n = \Omega SU(n)$.) By Proposition 8.2, $j_* p_r = n \cdot j_* \sigma_n$, and $j_* \sigma_n$ generates $H_{2n}(\Omega S_{2n+1})$. The spherical classes in $H_{2n}(\Omega_{n+1})$ are generated by $(n-1)!\, p_n$, by Theorem 6; hence their image under j_* in $H_{2n}(\Omega S_{2n+1})$ is generated by $n!\, j_* \sigma_n$. It follows by the Hurewicz isomorphism that the index of $j_* \pi_{2r}(\Omega_{n+1})$ in $\pi_{2n}(\Omega S_{2n+1})$ is n!. Now $\pi_{2n-1}(\Omega_{n+1}) = 0$, by (8.1). Hence, by the exact sequence of the fibering, $\pi_{2n-1}(\Omega_n) = Z/n!Z$. But $\pi_{2n-1}(\Omega_n) = \pi_{2n}(U(n))$.

Theorem 6 also has the following corollary, which is evident, in view of Proposition 8.3.

COROLLARY. *The spherical classes of dimension* 2n, *in* $B_{U(m)}$ $(m \geq 2n)$ *are divisible by precisely* $(n-1)!$.

An equivalent formulation:

The nth *Chern class of a* U(m) *bundle over* S_{2n} *is divisible by* $(n-1)!$.

Indeed the Chern classes of a GL(n, C) bundle over X are by definition preimages of classes in $B_{U(m)}$ $(m \geq n)$. Finally, and for a similar reason:

The kth *Pontryagin class of a* GL(n, R) *bundle over* S_{4k} *is divisible by* $(2k-1)!$.

9. THE ODD ORTHOGONAL GROUPS

In Section 5, a generating variety for SO(2n + 1) $(n > 2)$ was found to be the Grassmannian: $V_{2n-1} = SO(2n+1)/U(1) \times SO(2n-1)$. Using the fibering

$$SO(2n+1)/SO(2n-1) \rightarrow V_{2n-1},$$

we can easily determine the cohomology of V_{2n-1}. We find that if $u \in H^2(V_{2n-1})$ is a generator, then the classes

$$\{1, u, \cdots, u^{n-1}, u^n/2, \cdots, u^{2n-1}/2\} \tag{9.1}$$

span $H^*(V_{2n-1})$. Let $\sigma_i' \in H_{2i}(V_{2n-1}; Q)$ be determined by $u^i(\sigma_i') = 1$. In terms of these, a dual basis for (9.1), in $H_*(V_{2n-1})$, is

$$\{\sigma_0', \sigma_2', \cdots, \sigma_{n-1}', 2\sigma_n', \cdots, 2\sigma_{2n-1}'\} . \tag{9.2}$$

Let $g: V_{2n-1} \rightarrow \Omega' SO(2n+1)$ be the generating map, and set $\sigma_i = g_* \sigma_i'$. In view of Theorem 1, we have the following propositions.

PROPOSITION 9.1. *The algebra* $H_*(\Omega' SO(2n+1))$ $(n \geq 2)$ *is generated by the classes*

$$\{\sigma_0, \cdots, \sigma_{n-1}, 2\sigma_n, \cdots, 2\sigma_{2n-1}\}. \tag{9.3}$$

In terms of the rational classes σ_i, *the diagonal map is given by*

$$\Delta_* \sigma_i = \sum \sigma_\alpha \otimes \sigma_\beta \qquad (\alpha + \beta = i). \tag{9.4}$$

The dual statement reads as follows:

PROPOSITION 9.2. *Under* g^{**}, *the ring* $H^*(\Omega SO(2n+1))$ *is identified with the rational closure of the subring generated in* $\mathscr{S} H^*(V_{2n+1})$ *by the classes*

$$\{1, i(u), i(u^3), \cdots, i(u^{2n-1})\}. \tag{9.4}$$

This follows from Theorem 3 and the well-known fact that the algebra under consideration has primitive classes only in the dimensions of the form $4k - 2$ $(k = 1, \cdots, n)$.

