AN EXAMPLE OF A FUNCTION WITH A DISTORTED IMAGE

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The analogy between measure and measurability on the one hand, and category and possession of the Baire property on the other, is well known (see, for example, [3, pp. 49, 63, 225] and [5, p. 26]); one aspect of it, dealing with rectilinear sections of a plane set, will concern us here.

We shall consider exclusively sets of points in the plane P. Denote by X the set of all real numbers, and by R the set of all positive real numbers. Then

$$P = \{(x, y): x \in X, y \in X\}.$$

For every $x_0 \in X$, let $L_{x_0} = \{(x_0, y): y \in X\}$, and, for every $r \in R$, let

$$C_r = \{(x, y): x^2 + y^2 = r^2\}.$$

According to Fubini [1], if $E \subset P$ and E is (plane Lebesgue) measurable, then there exists a subset X_0 of X of (linear) measure zero such that, for every $x \in X - X_0$, the intersection $E \cap L_x$ is a measurable subset of L_x. If E is a subset of P of measure zero, then there exists a subset X_0 of X of measure zero such that, for every $x \in X - X_0$, the intersection $E \cap L_x$ is a subset of L_x of measure zero. According to Kuratowski and Ulam (see [4] or [3, pp. 223, 222]), if E is a subset of P that possesses the Baire property, then there exists a subset X_1 of X of first category such that, for every $x \in X - X_1$, the subset $E \cap L_x$ of L_x possesses the Baire property. If E is a subset of P of first category, then there exists a subset X_1 of X of first category such that, for every $x \in X - X_1$, the subset $E \cap L_x$ of L_x is of first category.

The converses of these results are false. If f(x) is a function of a real variable, the plane set $J(f) = \{(x, y): y = f(x), x \in X\}$ is called the (geometrical) image of the function f. Sierpiński has shown [6] that there exists a single-valued function whose image is not measurable, and Sierpiński and Zalcwasser have given an example (in [2, p. 85]) of a single-valued function whose image is not of first category and therefore [3, p. 229] does not possess the Baire property. Sierpiński has also proved [6] the existence of a nonmeasurable subset of P which intersects every (straight) line in at most two points.

Let $p \in P$ and $E \subset P$. We say that E is measurable at p if there exists a (circular, open) neighborhood N of p such that $E \cap N$ is a measurable subset of N; otherwise, E is said to be nonmeasurable at p. Similarly, E is of first category at p if there exists a neighborhood N of p such that $E \cap N$ is a subset of N of first category; otherwise, E is of second category at p.

THEOREM. *There exists a function* f(x) $(x \in X)$ *possessing the following properties:*

(a) f *and its inverse are single-valued,*

(b) f *maps the set of real numbers onto itself,*

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(c) *every line intersects* J(f) *in at most two points,*

(d) J(f) *is nonmeasurable at every point* $p \in P$,

(e) J(f) *is of second category at every point* $p \in P$.

Remark. Kuratowski has shown ([2, p. 84], [4, p. 250], [3, p. 229]) that, for every single-valued function f(x) ($x \in X$), the set P - J(f) is of second category at every point $p \in P$. With the aid of Fubini's theorem, it is easy to see that P - J(f) is not of measure zero at any point $p \in P$.

Proof of the theorem. By a circular perfect set we mean a set that is a (non-empty) perfect subset of C_r for some $r \in R$. Since there are $\aleph$ elements of R, and every C_r contains $\aleph$ perfect subsets, there are $\aleph$ circular perfect subsets all told, and the set consisting of all horizontal and vertical lines in the plane is also a set of $\aleph$ elements. Hence, the union of these two sets can be well-ordered to form a (transfinite) sequence

(1) $$Q_0, Q_1, \cdots, Q_\xi, \cdots \quad (\xi < \omega_\alpha),$$

where ω_α denotes the initial ordinal number of $Z(\aleph)$ (Greek letters will stand for ordinal numbers).

We define, by transfinite induction, two sequences of points, $\{a_\xi\}_{\xi<\omega_\alpha}$ and $\{b_\xi\}_{\xi<\omega_\alpha}$, as follows.

