THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS BELONGING TO A CLASS OF HOROCYCLIC GROUPS

Joseph Lehner

1. R. A. Rankin has recently coined the term "horocyclic group" as an English equivalent of the French "Fuchsian group of the first kind" and the German "*Grenzkreisgruppe*" ([8]). He calls such a group "real" if all substitutions of the group preserve the real axis, and "zonal" if the group contains translations. In this paper, we shall refer to real zonal horocyclic groups as "H-groups." An H-group, then, is a group Γ of linear transformations of a complex variable such that

(a) Γ is discontinuous in the upper half-plane but is not discontinuous at any point of the real axis,

(b) every transformation of Γ preserves the upper half-plane,

(c) Γ contains parabolic substitutions with fixed point ∞.

The main object of this paper is to determine, by the use of the circle method, the expansions of the Fourier coefficients of automorphic forms on H-groups of a certain class. The circle method has been employed by Rademacher and Zuckerman ([6], [7], [10], [11]) for the case where the H-group is the modular group or one of its subgroups. Here we develop the circle method for a class of H-groups defined by the following

(1.0) RESTRICTION: *A fundamental region of the* H-*group shall have exactly one parabolic cusp.*

(This implies that the fundamental region has a finite number of sides ([2], Thm. 16, p. 75).) As a consequence of this condition, there exists a number $h > 0$ such that the fundamental region with cusp at ∞ does not extend below the horizontal line at height h above the real axis.

The circle method is elementary in character; it uses only Cauchy's theorem and a careful dissection of the path of integration. Lacking an arithmetic characterization of the parabolic points of the H-group such as is available in the case of the modular group, we use the geometry of the fundamental region for the dissection of the path.

We treat entire automorphic forms of dimension r, that is, analytic functions of a complex variable z which are regular in the upper half-plane and satisfy there the functional equation

$$F(Vz) = \varepsilon(V)(-i(cz+d))^{-r}F(z), \tag{1.1}$$

for every transformation $Vz = (az+b)/(cz+d)$ of Γ. Here r is real, and the multiplier $\varepsilon(V)$ is independent of z and satisfies the condition $|\varepsilon(V)| = 1$. If $c \neq 0$, we assume that $c > 0$ and require $\arg(-i(cz+d))$ to lie between $-\pi/2$ and $\pi/2$, as a means of determining the branch of the many-valued function. If $c = 0$, we have, as we shall presently see, $Vz = z + m\lambda$, where m is an integer and $\lambda > 0$. Set $Sz = z + \lambda$ and

Received November 6, 1957.
Work performed under the auspices of the U.S. Atomic Energy Commission.

$$e(z) = e^{2\pi i z}.$$

Then we can write $(-i)^{-r}\,\varepsilon(S) = e(\alpha)$, where $0 \leq \alpha \leq 1$. From (1.1) we now get

$$F(Sz) = F(z+\lambda) = e(\alpha)\,F(z). \tag{1.2}$$

The relation (1.2) implies the existence of a Fourier expansion for $e(-\alpha z/\lambda)\,F(z)$,

$$e(-\alpha z/\lambda)\,F(z) = \sum_{m=-\mu}^{\infty} a_m e(mz/\lambda) = f(t) \qquad (t = e(z/\lambda)); \tag{1.3}$$

we assume that this expansion has only a finite number of terms with negative exponents. Since $F(z)$ is regular in $\Im z > 0$, the series converges absolutely in the same region.

By "automorphic form of dimension r on Γ," we shall mean a function, regular in the upper half-plane, satisfying (1.1) to (1.3).

Our main results are contained in three theorems which follow. For $c \in C$ and $c > 0$, put

$$A_{c,\nu}(m) = \sum_{d \in D_c} \varepsilon^{-1}(V_{c,d}) \cdot e\{[(m+\alpha)d - (\nu-\alpha)a]/c\lambda\} \qquad \left(V_{c,d} = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}\right), \tag{1.4}$$

$$L_c(m, \nu, r, \alpha) = \left(\frac{\nu-\alpha}{m+\alpha}\right)^{(r+1)/2} I_{r+1}\left(\frac{4\pi}{c\lambda}(\nu-\alpha)^{1/2}(m+\alpha)^{1/2}\right) \qquad (m+\alpha > 0), \tag{1.5}$$

$$L_c(0, \nu, r, 0) = \lim_{m+\alpha \to 0} L_c(m, \nu, r, \alpha) = \frac{(2\pi\nu/c\lambda)^{r+1}}{\Gamma(r+2)}, \tag{1.6}$$

where I_r is the Bessel function of the first kind with purely imaginary argument, and C is defined in (2.2) and D_c in (2.3). Let Γ be an H-group satisfying the restriction (1.0).