PROPOSITION 9.3. *The generators* $\{\sigma_i\}$ *of Proposition* 9.1 *satisfy the following relations:*

$$\sigma_i^2 - 2\sigma_{i-1} \cdot \sigma_{i+1} + \cdots \pm 2\sigma_0 \cdot \sigma_{2i} = 0 \qquad (i = 1, \cdots, n-1). \tag{9.6}$$

This formula could be proved from (9.5). We sketch a shorter proof: Let $\{p_i\}$ be the polynomials in the $\{\sigma_i\}$ determined by the recursion (8.10). In view of (9.4), these classes represent primitive elements. Because primitive classes can occur only in dimensions of the form $4k - 2$, the polynomials p_{2i} $(i = 1, \cdots, n-1)$ must represent 0. Thus $p_{2i} = 0$ $(i = 1, \cdots, n-1)$ is a system of relations for our generators. These are rather cumbersome, but they can be transformed into (9.6) by the following device. We identify the polynomial ring in the $\{\sigma_i\}$ with S_{2n-1}^{2n-1}, as in the proof of (8.3). Under this identification, p_i goes over into the even power sum $\Sigma_{2i}(x)$. Now these power sums generate rationally the first $n - 1$ elementary symmetric functions in the squares of the variables, and (9.6) is seen to be precisely the condition that these elementary functions represent 0 in our algebra.

The polynomials p_{2i-1} $(i = 1, \cdots, n)$ span the primitive subspace of our algebra over the rational numbers. In view of (9.3), this proves the following proposition.

PROPOSITION 9.4. *Let* p_i $(i = 1, \cdots, 2n-1)$ *be polynomials in the generators* $\{\sigma_i\}$, *defined recursively by*

$$p_k - p_{k-1} \cdot \sigma_1 + \cdots \pm k\sigma_k = 0 \qquad (k = 1, \cdots). \tag{9.7}$$

Then the primitive subspace of $H_*(\Omega SO(2n+1))$ *is spanned by*

$$\{p_1, p_3, \cdots, p_{2[n/2]-1}, 2p_{2[n/2]+1}, \cdots, 2p_{2n-1}\}. \tag{9.8}$$

Remarks. For n - 2, that is, for $\Omega' SO(5)$, we have generators $1, \sigma_1, 2\sigma_2, 2\sigma_3$. The only relation is $\sigma_1^2 = 2\sigma_2$. Hence $H_*(\Omega' SO(5))$ is a polynomial ring with generators σ_1 and $2\sigma_3$. For $n > 2$, the generators are $1, \sigma_1, \sigma_2, \cdots$, so that σ_1^2 is divisible by 2. As a consequence, the algebra is no longer a polynomial ring. This is in agreement with Borel's result [3] that Spin$(2n+1)$ has 2-torsion for $n > 2$.

10. THE EVEN ORTHOGONAL GROUPS

This section parallels the preceeding one. The generating variety for $\Omega' SO(2n+2)$ $(n \geq 1)$ is again the Grassmannian of oriented 2-planes in the space of $2n+2$ dimensions. We denote it by V_{2n}. If u is a generator of $H^2(V_{2n})$, then a base for $H^*(V_{2n})$ is given by

$$(10.1) \qquad \{1, u, \cdots, u^n, u^{n+1}/2, \cdots, u^{2n}/2; v\},$$

where v is a class in $H^{2n}(V_{2n})$ which is uniquely determined by the requirement that u^n and v span $H^{2n}(V_{2n})$, and that

$$(10.2) \qquad uv = u^{n+1}/2.$$

The square of v is given by:

$$(10.3) \qquad v^2 = \begin{cases} u^n v & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The formula (10.3) is obtained by considering the index $\tau(V_{2n})$ of V_{2n}. According to [2, Chapter VI, Section 24], $\tau(V_{2n}) = \sum (-1)^i \dim H_{2i}(V_{2n})$. This formula immediately implies (10.3).

Let $\sigma_i' \in H_{2i}(V_{2n}; Q)$ and $(\varepsilon' \in H_{2n}(V_{2n}; Q))$ be determined by the relations

$$u^i(\sigma_i') = 1, \quad u^n(\varepsilon') = 0, \quad w = u^n - 2v, \quad w(\sigma_n') = 0, \quad w(\varepsilon') = 1.$$

In terms of these rational classes, a basis for $H_*(V_{2n})$ is given by

$$(10.4) \qquad \{1, \sigma_1', \cdots, \sigma_{n-1}', \sigma_n' + \varepsilon, \sigma_n' - \varepsilon, 2\sigma_{n+1}', \cdots, 2\sigma_{2n}'\}.$$

Let $g\colon V_{2n} \to \Omega' SO(2n+2)$ be the generating map, and set $g_*\sigma_i = \sigma_i'$, $g_*\varepsilon' = \varepsilon$. The analogues of Propositions 9.1 to 9.4 then take the following form:

PROPOSITION 10.1. *For* $n > 1$, *the algebra* $H_*(\Omega' SO(2n+2))$ *is generated by the* g_**-image of the base* (10.4). *The diagonal map is given by:*

$$(10.5) \qquad \begin{cases} \Delta_*\sigma_i = \sum \sigma_\alpha \otimes \sigma_\beta & (\alpha + \beta = i), \\ \Delta_*\varepsilon = \varepsilon \otimes 1 + 1 \otimes \varepsilon, & \\ \Delta_*\sigma_{2n} = \sum \sigma_\alpha \otimes \sigma_\beta + (-1)^n \varepsilon \otimes \varepsilon & (\alpha + \beta = 2n), \end{cases}$$

