SOME SETS OF SUMS AND DIFFERENCES

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Denote by R the set of real numbers. If $X \subseteq R$ and $Y \subseteq R$, let

$$s(X, Y) = \{x + y: x \in X, y \in Y\},$$

$$d(X, Y) = \{x - y: x \in X, y \in Y\} \cup \{y - x: x \in X, y \in Y\},$$

and set $s(X, X) = s(X)$, $d(X, X) = d(X)$.

It is known and easy to prove that $s(X, Y) = d(X, Y) = R$ if $R - X$ and $R - Y$ are both of Lebesgue measure zero or both of first Baire category; in particular, $s(X) = d(X) = R$ if $R - X$ is of measure zero or of first category (for an exposition of results of this kind, see [2] and [3]). Suppose that $R - X$ is of measure zero and $R - Y$ is of first category; we shall show that it is possible to have

$$s(X, Y) = d(X, Y) \neq R.$$

Let A be the set of algebraic numbers, L the set of Liouville numbers, and S, T, U the sets of S-, T-, U-numbers, respectively, in Mahler's classification of transcendental numbers (see [4]). No two of the sets A, S, T, U have any elements in common, and $L \subset U$.

LEMMA. L *is a residual subset of* R.

Proof. For every natural number n, define E_n to be the set of numbers ξ such that, for all integers p and q with $q > 1$,

$$\left| \xi - \frac{p}{q} \right| \geq \frac{1}{q^n}.$$

If I is any open interval, it contains a rational number p/q (where p and q are integers and $q > 1$), and the intersection of I with the interval $(pq^{-1} - q^{-n}, pq^{-1} + q^{-n})$ contains no point of E_n. This means that E_n is nowhere dense, so that if all the rational numbers as well as the elements of $\bigcup_{n=1}^{\infty} E_n$ are removed from R, a residual set remains. If ξ belongs to this residual set, then ξ is irrational, and, for every natural number n, there are integers p and q with $q > 1$, such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^n},$$

which implies that ξ is a Liouville number.

Mahler has proved that $R - S$ is of measure zero, so that, in a metrical sense, most numbers are S-numbers. The lemma shows, however, that, in a topological sense, most numbers are Liouville numbers. If T should prove to be nonempty, it is nevertheless small in comparison with S and U in the sense that it is both of first category and measure zero. According to the results cited in the second

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paragraph above, s(X) = d(X) = R if X = S (actually, if X is that subset of S for which the "type" θ satisfies the condition $1 \leq \theta \leq 2$; see [4, p. 86]) or X = L (Mahler ascribes to Erdös the recognition of the fact (of which there is a simple direct proof by means of decimals) that every real number is the sum of two Liouville numbers).

THEOREM. *There exist sets* $X \subset R$ *and* $Y \subset R$ *such that* $R - X$ *is of measure zero,* $R - Y$ *is of first category, and* $R - s(X, Y)$ *and* $R - d(X, Y)$ *are equal and everywhere dense.*

Proof. Let X = S and Y = U. Then $A \subseteq R - s(X, Y)$ and $A \subseteq R - d(X, Y)$; for if $x \in X$, $y \in Y$, $a \in A$ and $\pm x \pm y = a$, then the numbers x and y are algebraically dependent, and they must therefore belong either both to S or both to U [4, p. 69], which contradicts our assumption. Furthermore, since $-y \in U$ if $y \in U$ and $-x \in S$ if $x \in S$, s(X, Y) = d(X, Y).

Remark. It follows from [1] that, if E is any enumerable subset of R, there exist sets $X \subset R$ and $Y \subset R$ such that R - X is of measure zero, R - Y is of first category, $s(X, Y) \subseteq R - E$, and $d(X, Y) \subseteq R - E$.

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