THEOREM 1. *Let* $F(z)$ *be an automorphic form of dimension* $r > 0$ *on* Γ. *Then the Fourier coefficients* a_m *with* $m \geq 0$ *are given in terms of those with* $m < 0$ *by the formula*

$$a_m = (2\pi/\lambda) \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\substack{c \in C \\ c > 0}} c^{-1} A_{c,\nu}(m)\, L_c(m, \nu, r, \alpha) \qquad (m \geq 0). \tag{1.7}$$

THEOREM 2. *If* $F(z)$ *is an automorphic form of dimension zero on* Γ *with Fourier coefficients* a_m, *then*

$$a_m = (2\pi/\lambda) \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\substack{c \in C \\ 0 < c < \beta\sqrt{m}}} c^{-1} A_{c,\nu}(m)\, L_c(m, \nu, 0, \alpha) + O(1) \qquad (m \geq 1), \tag{1.8}$$

where β *is any positive constant.*

As an immediate consequence of Theorem 2, we have

THEOREM 3. *Let* G(z) *be an automorphic form of dimension* -2 *with Fourier coefficients* b_m *which is, moreover, the derivative of a form* F(z) *(of dimension zero). Then*

$$b_m = (2\pi/\lambda)^2 i \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\substack{c\in C \\ 0<c<\beta\sqrt{m}}} c^{-1} A_{c,\nu}(m)\, L_c(m, \nu, 0, \alpha) + O(m) \qquad (m \geq 1).$$

Theorem 1 was obtained by Petersson [5] by different methods. Theorems 2 and 3 are believed to be new.

A remark on forms of dimension $r < -2$ is presented in Section 7.

2. Let Γ be an H-group. The elements of Γ may be represented by unimodular matrices $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; a, b, c, d are real, since Γ leaves the real axis invariant. Since we assume that $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$, the matrix $-V = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in \Gamma$ whenever $V \in \Gamma$. The matrices $\pm V$ correspond to the linear transformation

$$Vz = (az + b)/(cz + d),$$

and they may be identified with it. In all of the paper except Section 7, we shall use, as the representative of the transformation Vz, the matrix V for which $c > 0$, and for which $a > 0$ when $c = 0$.

The subgroup of Γ consisting of all V which preserve ∞, that is, in which $c = 0$, is known (see [4], p. 33) to be a cyclic group generated by a translation

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad (\lambda > 0). \tag{2.1}$$

Being discontinuous, Γ is discrete; there is no sequence of different $V_n \in \Gamma$ which tends to the identity. Let C be the set of third coefficients in the elements of Γ, that is, let

$$C = \left\{ x \,\middle|\, \exists \begin{pmatrix} \cdot & \cdot \\ x & \cdot \end{pmatrix} \in \Gamma \right\}; \tag{2.2}$$

similarly for A, B, D. Using the discreteness of Γ, Petersson ([4], p. 34) proved that there is no sequence of different $c_n \in C$ such that $c_n \to 0$. Using the same property, we can show that C is a discrete set.

LEMMA 1. *The set* C *is discrete.*

Suppose otherwise; then there exists a sequence of different c_n in

$$V_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \Gamma \qquad (n = 1, 2, \cdots),$$

with $c_n \to c$ (c finite) as $n \to \infty$. We recall that $c_n \geq 0$. Now

$$S^p V S^q = \begin{pmatrix} a + p\lambda c & \\ c & d + q\lambda c \end{pmatrix},$$

so we may assume that

$$1 \leq a_n < 1 + \lambda c_n, \quad 1 \leq d_n < 1 + \lambda c_n.$$

Hence, $\{a_n\}$, $\{d_n\}$ are bounded, so for a certain subsequence (denoted by the same subscript), we have

$$c_n \to c, \quad a_n \to a, \quad d_n \to d,$$

with $1 \leq a \leq 1 + \lambda c$ and $1 \leq d \leq \lambda c$. Now, if $c \neq 0$, we have immediately

$$\beta = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_n d_n - 1}{c_n} = \frac{ad - 1}{c}.$$

If $c = 0$, we note that for large n,

$$0 \leq b_n < \frac{(1 + \lambda c_n)^2 - 1}{c_n} = 2\lambda + \lambda^2 c_n < 3\lambda.$$

Thus $\{b_n\}$ is bounded, and on a further subsequence, $b_n \to b$.

In either case, then, $V_n \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on some subsequence, where

$$ad - bc = \lim_{n\to\infty} (a_n d_n - b_n c_n) = 1.$$

It follows that $U_n = V_n V_{n+1}^{-1} \to I$. If only a finite number of V were different, we should have $U_n = I$ $(n > N)$, that is, $V_n = V_{n+1}$ $(n > N)$. But this contradicts the assumption that the c_n are all distinct. Hence Γ is not discrete. This contradiction completes the proof.

For $c \in C$, let

$$D_c = \left\{ d \in D \;\middle|\; \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma,\ 0 \leq -d < c\lambda \right\}. \tag{2.3}$$

As a corollary of Lemma 1, we have

LEMMA 2. *For each positive* $c \in C$, *the set* D_c *is finite.*

Let D_c^* be the set of $d \in D$ such that $\begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$. The lemma will follow if we can show that D_c^* is discrete. If D_c^* is not discrete, there exists a sequence of different $d_n \to d \neq \infty$ for which $V_n = \begin{pmatrix} \cdot & \cdot \\ c & d_n \end{pmatrix} \in \Gamma$. This contradicts the discreteness of Γ, by the same argument as was used in Lemma 1.

