ON THE IMPOSSIBILITY OF FIBRING CERTAIN MANIFOLDS BY A COMPACT FIBRE

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1. INTRODUCTION

By a proper fibration of a space we shall understand a fibration in which the fibre is not a single point. It is well known that a Euclidean space does not admit a proper fibration by a compact fibre [2], [6]. It is the purpose of this note to extend this result to a wider class of spaces.

THEOREM 1. *Let* W^n *be an open, simply connected manifold whose one-point compactification is again a manifold; then* W^n *does not admit a proper fibration by a compact fibre.*

To see that a hypothesis implying orientability is needed, we observe that the manifold obtained by removing a point from the real projective plane admits the cyclic group Z_p, for every odd prime, as a fixed-point free group of transformations.

To prove the theorem, we first examine the case in which the fibre is connected. By means of a theorem of Spanier and Whitehead [7], we show that the fibre is an H-space. With the help of Borel's principal algebraic theorem in [1], we show further that the fibre is a rational cohomology sphere. An argument based on the Gysin sequence shows that this is impossible. We are then reduced to showing that a finite group cannot act freely on W^n, and this follows from a result of Mostow, which, incidentally, suggested our note.

2. PRELIMINARIES

A space W^n whose one-point compactification is a manifold is said to be locally Euclidean at infinity. Given such a W^n, for every open set $U \subset W^n$ with $\overline{U}$ compact, there is an open set V such that $\overline{U} \subset V$, $\overline{V}$ is compact, and $W^n - V$ is homeomorphic to a closed n-cell with the origin removed.

We shall denote by M^n the compact manifold obtained by adjoining the point at infinity to W^n, and by $p \in M^n$, the added point. By $[W^n, B, F; \pi]$ we denote a proper fibration of W^n over the base space B with compact fibre F and projection map π. By the term *fibration* we shall mean local product structure. Given $[W^n, B, F; \pi]$, we denote by $\hat{B}$ the one-point compactification of B, by $g \in \hat{B}$ the added point, and by $\hat{\pi}: M^n \to \hat{B}$ a map for which

$$\hat{\pi} \,|\, M^n - p = \pi, \qquad \pi(p) = g.$$

It will be useful to regard $[M^n, p, \hat{B}, g, F; \hat{\pi}]$ as a singular fibration [5].

It is immediately seen from the local product structure that in a fibration of a manifold, such as $[W^n, B, F; \pi]$, both the fibre and the base are well behaved with

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respect to local homotopy and homology properties. Every component of the fibre F as well as of the base B is homotopy locally connected; furthermore, every component of F as well as of B is a locally orientable generalized manifold. Since W^n is orientable, every component of F is orientable. Thus we may treat the fibre and the base without concern for local pathologies.

Let us recall some facts from [1] and [7]. An arcwise connected space X is an H-space if it admits a multiplication with a two-sided identity [7].

THEOREM (Spanier and Whitehead [7]). *If* $[X, Y, F; \pi]$ *is a fibre space in which the fibre* F *is contractible to a point in* X, *then* F *is an* H-*space.*

Note that in a fibre space any fibre is deformable into any other, so that if one fibre is contractible, then all fibres are contractible. The H-space is like a group, with respect to its rational cohomology algebra [1, p. 142]. For our purposes, we may assume that the H-space is a compact, connected, generalized orientable manifold, so that its rational cohomology algebra is an exterior algebra on a finite number of odd-dimensional generators. Let P denote a finite-dimensional vector space over the rationals which is graded by odd degrees; by $\wedge P$ we denote the exterior algebra generated by P. By H we denote a graded anticommutative algebra with unit over Q, and we shall assume that $H^s = 0$ for all sufficiently large s. The following is an immediate corollary to Borel's principal theorem [1, p. 157].