These generators are related by the equations

$$(10.6) \qquad \sigma_i^2 - 2\sigma_{i-1} \cdot \sigma_{i+1} + \cdots \pm 2\sigma_0 \cdot \sigma_{2i} = 0 \qquad (1 \leq i \leq n),$$

$$(10.7) \qquad (\sigma_n + \varepsilon)(\sigma_n - \varepsilon) - 2\sigma_{n-1} \cdot \sigma_{n+1} + \cdots \pm 2\sigma_0 \cdot \sigma_{2n} = 0.$$

The primitive subspace of the algebra is spanned by

$$(10.8) \qquad \{p_1, p_3, \cdots, p_{n-2}, p_{n+\varepsilon}, 2\varepsilon, 2p_{n+2}, \cdots, 2p_{2n-1}\} \qquad (n \textit{ odd}),$$

(10.9) $\{p_1, p_3, \cdots, p_{n-1}, 2\varepsilon, 2p_{n+1}, \cdots, 2p_{2n-1}\}$ (n *even*),

where the polynomials $\{p_i\}$ *are determined by the recursion* (9.7). *Under* g^{**}, *the ring* $H^*(\Omega' SO(2n+2))$ *is identified with the rational closure of the subring of* $\mathscr{P}H^*(V_{2n})$ *generated by the classes*

(10.10) $\{1, i(u), i(u^3), \cdots, i(u^{2n-1}); i(v)\}$.

The derivation of these formulae is facilitated by the following fact.

PROPOSITION 10.2. *The inclusion* $i: SO(2n+1) \subset SO(2n+2)$ *induces an isomorphism of* $H_*(\Omega' SO(2n+1))$ *onto the subring of* $H_*(\Omega' SO(2n+2))$ *generated by*

(10.11) $\{1, \sigma_1, \cdots, \sigma_{n-1}, 2\sigma_n, \cdots, 2\sigma_{2n-1}\}$.

This proposition, in turn, follows immediately from the commutativity of the generating maps with the respective inclusions.

An obvious problem now is to compare the algebras $A_n = H_*(\Omega' SO(n))$ for different values of n. Let i_* denote the homomorphism $A_n \to A_{n+1}$ induced by the standard inclusion. According to (10.2), i_* injects A_{2n+1} into A_{2n+2} $(n \geq 2)$, and the quotient algebra is a polynomial ring with one generator of dimension 2n. *We assert that the extension is nontrivial if* n *is even, and trivial if* n *is odd.* To see this, let B_{2n} be the polynomial ring $Z[\alpha]$ (dim $\alpha = 2n$) with diagonal map $\Delta_*\alpha = 1 \otimes \alpha + \alpha \otimes 1$. If the extension in question is to be nontrivial, there must exist a homomorphism $\lambda: B_{2n} \to A_{2n+2}$ such that the induced homomorphism $i_* \otimes \lambda: A_{2n+1} \otimes B_{2n} \to A_{2n+2}$ is bijective. If n is even, a prospective λ must take α into an integral multiple of 2ε. (By (10.9), this is the only available primitive class.) But then the image of $i_* \otimes \lambda$ can not contain the class $\sigma_n + \varepsilon$. In the odd case, we can set $\lambda(\alpha) = p_n + \varepsilon$. It is then easy to see, in view of (10.11), that the image of $i_* \otimes \lambda$ contains all the generators of A_{2n+2}. Over the rational numbers, $i_* \otimes \lambda$ is clearly injective. Hence, in this case, $i_* \otimes \lambda: A_{2n+1} \otimes B_{2n} \to A_{2n+2}$ is bijective.

11. TWO OTHER EXAMPLES

So far, we have never had to use Theorem 6 to determine the Hopf algebras under consideration. This is not true of the algebra $H_*(\Omega Sp(n))$ $(n \geq 3)$.

Because Sp(n) has no torsion, the Borel description of the generating variety $V_n = Sp(n)/U(n)$ is valid over the integers. With respect to a suitable base of $H^1(T)$ on a maximal torus T of Sp(n), say $x_1, \cdots, x_n$, the Borel description gives $H^*(V_n)$ as $S(x_1, \cdots, x_n)/S^+(x_1^2, \cdots, x_n^2)$, where $S(x_1, \cdots, x_n)$ denotes the symmetric polynomials of $Z[x_1, \cdots, x_n]$, and where $S^+(x_1^2, \cdots, x_1^2)$ denotes the ideal in $S(x_1, \cdots, x_n)$ generated by the elementary symmetric functions in the x_i^2, of positive degree (see [1]).