We shall now normalize the group Γ as follows. Choose $V_0 = \begin{pmatrix} \cdot & \cdot \\ c_0 & \cdot \end{pmatrix} \in \Gamma$ in such a way that $c_0 > 0$ and $|c_0|$ is minimal. Let $L = \begin{pmatrix} c_0^{1/2} & 0 \\ 0 & c_0^{-1/2} \end{pmatrix}$, and let $\Gamma' = L\Gamma L^{-1}$ be the transformed group with elements $V' = \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix}$. Note that

$S' = LSL^{-1} = \begin{pmatrix} 1 & c_0\lambda \\ 0 & 1 \end{pmatrix}$, so that Γ' is an H-group. It follows that Lemmas 1 and 2 hold for Γ' also.

Now $c' = c/c_0$; therefore,

$$|c'| \geq 1 \quad \text{if } c' \neq 0. \tag{2.4}$$

There are elements of Γ' for which $c' = 1$, for example, $LV_0L^{-1} = \begin{pmatrix} a_0+d_0 & -1 \\ 1 & 0 \end{pmatrix}$. The effect of the normalization is to make the smallest nonzero value of the third coefficient in the transformations of the group numerically equal to 1.

It is the transformed group which we shall study. From now on, we shall drop the primes, and denote the transformed group by Γ, its elements by $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its generating translation by $S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ $(\lambda > 0)$.

3. We now introduce the restriction on Γ that was stated in Section 1. Since Γ has exactly one equivalence class of parabolic points which, by the definition of an H-group, must contain the point ∞, we see that all parabolic points of Γ are equivalent to ∞. Therefore every parabolic point is of the form $-d/c$, where $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. In this representation, c and d are unique. For suppose that $-d'/c'$ is the same parabolic point as $-d/c$, and let $V' = \begin{pmatrix} \cdot & \cdot \\ c' & d' \end{pmatrix}$. Then $V'V^{-1}$ preserves ∞ and so is equal to S^m. It follows that $V' = S^m V$, which implies that $c = c'$, $d = d'$.

Let $z = x + iy$ be a complex variable. We choose a closed fundamental region (FR) of Γ, bounded laterally by portions of the vertical lines $x = 0$ and $x = \lambda$, and below by arcs of isometric circles $|cz + d| = 1$ $(c > 0)$; see [2], Section 35. We have already remarked that FR is bounded by a finite number of arcs ([2], p. 75).

Let R be the closed region which is the union of FR and all its translates by integral multiples of λ. Since $|c| \geq 1$ when $c \neq 0$ (see (2.4)), it follows that the radii of the isometric circles $(1/|c|)$ do not exceed unity. Hence

$$y \geq 1 \quad \text{implies} \quad z \in R. \tag{3.1}$$

Also, FR does not extend below a horizontal line of height h above the real axis (see Section 1), so

$$y < h \quad \text{implies} \quad z \notin R. \tag{3.2}$$

We see, incidentally, that

$$h < 1. \tag{3.2a}$$

We shall now describe a dissection of a line segment which will be used later for purposes of integration. Let L_N be the line segment

$$L_N\colon\ 0 \leq x < \lambda, \quad y = y_0 = N^{-2}h^{-1}, \tag{3.3}$$

where $N > h^{-1}$ is arbitrary. Consider the sets

$$I_{c,d} = \{z \in L_N \mid V_{c,d}\, z \in R\}, \tag{3.4}$$

where $V_{c,d} = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}$. (The particular $V_{c,d}$ we choose makes no difference. For $V'_{c,d} = S^m V_{c,d}$ with some integer m, and so $V'_{c,d}$ maps a point into R if and only if $V_{c,d}$ does.) Since L_N is contained in a compact subset of the upper half-plane, it is covered by a finite number of copies of FR, each of which has exactly one real cusp, $-d/c$. Hence $I_{c,d}$ is empty except for a finite set of pairs (c,d). Calling this finite set M, we have

$$L_N = \bigcup_{(c,d)\in M} I_{c,d}. \tag{3.5}$$

We see easily that each $I_{c,d}$ is the union of a finite number of closed intervals, for $V_{c,d}$ maps L_N onto the arc of a circle which, because FR has a finite number of sides, intersects R in a finite number of arcs. Denote by $|I_{c,d}|$ the measure of $I_{c,d}$. The sets $I_{c,d}$ do not overlap, except possibly at their end points; for no point can be mapped into the interior of R by two $V_{c,d}$ with different (c,d). This implies

$$\sum_{(c,d)\in M} |I_{c,d}| = \lambda. \tag{3.6}$$

4. Now let $F(z)$ be an automorphic form on Γ of the type described in Section 1. Let C_N be the circle $|t| = \exp(-2\pi/N^2 h\lambda)$. By applying Cauchy's theorem to (1.3), we have

$$\lambda a_m = \frac{\lambda}{2\pi i}\int_{C_N} \frac{f(t)}{t^{m+1}}\,dt = \int_{L_N} e(-\alpha z/\lambda)\, F(z)\, e(-mz/\lambda)\,dz = \sum_{(c,d)\in M}\int_{I_{c,d}} \cdots, \tag{4.1}$$