THEOREM (Borel). *If* $\{E_r^{s,t}\}$ *is a canonical spectral sequence with*

$$E_2^{s,t} \simeq H^s \otimes (\wedge P)^t,$$

and if $E_\infty^i = 0$ *for all* $0 < i \neq n$, *while* $E_\infty^n \simeq Q$, *then*

(a) $\wedge P$ *has a single odd-dimensional generator,*

(b) H *is a truncated polynomial ring on a single generator.*

The generator of H is the image by transgression of the generator of $\wedge P$. We shall use the algebraic form of the theorem as it is stated here. The topological analogue concerns a fibration of a sphere by an H-space. The theorem asserts that the fibre is a rational cohomology sphere.

Henceforth, this note deals only with locally compact, metric spaces. By $H^i(X; A)$ we denote the Čech-Alexander cohomology groups of X based on cochains with arbitrary closed supports; and by $H_c^i(X; A)$, the cohomology based on cochains with compact supports [3, page 15-03].

3. FIBRATIONS OF LOCALLY COMPACT SPACES

For a locally compact space X, let A(X) denote all closed subsets of the form X - V, where $V \subset X$ is an open set with $\overline{V}$ compact. The family A(X) may be indexed by B(X), the directed set of open subsets with compact closure. If $V \subset U$, then $X - V \supset X - U$. Consider all exact sequences of the form

$$\to H^i(X, X - V; A) \xrightarrow{j^*} H^i(X; A) \xrightarrow{i^*} H^i(X - V; A) \xrightarrow{\delta^*} H^{i+1}(X, X - V; A) \to$$

with $X - V \in A(X)$. These exact sequences constitute a system of direct limits indexed by B(X); passing to the direct limit, we obtain the exact sequence

$$H_c^i(X; A) \xrightarrow{j^*} H^i(X; A) \xrightarrow{i^*} I^i(X; A) \xrightarrow{\delta^*} H_c^{i+1}(X; A)$$

$$(I^i(X; A) = \text{dir lim } H^i(X - V; A), \ X - V \in A(X)) .$$

We have used the fact that direct limits preserve exactness. Now these groups $I^i(X; A)$ may also be defined directly. Let $\{C^i(X; A)\}$ be the Alexander-Wallace-Spanier (AWS) cochains of X, and $\{C_c^i(X; A)\}$, the AWS cochains of compact support; then the groups $I^i(X; A)$ are the cohomology groups of $\{C^i(X; A)/C_c^i(X; A)\}$. We shall regard $I^i(X; A)$ as the cohomology groups of the ideal boundary of X.

Let X and Y be two locally compact spaces, and let $f\colon X \to Y$ be a proper mapping; then f induces a commutative diagram:

$$\begin{array}{ccccccccc} \to H_c^i(X; A) & \xrightarrow{j^*} & H^i(X; A) & \xrightarrow{i^*} & I^i(X; A) & \xrightarrow{\delta^*} & H_c^{i+1}(X; A) & \to . \\ \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & \\ \to H_c^i(Y; A) & \xrightarrow{j^*} & H^i(Y; A) & \xrightarrow{i^*} & I^i(Y; A) & \xrightarrow{\delta^*} & H_c^{i+1}(Y; A) & \to \end{array}$$

This may be seen immediately from the second definition of $I^i(X; A)$; but it also follows from the first definition and the observation that the open sets in X of the form $f^{-1}(V)$, with $V \in B(Y)$, form a cofinal sequence in B(X). We may take the direct limit of

$$f_V^*\colon H^i(Y - V; A) \to H^i(X - f^{-1}(V); A)$$

for the definition of $f^*\colon I^i(Y; A) \to I^i(X; A)$.

We shall say that X is arcwise connected at infinity if for each open set $V \in B(X)$ there exists a $U \in B(X)$, with $\overline{V} \subset U$ and $X - U$ arcwise connected. The space X is LC^n at infinity if for each $V \in B(X)$ for which $X - V$ is arcwise connected, there exists a $U \in B(X)$ with $\overline{V} \subset U$ such that $X - U$ is arcwise connected and such that the homomorphism $\pi_i(X - U) \to \pi_i(X - V)$ is trivial for $i \leq n$.