If Q_0 is a circular perfect set, let a_0, b_0 be any two distinct points of Q_0. If Q_0 is a horizontal or a vertical line, there exists an $r \in R$ such that C_r and Q_0 intersect in two points: call one of these points a_0, the other, b_0.

Let $0 < \gamma < \omega_\alpha$, and suppose that the sequences of points $\{a_\xi\}_{\xi<\gamma}$ and $\{b_\xi\}_{\xi<\gamma}$ have been defined so that, for every $\beta < \gamma$,

(i) $\{a_\xi\}_{\xi\le\beta}$ and $\{b_\xi\}_{\xi\le\beta}$ have no point in common,

(ii) if $\xi \le \beta$, there exists an $r \in R$ such that C_r contains both a_ξ and b_ξ,

(iii) no horizontal or vertical line contains more than one point of $\{a_\xi\}_{\xi\le\beta}$,

(iv) no line contains more than two points of $\{a_\xi\}_{\xi\le\beta}$.

Then it is evident that (i) to (iv) hold if "$\xi \le \beta$" is replaced therein by "$\xi < \gamma$." Let S_γ be the set consisting of all horizontal and vertical lines that contain a point of $\{a_\xi\}_{\xi<\gamma}$ and all lines that contain two points of $\{a_\xi\}_{\xi<\gamma}$. Since $\{a_\xi\}_{\xi<\gamma}$ contains fewer than $\aleph$ points, S_γ contains fewer than $\aleph$ lines.

Suppose first that Q_γ is a circular perfect set. If it contains a point of $\{a_\xi\}_{\xi<\gamma}$, we define a_γ to be this point. Otherwise, Q_γ contains a point—call it a_γ—which is not a term of $\{b_\xi\}_{\xi<\gamma}$ and does not lie on any line belonging to S_γ, because Q_γ contains $\aleph$ points and every line in S_γ intersects Q_γ in at most two points. Let b_γ be any point of Q_γ that does not belong to $\{a_\xi\}_{\xi\le\gamma}$.

Now suppose that Q_γ is a horizontal or a vertical line. If, for some $\delta < \gamma$, $a_\delta \in Q_\gamma$, define a_γ to be a_δ and b_γ to be b_δ. If Q_γ contains no point of $\{a_\xi\}_{\xi<\gamma}$, then obviously $Q_\gamma \notin S_\gamma$. Hence, there exists an $r \in R$ such that C_r intersects Q_γ in two points neither of which is a term of $\{a_\xi\}_{\xi<\gamma}$ or $\{b_\xi\}_{\xi<\gamma}$ and neither of which lies on a line in S_γ; call one of these two points a_γ, the other, b_γ.

It is easy to verify that (i) to (iv) hold if "β" is replaced therein by "γ", and so the sequences $\{a_\xi\}_{\xi<\omega_\alpha}$ and $\{b_\xi\}_{\xi<\omega_\alpha}$ are well-defined. Let A, respectively B, be the set of points that are terms of $\{a_\xi\}_{\xi<\omega_\alpha}, \{b_\xi\}_{\xi<\omega_\alpha}$. It is clear from the way in which we defined these sequences and (1), that we can draw the following conclusions. Every horizontal line and every vertical line intersects A in precisely one point. Hence, A is the image J(f) of a function f satisfying (a) and (b); (c) is also immediately evident. To prove (d) and (e), we first observe that, since $B \subset P - A$, every circular perfect set is neither a subset of A nor of P - A. It follows [3, p. 423] that, for every $r \in R$ and any subarc K of C_r, $A \cap K$ is a nonmeasurable subset of K of second category. Now suppose that p is any point of P, and that A is either measurable or of first category, at p. Then there exists a neighborhood N of p such that $A \cap N$ is either a measurable subset of N, or of first category. By an obvious modification of the Fubini or the Kuratowski-Ulam theorem stated above, there exists an $r \in R$ for which $C_r \cap N$ is a (nonempty) arc, call it K, such that $A \cap K$ is either a measurable subset of K, or a subset of K of first category, which we have just seen is impossible. Therefore (d) and (e) are true, and the proof is complete.

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