the integrand being the same in the last two integrals. On each set $I_{c,d}$ we apply the transformation formula (1.1) with $V = V_{c,d}$, and in the result we introduce the Fourier series (1.3) for $F(z')$, where

$$z' = V_{c,d}\, z = x' + iy'.$$

We then get

$$\begin{aligned}\lambda a_m = {} & \sum_{(c,d)\in M} \varepsilon^{-1}\int_{I_{c,d}} (-i(cz+d))^r \sum_{\nu=-\mu}^{-1} a_\nu e\{[(\nu+\alpha)z' - (m+\alpha)z]/\lambda\}\,dz \\ & + \sum_{(c,d)\in M} \varepsilon^{-1}\int_{I_{c,d}} (-i(cz+d))^r \sum_{\nu=0}^{\infty} a_\nu e\{[(\nu+\alpha)z' - (m+\alpha)z]/\lambda\}\,dz = S_1 + S_2.\end{aligned} \tag{4.2}$$

The choice of $V_{c,d}$ affects the value of $\varepsilon = \varepsilon(V_{c,d})$. But from (1.1), we easily deduce that $\varepsilon(V'_{c,d}) = \varepsilon(S^m V_{c,d}) = e(\alpha m)\,\varepsilon(V_{c,d})$. This makes it clear that the expression $\varepsilon^{-1} e\{(\nu+\alpha)\, z'/\lambda\}$ is independent of the choice of $V_{c,d}$.

When $z \in I_{c,d}$, we have $z' \in R$ by (3.4), so $y' \geq h$ by (3.2). Also, $y = y_0$ by (3.3). Now

(4.3)
$$y' = \frac{y_0}{|cz + d|^2} = \frac{y_0}{(cx + d)^2 + c^2 y_0^2}.$$

Hence, $|cz + d|^2 = y_0/y' \leq N^{-2}h^{-2}$. We therefore have the estimates

(4.4)
$$y' \geq h, \qquad |cz + d| \leq N^{-1}h^{-1} \qquad (z \in I_{c,d}).$$

With these estimates, we get

$$|S_2| \leq O(N^{-r}) \sum_{\nu=0}^{\infty} |a_\nu| \exp\{-2\pi(-|m + \alpha|/N^2 h + (\nu + \alpha)h)/\lambda\} \sum_{(c,d)\in M} |I_{c,d}|,$$

where the O-symbol indicates a constant independent of m and N. The inner sum is finite, by (3.6). Since $h > 0$, the infinite series converges (see (1.3)); we have then

$$S_2 = O(N^{-r} \exp 2\pi|m + \alpha|/N^2 h\lambda).$$

This error term will occur repeatedly, and we shall denote it by

$$E_{m,N} = O(N^{-r} \exp 2\pi|m + \alpha|/N^2 h\lambda).$$

In S_1, we break up the range of (c,d). Set $M = M_1 \cup M_2$, where

(4.5)
$$M_1 = \{(c, d) \in M \mid 0 < c < Nh/2\}.$$

Denote the corresponding parts of S_1 by T_1 and T_2. In T_2, we still have the estimates (4.4) and, in addition, by (4.3),

(4.6)
$$y' \leq y_0/c^2 y_0^2 = 1/c^2 y_0 < 4N^{-2}h^{-2}\cdot N^2 h = 4h^{-1}.$$

If M_2 is empty, we have, of course, $T_2 = 0$; otherwise we get from (4.2)

$$|T_2| \leq O(N^{-r}) \sum_{\ell=1}^{\mu} |a_{-\ell}| \exp\{8\pi\ell/h\lambda + 2\pi|m + \alpha|/N^2 h\lambda\} = E_{m,N}.$$

Putting these results together, we get

(4.7)
$$\lambda a_m = \sum_{(c,d)\in M} \varepsilon^{-1} \int_{I_{c,d}} (-i(cz + d))^r \sum_{\ell=1}^{\mu} a_{-\ell}\, e\{-[(m + \alpha)z + (\ell - \alpha)z']/\lambda\}\, dz + E_{m,N}.$$

5. In order to make further progress, let us study the sets $I_{c,d}$ more closely. When $(c, d) \in M_1$, we have $c < Nh/2$, and

$$\Im V_{c,d}\left(-\frac{d}{c} + iy_0\right) = y_0/c^2 y_0^2 > 4N^{-2}h^{-2}\cdot N^2 h = 4h^{-1} > 1,$$

in view of (3.2a). Hence $I_{c,d}$ contains the point $-d/c + iy_0$, and, by continuity, it contains a largest closed interval $J_{c,d}$ which includes that point. The endpoints of $J_{c,d}$

are determined by considering the map of L_N by $V_{c,d}$. This is a circle K which definitely intersects the interior of R and which leaves R for the first time at two points. The inverse images of these two points are the endpoints of $J_{c,d}$. Therefore, if we write

$$I_{c,d} = J_{c,d} + J'_{c,d},$$

then certainly

$$y' < 1 \qquad (z \in J'_{c,d}); \tag{5.1}$$

for K is definitely below the line $y = 1$ as soon as z' leaves R (see (3.1)). On $J_{c,d}$, we have the estimate (4.4), since $J_{c,d} \subset I_{c,d}$.