We remark that if $n \geq 2$ and the manifold W^n is locally Euclidean at infinity, then $I^i(W^n, A) \simeq H^i(S^{n-1}; A)$; also, W^n is LC^{n-2}. Furthermore, since W^n is orientable, the homomorphism $i^*\colon H^{n-1}(W^n; A) \to I^{n-1}(W^n; A)$ is trivial.

Let us consider a fibration $[X, Y, F; \pi]$ with compact fibre. For any open set $V \in B(Y)$ there is an induced bundle $[X - \pi^{-1}(V), Y - V, F; \pi]$ and a spectral sequence $\{E_r^{s,t}(V)\}$, with

$$E_2^{s,t}(V) \simeq H^s(Y - V; H^t(F; A)) ,$$

whose E_∞-term is associated with $H^*(X - \pi^{-1}(V); A)$. Now elements of the form $\pi^{-1}(V)$ with $V \in B(Y)$ are cofinal in B(X). The bundles $[X - \pi^{-1}(V), Y - V, F; \pi]$ are directed by B(Y); the same is true of the spectral sequences $\{E_r^{s,t}(V)\}$ that make up a natural direct limit system. We may define

$$I_r^{s,t} \approx \text{dir lim } E_r^{s,t}(V)$$

and so obtain a new spectral sequence. Clearly, we have

LEMMA 3.1. *Given a fibration by a compact fibre* F, *there is a spectral sequence* $\{I_r^{s,t}\}$, *with*

$$I_2^{s,t} = \text{dir lim } H^s(Y - V; H^t(F; A)),$$

whose E_∞*-term is associated with* $I^*(X; A)$.

Suppose that $V \in B(Y)$ is such that $Y - V$ is arcwise connected and

$$\pi_1(Y - V) \to \pi_1(Y)$$

is trivial. Then, in the induced bundle $[X - \pi^{-1}(V), Y - V, F; \pi]$, the group $\pi_1(Y - V)$ acts trivially on the cohomology of the fibre; thus, for a field k of coefficients,

$$E_2^{s,t}(V) \simeq H^s(Y - V; k) \otimes H^t(F; k).$$

If Y is LC^1 at infinity, there is a cofinal set $\{V\}$ in $B(Y)$ for which the homomorphism $\pi_1(Y - V) \to \pi_1(Y)$ is trivial.

LEMMA 3.2. *If, in the fibration* $[X, Y, F; \pi]$ *with compact fibre, the space* Y *is* LC^1 *at infinity, then the spectral sequence of Lemma* 3.1 *has an initial term of the form*

$$I_2^{s,t} \simeq I^s(Y; k) \otimes H^t(F; k), \tag{1}$$

where k *is a field.*

We observe without proof that if in $[X, Y, F; \pi]$ the space X is LC^1 at infinity and F is connected, then Y is also LC^1 at infinity.

We shall now introduce the analogue of the spectral sequence of a finite covering for the ideal boundary. Let G denote a finite group, and let $H^i(G; A)$ denote the cohomology groups of the group G with coefficients in the G-module A (see [3], [4]). Let (G, X) denote the action of G as a group of covering transformations on X, and let X/G denote the quotient space of (G, X). There is a spectral sequence (see [3], [4]), with

$$E_2^{s,t} \simeq H^s(G; H^t(X; A)),$$

whose E_∞-term is associated with $H^*(X/G; A)$. Since the natural map $\pi: X \to X/G$ is proper, a limiting process again yields a spectral sequence, with

$$E_2^{s,t}(G) \simeq H^s(G; I^t(X; A)), \tag{2}$$

whose E_∞-term is associated with $I^*(X/G; A)$.