Let $S(x_1^2, \cdots, x_n^2)$ be the invariant polynomials in x_i^2. This ring describes $H^*(B_{Sp(n)})$. According to Theorem 4, we have to find the image of the composition

$$S(x_1^2, \cdots, x_n^2) \xrightarrow{i} S(x_1, \cdots, x_n) \xrightarrow{\theta} S(x_1, \cdots, x_n) \to H\ (V_n),$$

where i denotes the natural inclusion, while θ denotes the derivation corresponding to the generating variety. In this case, θ is seen to be $\frac{1}{2}\sum \partial/\partial x^i$. Let Σ_i $(i = 1, \cdots, n)$

denote the class of $x_1^i + \cdots + x_n^i$ in $H^*(V_n)$. Then it is easily verified that the image in question is additively generated by $\Sigma_1, \Sigma_3, \cdots, \Sigma_{2n-1}$. Since these elements are not divisible, we conclude *that the double suspension takes* $H^*(B_{Sp(n)})$ *onto the primitive elements of* $H^*(\Omega Sp(n))$. Theorems 1 and 4 now imply

PROPOSITION 11.1. *The generating map identifies* $H^*(\Omega Sp(n))$ $(n \geq 3)$ *with the rational closure of the ring generated in* $\mathscr{P} H^*(V_n)$ *by the classes*

$$\{1, i(\Sigma_1), i(\Sigma_3), \cdots, i(\Sigma_{2n-1})\}. \tag{11.1}$$

This is hardly a palatable description. For instance, I do not know how (11.1) implies (as it must) that $H_*(\Omega Sp(n))$ is a polynomial ring. However, this complexity is not entirely artificial. In evidence of this fact, here is the diagonal map for $n = 3$:

In suitable generators u_1, u_3, u_5 $(\dim u_i = 2i)$,

$$H_*(\Omega Sp(3)) = Z[u_1, u_3, u_5], \tag{11.2}$$

and the diagonal map is given by

$$\begin{cases} \Delta_* u_1 = u_1 \otimes 1 + 1 \otimes u_1, \\ \Delta_* u_3 = u_3 \otimes 1 + u_1^2 \otimes u_1, \text{ and so forth,} \\ \Delta_* u_5 = u \otimes 1 + 2u_3 \cdot u_1 \otimes u_1 + (u_1^3 + u_5) \otimes u_1^2, \text{ and so forth.} \end{cases} \tag{11.3}$$

The exceptional group G_2. The generating variety V for the group G_2 has dimension 10. By using the methods of [4], one finds that if $x \in H^2(V)$ is a generator, then $\{1, x, x^2/3, x^3/3, x^4/3^2 \cdot x^5/3^2 \cdot 2\}$ is an additive base for $H^*(V)$. We let $\sigma_i^! \in H_{2i}(V)$ be the corresponding dual base, and set $g_* \sigma_i^! = \sigma_i$, where g denotes the generating map. By Theorem 1, these elements generate $H_*(\Omega G_2)$. The primitive elements of ΩG_2 occur in dimensions 2 and 10. Hence $g^* \mathfrak{P}^* = \{x, x^5/3^2 \cdot 2\}$. It now follows by Proposition 6.2 that

(11.4) $H^*(\Omega G_2)$ *is the rational closure of the ring generated by* $i(x)$ *and* $i(x^5)$ *in* $\mathscr{P} H^*(V)$.

This result implies the following relations between the $\{\sigma_i\}$: $2\sigma_2 - 3\sigma_1^2 = 0$, $2\sigma_3 - \sigma_1^3 = 0$, $4\sigma_4 - 3\sigma_1^4 = 0$. It follows that if we set $u = \sigma_1$, $v = \sigma_2 - \sigma_1^2$ and $w = \sigma_5$, then these three generate $H_*(\Omega G_2)$, and $2v = u^2$. Hence

$$H_*(\Omega G_2) = Z[u, v, w] / \{2v - u^2\}. \tag{11.5}$$

Finally, the diagonal map is found to be

$$\begin{cases} \Delta_* u = u \otimes 1 + 1 \otimes u, \\ \Delta_* v = v \otimes 1 + u \otimes u + 1 \otimes v, \\ \Delta_* w = w \otimes 1 + 2v^2 \otimes u + 6uv \otimes v + 6v \otimes uv + 2u \otimes v^2 + 1 \otimes w. \end{cases} \tag{11.6}$$

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