The above argument shows, incidentally, that M_1 consists of *all* pairs (c, d) in the range $0 < c < Nh/2$. For each such pair does map a point, namely, $-d/c + iy_0$ (and an interval surrounding the point), into R. Moreover, the sets $\{J_{c,d}, J'_{c,d}\}$ ($(c, d) \in M_1$) are obviously nonoverlapping; hence

$$\sum_{(c,d)\in M_1} |J_{c,d}| < \lambda, \qquad \sum_{(c,d)\in M_1} |J'_{c,d}| < \lambda. \tag{5.2}$$

We break the sum in (4.7) into two parts, calling U_1 the sum in which the integral is extended over $J_{c,d}$, and U_2 the sum in which the path of integration is $J'_{c,d}$. For U_2, we have the estimate $E_{m,N}$ obtained in the same way as that for T_2, since (5.1) is essentially the same as (4.6), and (4.4) holds in both cases; also, (5.2) plays the role of (3.6). This gives

$$\lambda a_m = \sum_{(c,d)\in M_1} \varepsilon^{-1} \sum_{\nu=1}^{\mu} a_{-\nu} \int_{J_{c,d}} (-i(cz + d))^r e\{-[(\nu + \alpha)z' + (m + \alpha)z]/\lambda\} dz + E_{m,N}, \tag{5.3}$$

where we have interchanged the order in the integral and the finite sum.

The integrals in (5.3) must be evaluated in closed form, to yield our final result. For this purpose, we need precise inequalities on the length of $J_{c,d}$. Let

$$z = -d/c + \xi + iy_0 \qquad (-\theta'_{c,d} \leq \xi \leq \theta''_{c,d}) \tag{5.4}$$

when $z \in J_{c,d}$. Let the endpoints of $J_{c,d}$ be denoted by z', z''. The point $V_{c,d}[z_1]$, where z_1 is either z' or z'', has an imaginary part lying between h and 1, that is,

$$h \leq y_0[c^2(\theta^2 + y_0^2)]^{-1} \leq 1 \qquad (\theta = \theta' \text{ or } \theta = \theta'').$$

This leads directly to the desired inequalities:

$$\frac{1}{2h^{1/2}cN} < \theta'_{c,d},\ \theta''_{c,d} < \frac{1}{hcN} \qquad ((c, d) \in M_1). \tag{5.5}$$

6. If we make the change of variable $-i(cz + d) = cw$, each individual integral in (5.3) takes the form

$$(6.1)\qquad e\{[(m+\alpha)d - (\nu-\alpha)a]/c\lambda\}c^r \times \frac{1}{i}\int_{y_0-i\theta''}^{y_0+i\theta'} w^r \exp\{(2\pi/\lambda)((m+\alpha)w + (\nu-\alpha)/c^2w)\}\,dw.$$

If we replace our N by $N/\sqrt{h}$, then the factor following the symbol $\times$ becomes the $I_c(m, \nu)$ in formula (4.22) of [7]. Our θ', θ'' satisfy the same inequalities as the θ', θ'' in [7], with slightly different constants (see the line following (3.4) in [7]). An examination of the developments of [7] shows that this difference does not affect the final result. When $m + \alpha > 0$, this result is ([7], pp. 440-442)

$$(6.2)\qquad I_c(m, \nu) = 2\pi c^{-r-1} L_c(m, \nu, r, \alpha) + c^{-r-1} N^{-1} E_{m,N},$$

where we have used the notation of (1.5).

Now, from (5.5), we deduce that

$$(6.3)\qquad \sum_{(c,d)\in M_1} \frac{1}{cN} < h^{1/2} \sum_{(c,d)\in M_1} (\theta'_{c,d} + \theta''_{c,d}) = h^{1/2} \sum_{(c,d)\in M_1} |J_{c,d}| < h^{1/2}\lambda.$$

Inserting (6.2) in (5.3), we get

$$\lambda a_m = 2\pi \sum_{(c,d)\in M_1} \varepsilon^{-1} \sum_{\nu=1}^{\mu} a_{-\nu}\, e\{[(m+\alpha)d - (\nu-\alpha)a]/c\lambda\}c^{-1} L_c(m, \nu, r, \alpha) + O\Big(E_{m,N} \sum_{(c,d)\in M_1} \frac{1}{cN}\Big) + E_{m,N}.$$

Since, by the remark following (5.1),

$$\sum_{(c,d)\in M_1} = \sum_{\substack{0<c<Nh/2\\ c\in C}} \sum_{d\in D_c},$$

we have, from (1.4) and (6.3) and with the interchange of order in the finite sums,

$$(6.4)\qquad \lambda a_m = 2\pi \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\substack{0<c<Nh/2\\ c\in C}} c^{-1} A_{c,\nu}(m)\, L_c(m, \nu, r, \alpha) + E_{m,N}.$$

Now suppose $r > 0$. Keep m fixed and let $N \to \infty$. Then $E_{m,N} \to 0$. For $m + \alpha > 0$, (6.4) goes over into (1.7).