Let K be a G-free acyclic complex. Given (G, X), we form the auxiliary transformation group $(G, K \times X)$, where $g(y, x) = (gy, gx)$, and the natural diagram

$$\begin{array}{ccc} & (K \times K)/G & \\ \swarrow^{\alpha} & & \downarrow^{\beta} \\ K/G & & X/G \end{array}$$

Since $\beta^*: H^i(X/G; A) \simeq H^i((K \times X/G); A)$, the formula $\eta^* = \beta^{*-1}\alpha^*$ induces the characteristic homomorphism

$$\eta^*: H^i(G; A) \to H^i(X/G; A).$$

A second characteristic homomorphism $\nu^*: H^i(G; A) \to I^i(X/G; A)$ can be defined for (G, X), by means of direct limits, in such a way that the diagram

$$\begin{array}{ccc} H^i(X/G; A) & \xrightarrow{i^*} & I^i(X/G; A) \\ {\scriptstyle \eta^*} \nwarrow & & \nearrow {\scriptstyle \nu^*} \\ & H^i(G; A) & \end{array}$$

is commutative.

4. THE THEOREM FOR A CONNECTED FIBRE

We shall prove the theorem under the assumption that the fibre is connected. Returning to the map $\hat{\pi}: M^n \to \hat{B}$, we choose a closed n-cell $e^n \subset M^n$ with origin p. Since M^n is compact, there is a closed neighborhood $N_g \subset \hat{B}$ such that $\hat{\pi}^{-1}(N_g)$ lies in the interior of e^n.

LEMMA 4.1. *No fibre in* $\hat{\pi}^{-1}(N_g)$ *can separate* p *from the boundary of* e^n.

Let us assume the contrary. Then dim F = n - 1, and B is an open unit interval, while $\hat{B}$, its one-point compactification, is a circle. The map $\hat{\pi}: M^n \to S^1$ is open, onto and monotonic; thus $p = \hat{\pi}^{-1}(g)$ is a local cut point of M^n, which is impossible for $n \geq 2$.

LEMMA 4.2. *If* F *is connected and the fibring* $[W^n, B, F; \pi]$ *is compact, then* F *is an* H-*space*.

By Lemma 4.1, the fibre lying in the cell e^n does not separate the origin from the boundary; thus it may be contracted, in the complement of the origin. As observed earlier, $I^i(W^n; Q) \simeq H^i(S^{n-1}; Q)$. We may apply the result of Borel, referred to in Section 2, to the spectral sequence (1) of Lemma 3.2.

LEMMA 4.3. *If* F *is connected and the fibring* $[W^n, B, F; \pi]$ *is compact, then* F *is a rational cohomology* r-*sphere* (r *odd*), *and* I*(B; Q) *is a truncated polynomial ring with an* (r + 1)-*dimensional generator* C^{r+1}.

We now study $H^s(B; H^r(F; Q))$. By hypothesis, $\pi_1(W^n) = 0$ and F is connected; therefore $\pi_1(B) = 0$, and it follows that

$$H^s(B; H^r(F; Q)) \simeq H^s(B; Q).$$

Actually, the hypothesis $\pi_1(W^n) = 0$ is slightly stronger than needed; but we have no simple substitute condition. We may write down the parallel Gysin sequences [4, p. 328] for $[W^n, B, F; \pi]$ and $\{I_r^{s,t}\}$,

$$\begin{array}{ccccccccc} \to H^{n-r-2}(B; Q) & \to & H^{n-1}(B; Q) & \to & H^{n-1}(W^n; Q) & \xrightarrow{h^*} & H^{n-r-1}(B; Q) & \to. \\ \downarrow & & \downarrow & & i^* \downarrow & & i^* \downarrow & \\ \to I^{n-r-2}(B; Q) & \to & I^{n-1}(B; Q) & \to & I^{n-1}(W^n, Q) & \xrightarrow{h^*} & I^{n-r-1}(B; Q) & \to \end{array}$$