If $r = 0$, choose $N = \beta\sqrt{m}$. Then $E_{m,N} = O(1)$, and we get (1.8).

We still have to treat the case $m + \alpha = 0$, which is equivalent to $m = \alpha = 0$. Our developments are valid up to and including (6.1). We see directly that the integral in (6.1) is a continuous function of m and α, for $m \geq 0$, $\alpha \geq 0$. Hence we have, from (6.2),

$$\lim_{m,\alpha\to 0} I_c(m, \nu) = \lim_{m,\alpha\to 0} 2\pi c^{-r-1} L_c(m, \nu, r, \alpha) + c^{-r-1} N^{-1} O(N^{-r})$$

$$= 2\pi c^{-r-1} L_c(0, \nu, r, 0) + c^{-r-1} N^{-1} O(N^{-r}),$$

where $L_c(0, \nu, r, 0)$ is the quantity defined in (1.6). The remainder of the argument is the same as before, and leads to the formula (1.7) for $m = \alpha = 0$.

The following result is an easy corollary of Theorem 1.

THEOREM 4. *If* $F(z)$ *is an automorphic form on* Γ, *of dimension* $r > 0$ *and finite at* ∞, *then* $F(z)$ *vanishes identically.*

For the hypothesis states that $a_{-1} = \cdots = a_{-\mu} = 0$. Then (1.7) shows that $a_m = 0$ for $m \geq 0$.

7. The condition $r > 0$, or at least the condition $r \geq 0$, is essential in the developments of the preceding sections. However, in certain cases, it is possible to extend the foregoing results to some negative values of r.

Let

$$r < -2. \tag{7.1}$$

Let $F(z)$ be an automorphic form of dimension r on Γ, where Γ is still an H-group satisfying the restriction (1.0). Then $F(z)$ satisfies the transformation equations (1.1), and has a polar signularity (1.3) of order μ at ∞, where the coefficients of the "principal part," $a_{-1}, \cdots, a_{-\mu}$, are given.

Set up the function

$$G(z) = \sum_{m=-\mu}^{\infty} a_m e((m + \alpha)z/\lambda), \tag{7.2}$$

with the a_m given by (1.7). The absolute convergence of the series in $\Im z > 0$ is clear, since we have the asymptotic estimate

$$\begin{aligned} a_m &\sim (2\lambda)^{-1/2} \sum_{\nu=1}^{\mu} a_{-\nu} A_{1,\nu}(m)\, (\nu - \alpha)^{1/4+r/2} (m + \alpha)^{3/4-r/2} \exp\{4\pi(\nu-\alpha)^{1/2}(m+\alpha)^{1/2}/\lambda\} \\ &= O(m^{s/2} \exp\{8\pi\mu\sqrt{m}/\lambda\}) \qquad (m \to \infty), \end{aligned} \tag{7.3}$$

where we have put

$$s = -r > 2. \tag{7.4}$$

This asymptotic expression is readily obtained from (1.7) by means of the well-known asymptotic formula for $I_r(z)$ ([10], p. 373).

Remark. Let $\Gamma = \Gamma(\lambda)$, the group generated by $S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and let $r = 0$, $\varepsilon(V) = 1$ for $V \in \Gamma(\lambda)$, and $\mu = 1$, $a_{-1} = 1$. Then the expression (7.3) for a_m reduces to

$$a_m \sim \frac{\exp(4\pi\sqrt{m}/\lambda)}{\sqrt{2\lambda}\, m^{3/4}} \qquad (m \to \infty)$$

as given in a previous paper ([3], p. 244). I wish to point out that the proof given there is not correct in detail. A valid argument may be obtained by specializing the reasoning of Sections 3 to 6 to the case considered in [3].

We shall now verify directly that $G(z)$ satisfies the transformation equation (1.1).

Let $\alpha > 0$. Define

$$G_{\nu,\alpha}(z) = e(-(\nu-\alpha)z/\lambda) + \sum_{m=0}^{\infty} a_m^{(\nu)} e((m+\alpha)z/\lambda) \qquad (\nu = 1, 2, \cdots, \mu), \tag{7.5}$$

where $a_m^{(\nu)}$ is obtained from a_m of (1.7) by setting

$$a_{-\ell} = \begin{cases} 0 & (\ell \neq \nu,\ 1 \leq \ell \leq \mu), \\ 1 & (\ell = \nu). \end{cases} \tag{7.6}$$

Then

$$G(z) = \sum_{\nu=1}^{\mu} a_{-\nu} G_{\nu,\alpha}(z), \tag{7.7}$$

since a_m is a linear combination of $a_{-1}, \cdots, a_{-\mu}$. Since the automorphic forms of given dimension and given multiplier system on a group Γ form a linear set, we have only to verify the transformation formula for $G_{\nu,\alpha}$ $(\nu = 1, 2, \cdots, \mu)$.

We shall rearrange the series of (7.5) by means of the Lipschitz formula ([1], p. 206)

$$\{(2\pi)^\beta/\Gamma(\beta)\} \sum_{m=0}^{\infty} (m+\alpha)^{\beta-1} \exp\{-2\pi(m+\alpha)t\} = \sum_{q=-\infty}^{\infty} e(q\alpha)\cdot(t+qi)^{-\beta}, \tag{7.8}$$

valid for $\beta > 0$, $\Re t > 0$, $0 < \alpha < 1$, where $|\arg(t+qi)| < \pi/2$. We insert $a_m^{(\nu)}$ in (7.5), expand the Bessel functions in power series, and use (7.8). All rearrangements of order of summation are justified by absolute convergence. The result is:

$$\begin{aligned} G_{\nu,\alpha}(z) &= e(-(\nu-\alpha)z/\lambda) + \sum_{c,d} \varepsilon^{-1}(V_{c,d})(-i(cz+d+qc\lambda))^{-s} \\ &\quad \cdot \sum_{\ell=0}^{\infty} \frac{(2\pi i(\nu-\alpha)/c\lambda)^\ell}{\ell!} \sum_{q=-\infty}^{\infty} e(-(\nu-\alpha)(a-qc\lambda)/c\lambda)\cdot(cz+d+qc\lambda)^{-\ell}. \end{aligned} \tag{7.9}$$

Here $c > 0$, $c \in C$ and $d \in D_c$.

As we saw in the remark following (4.2), the quantity $\varepsilon^{-1}(V_{c,d})\, e(-(\nu - \alpha)a/c\lambda)$ is independent of the choice of a and b in $V_{c,d}$. Hence, in the innermost sum of (7.9), we may replace $a - qc\lambda$ by a, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Now

$$\begin{pmatrix} a & \\ c & d + qc\lambda \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & q\lambda \\ 0 & 1 \end{pmatrix} \in \Gamma,$$

so we may take $V_{c,d+qc\lambda} = \begin{pmatrix} a & \cdot \\ c & d + qc\lambda \end{pmatrix}$. Then, if we perform the summation on q, $d + c\lambda$ will run over all the values such that $c > 0$, $d \leq 0$ and $\begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$. Let us denote by S' a complete system of $V \in \Gamma$ such that

(i) no two V have the same second row,

(ii) $c \geq 0, \quad d \leq 0.$

The only element of S' with $c = 0$ is I.

From (7.9) we now have

$$(7.10)\quad \begin{aligned} G_{\nu,\alpha}(z) &= e(-(\nu - \alpha)z/\lambda) + \sum_{V\in S', V\neq I} \varepsilon^{-1}(V)\,(-i(cz + d))^{-s}\, e(-(\nu - \alpha)a/c\lambda) \\ &\quad \cdot \sum_{\ell=0}^{\infty} (2\pi i(\nu - \alpha)/c\lambda(cz + d))^{\ell}/\ell\,! \\ &= e(-(\nu - \alpha)z\ \lambda) + \sum_{V\in S', V\neq I} e\left\{(\nu - \alpha)\left(-\frac{a}{c} + \frac{1}{c(cz + d)}\right)/\lambda\right\} \\ &\quad \cdot \varepsilon^{-1}(V)\,(-i(cz + d))^{-s} \\ &= \sum_{V\in S'} e((\nu - \alpha)Vz/\lambda)\,\varepsilon^{-1}(V)\,(-i(cz + d))^{r}. \end{aligned}$$

We now wish to admit V with $c < 0$. Again from (1.1), we have

$$F(-Vz) = \varepsilon(-V)\,(-i(-cz - d))^{-r}\, F(z),$$

and so, by comparison,

$$(7.11)\quad \varepsilon(-V)\,(-i(-cz - d))^{-r} = \varepsilon(V)(-i(cz + d))^{-r}.$$

This shows that the terms in the series of (7.11) are invariant under $V \to -V$; thus we can write

$$(7.12)\quad G_{\nu,\alpha}(z) = \frac{1}{2} \sum_{V\in S} e((\nu - \alpha)Vz/\lambda)\, \varepsilon^{-1}(V)\,(-i(cz + d))^{r},$$

where S is obtained from S' by dropping the restriction $c \geq 0$, that is, where S is a complete system of elements of Γ with different second row. Note that both I and -I appear in (7.12).

Let $L = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma$. If $c_1 = 0$, we have $L = S^m$. From the definition (7.2), we obtain directly

$$G(Lz) = e(m\alpha)\, G(z). \tag{7.13}$$

Hence we assume $c_1 \neq 0$. Then VL clearly runs over the set S if V does. It follows from (7.12) that

$$G_{\nu,\alpha}(Lz) = \frac{1}{2} \sum_{V \in S} e((\nu - \alpha)VLz/\lambda)\ \varepsilon^{-1}(V)\, (-i(cLz + d))^r. \tag{7.14}$$

But (1.1) gives, for $VL = \begin{pmatrix} \cdot & \cdot \\ c_2 & d_2 \end{pmatrix}$,

$$\begin{aligned} F(VLz) &= \varepsilon(VL)\, (-i(c_2 z + d_2))^{-r}\, F(z) \\ &= F(V[Lz]) = \varepsilon(V)\, (-i(cLz + d))^{-r}\, F(Lz) \\ &= \varepsilon(V)\, (-i(cLz + d))^{-r}\, \varepsilon(L)\, (-i(c_1 z + d_1))^{-r}\, F(z); \end{aligned}$$

hence,

$$\varepsilon(VL)\, (-i(c_2 z + d_2))^{-r} = \varepsilon(V)\, \varepsilon(L)\, (-i(cLz + d))^{-r}\, (-i(c_1 z + d_1))^{-r}.$$