The homomorphism $i_1^*: H^i(B; Q) \to I^i(B; Q)$ $(i \geq 0)$ is onto; for the image by transgression of the fundamental class of the fibre in $[W^n, B, F; \pi]$ must map into the generator C^{r+1} of I*(B; Q). For $r \geq 1$, dim $B \leq n - 1$, and therefore $H^n(B; Q) = 0$ and $I^{n-1}(B; Q) = I^n(B; Q) = 0$. Thus the homorphism

$$h^*: H^{n-1}(W^n; Q) \to H^{n-r-1}(B; Q)$$

is onto, and $i_1^*: H^{n-r-1}(B; Q) \to I^{n-r-1}(B; Q)$ is also onto; but since W^n is orientable, the homomorphism $i^*: H^{n-1}(W^n; Q) \to I^{n-1}(W^n; Q)$ is trivial; since also $i_1^* h^* = h^* i^*$, it follows that $I^{n-1}(W^n; Q) = 0$, which contradicts the fact that $I^{n-1}(W^n; Q) \approx Q$. This proves Theorem 1 under the assumption that F is connected.

5. THE THEOREM IN GENERAL

To complete the argument, we shall prove a result communicated to us by Mostow.

THEOREM (Mostow). *If* (Z_p, M^n) *is a transformation group on a compact manifold, orientable* mod p, *then* Z_p *can not have exactly one fixed point.*

Suppose this is false; then, by removing the one fixed point, we get a fixed-point free transformation group (Z_p, W^n) on a manifold which is orientable mod p and locally Euclidean at infinity. The quotient space W^n/Z_p will also be orientable mod p, so that the homomorphism

$$i^*: H^{n-1}(W^n/Z_p; Z_p) \to I^{n-1}(W^n/Z_p; Z_p)$$

is trivial. Using the characteristic homomorphism as introduced in the conclusion of Section 3, we see that the homomorphism

$$\nu^*: H^{n-1}(Z_p; Z_p) \to I^{n-1}(W^n/Z_p; Z_p)$$

is trivial. On the other hand, by using the spectral sequence (2) of Section 3, together with the fact that $I^i(W^n; Z_p) \approx H^i(S^{n-1}; Z_p)$, we immediately show [4] that

$$\nu^*: H^i(Z_p; Z_p) \simeq I^i(W^n/Z_p; Z_p) \qquad (0 \leq i \leq n-1),$$

which is a contradiction, since $H^{n-1}(Z_p; Z_p) \approx Z_p$.

Corollary. *If* W^n *is an open, orientable manifold which is locally Euclidean at infinity, then* W^n *does not admit a nontrivial finite group of transformations without fixed points.*

If $[W^n, B, F; \pi]$ is a compact fibration, and F is compact, it follows from Section 4 that F consists of a finite number of points, and from our corollary, that F is a single point, hence not proper. This concludes our proof of Theorem 1.

6. SINGULAR FIBRATIONS

Our methods may be used to generalize some results in [5] about singular fibrations. We shall state the extensions without proof. By $[(X, A), (Y, B), F; \pi]$ we denote an open onto map $\pi: (X, A) \to (Y, B)$ satisfying

(i) $\pi^{-1}(B) = A$,

(ii) $\pi | A \to B$ is a homeomorphism,

(iii) $\pi | X - A \to Y - B$ is a fibre mapping with fibre F.

The map $[(X, A), (Y, B), F; \pi]$ is called a singular fibration with fibre F and singular set A. Since Lemma 4.3 is of a local nature, we may state

THEOREM 2. *If* $[(M^n, A), (Y, B), F; \pi]$ *denotes a singular fibration of the compact manifold* M^n, *by a fibre* F *of dimension less than* $n - 1$, *and if the singular set* A *contains an isolated point, then the fibre is an odd-dimensional rational cohomology sphere.*

COROLLARY. *If* $[(M^n, A), (Y, B), F; \pi]$ *is a singular fibration of a compact, simply connected manifold by a fibre of dimension less than* $n - 1$, *and if the singular set consists of a finite number of points, then the number of points equals* $\chi(M^n)$.

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