Using this in (7.14), we get, with $VL = W$,

$$\begin{aligned} G_{\nu,\alpha}(Lz) &= \frac{1}{2} \varepsilon(L)\, (-i(c_1 z + d_1))^{-r} \sum_{W \in S} e((\nu - \alpha)Wz/\lambda)\ \varepsilon^{-1}(W)\, (-i(c_2 z + d_2))^{r} \\ &= \varepsilon(L)\, (-i(c_1 z + d_1))^{-r}\, G_{\nu,\alpha}(z). \end{aligned} \tag{7.15}$$

Taken together, (7.14) and (7.15) state that $G_{\nu,\alpha}(z)$ satisfies the transformation formula (1.1).

It now follows from (7.7) that G(z) is an automorphic form on Γ of dimension r. From (7.2), we see that G(z) and F(z) have the same principal part at ∞, and, in fact, that $G(z) - F(z) \to 0$ as $z \to \infty$, since $\alpha > 0$. Hence

$$G(z) = F(z) + H(z),$$

where H(z) is a cusp form, that is, a form which vanishes at the parabolic vertex of Γ. This argument justifies the case $\alpha > 0$ of

THEOREM 5. *Let* F(z) *be an automorphic form on* Γ *of dimension* $r < -2$, *and let* $\alpha > 0$. *Then the Fourier coefficients of* F(z) *differ from the* a_m $(m \geq 0)$ *of* (1.7) *by the coefficients of a cusp form. If* $\alpha = 0$, *the Fourier coefficients are determined in the same way for* $m \geq 1$.

When $\alpha = 0$, the situation is somewhat different, since there are forms, not identically zero, which are merely bounded at ∞. Hence we define, in addition to the $G_{\nu,o}$ of (7.5), the function

$$G_o(z) = 1 + \sum_{m=1}^{\infty} a_m^{(o)} e(mz/\lambda),$$

where $a_m^{(o)}$ is obtained from a_m of (1.7) by letting $\nu - \alpha \to 0$. This gives

$$a_m^{(o)} = \{(2\pi/\lambda)^s m^{s-1}/\Gamma(s)\} \sum_{\substack{c \in C \\ c > 0}} c^{-s} \sum_{d \in D_c} \varepsilon^{-1}(V_{c,d})\, e(md/c\lambda) \qquad (s = -r).$$

Lipschitz's formula becomes

$$\{(2\pi)^{\beta}/\Gamma(\beta)\} \sum_{m=1}^{\infty} m^{\beta-1} \exp(-2\pi t) = \sum_{q=-\infty}^{\infty} (t + qi)^{-\beta} \qquad (\beta > 1,\ \Re t > 0).$$

Proceeding as before, we find that

$$G_o(z) = \frac{1}{2} \sum_{V \in S} \varepsilon^{-1}(V)\, (-i(cz + d))^{r}.$$

Thus, exactly as in the previous case, we prove that G_o is an automorphic form on Γ.

Now

$$F(z) - \sum_{\nu=o}^{\mu} a_{-\nu} G_{\nu,o}(z)$$

is automorphic on Γ and vanishes at ∞, hence is a cusp form. This completes the proof of Theorem 5.

REFERENCES

1. L. E. Dickson, *Studies in the theory of numbers*, Univ. of Chicago Press, 1930.

2. L. R. Ford, *Automorphic functions*, McGraw-Hill, 1929.

3. J. Lehner, *Note on the Schwarz triangle functions*, Pacific J. Math. 4 (1954), 243-249.

4. H. Petersson, *Zur analytischen Theorie der Grenzkreisgruppen, I*, Math. Ann. 115 (1938), 23-67.

5. ———, *Konstruktion der Modulformen und der zu gewissen Grenzkreisgruppen gehörigen automorphen Formen von positiver reeller Dimension und die vollständige Bestimmung ihrer Fourierkoeffizienten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1950, 417-494.

6. H. Rademacher, *On the expansion of the partition function in a series*, Ann. of Math. (2) 44 (1943), 416-422.

7. H. Rademacher and H. S. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. (2) 39 (1938), 433-462.

8. R. A. Rankin, *On horocyclic groups*, Proc. Lond. Math. Soc. (3) 4 (1954), 219-234.

9. E. T. Whittaker and G. H. Watson, *A course of modern analysis*, Fourth Edition, reprinted, Cambridge, 1940.

10. H. S. Zuckerman, *On the coefficients of certain modular forms belonging to subgroups of the modular group*, Trans. Amer. Math. Soc. 45 (1939), 298-321.

11. ———, *On the expansions of certain modular forms of positive dimension*, Amer. J. Math. 62 (1940), 127-152.

Los Alamos Scientific Laboratory
and
Michigan State University