THE NUMBER OF ORIENTED GRAPHS

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This note is a continuation of the paper [2], whose notation and terminology will be used. A graph G is *oriented* if to each line ab in G precisely one of the two orientations $\overrightarrow{ab}$ and $\overrightarrow{ba}$ is assigned. We note that not all directed graphs are oriented graphs; for in the latter each pair of points is joined by at most one directed line, while in the former there may be two such directed lines, one in each direction. Thus an oriented graph is a one-dimensional oriented simplicial complex. Two oriented graphs are *isomorphic* if there is a one-to-one correspondence between their point sets which preserves directed lines. Let θ_{pq} be the number of (nonisomorphic) oriented graphs with p points and q lines, and let

$$\theta_p(x) = \sum_{q=0}^{p(p-1)/2} \theta_{pq} x^q \tag{1}$$

be the counting polynomial for oriented graphs with p points. Our object is to obtain a formula for $\theta_p(x)$. As in [2], we use the enumeration methods of Pólya [3].

Davis [1] has recently counted several kinds of binary relations on p objects. His results include a formula for asym (p), the number of nonisomorphic asymmetric relations defined on p objects (a *relation* is a set of ordered couples; it is *asymmetric* if it is both irreflexive and antisymmetric). Thus Davis' number asym (p) is, in our notation $\Sigma_{q=0}^{p(p-1)/2} \theta_{pq}$, that is, the total number or oriented graphs with p points; therefore the formula which we shall obtain is a refinement of that of Davis.

In the framework of Pólya's Theorem (see the Hauptsatz of [3] or [2, Section 2]), an oriented graph of p points is a *configuration* whose figures are the p(p - 1)/2 pairs of points. The *content* of a figure is the number of directed lines it contains, and it is thus zero or one. However, the pair of vertices a and b in a figure can occur in two different directed lines $\overrightarrow{ab}$ and $\overrightarrow{ba}$. Therefore the *figure counting series* $\phi(x)$ for oriented graphs is

$$\phi(x) = 1 + 2x, \tag{2}$$

since there is one figure of content zero and two of content one. In order to apply Pólya's Theorem, it remains to find the cycle index of the configuration group Q_p. This group is a permutation group of degree p(p - 1)/2; but as an abstract group it is isomorphic with S_p, the symmetric group of degree p.

The *cycle index* of S_p is

$$Z(S_p) = \frac{1}{p!} \sum_{(j)} \frac{p!}{1^{j_1} j_1!\, 2^{j_2} j_2! \cdots p^{j_p} j_p!} f_1^{j_1} f_2^{j_2} \cdots f_p^{j_p}\,, \tag{3}$$

where the letters f_k are variables and where the summation is taken over all partitions $(j_1, j_2, \cdots, j_p)$ of p satisfying

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(4) $$1j_1 + 2j_2 + \cdots + pj_p = p.$$

The term of $Z(Q_p)$ corresponding to the general term $f_1^{j_1} f_2^{j_2} \cdots f_p^{j_p}$ of $Z(S_p)$ is derived in a manner analogous to that for ordinary graphs [2, (10)], with one essential modification. The change is this: if in a cycle of a permutation of Q_p induced by a permutation of S_p a pair of points occurs twice (once in each order), then the cycle is deleted; for example, the cycle (2 3) induces the product (2 3) (3 2), and this must be cancelled. To illustrate,

$$Z(S_3) = \frac{1}{6}(f_1^3 + 3f_1 f_2 + 2f_3)$$

induces the cycle index

$$Z(Q_3) = \frac{1}{6}(e_1^3 + 3e_2 + 2e_3),$$

where we use the letters e_k as indeterminates in $Z(Q_p)$; and corresponding to

$$Z(S_4) = \frac{1}{24}(f_1^4 + 6f_1^2 f_2 + 8f_1 f_3 + 3f_2^2 + 6f_4),$$

we have

$$Z(Q_4) = \frac{1}{24}(e_1^6 + 6e_1 e_2^2 + 8e_3^2 + 3e_2^2 + 6e_4).$$

The contribution to $Z(Q_p)$ from $f_1^{j_1} f_2^{j_2} \cdots f_p^{j_p}$ can be separated into two independent parts which are then multiplied to yield the result. The first part comes from the pairs of points lying on cycles of permutations of S_p of the same length; the second part from all remaining pairs.

If one divides the first part into the contributions from the odd and the even cycles, then (exactly as for ordinary graphs) the odd cycles yield

(5') $$f_{2n+1}^{j_{2n+1}} \rightarrow e_{2n+1}^{t_{2n+1}}, \quad \text{where } t_{2n+1} = nj_{2n+1} + (2n+1)\binom{j_{2n+1}}{2};$$

however, for oriented graphs, the even cycles yield

(5'') $$f_{2n}^{j_{2n}} \rightarrow e_{2n}^{t_{2n}}, \quad \text{where } t_{2n} = (n-1)\, j_{2n} + 2n\binom{j_{2n}}{2},$$

and this is different from the corresponding expression for graphs. Obviously, (5') and (5'') can be combined to obtain the following transformation, which is independent of the parity of the subscript:

(5) $$f_k^{j_k} \rightarrow e_k^{t_k}, \quad \text{where } t_k = \left[\frac{k-1}{2}\right] j_k + k\binom{j_k}{2}.$$

Finally, the contribution from cycles of different length not already obtained is (as for linear graphs)

$$f_r^{j_r} f_s^{j_s} \rightarrow e_{[r,s]}^{j_r j_s (r,s)}, \tag{6}$$

where (r, s) and [r, s] denote the g. c. d. and l. c. m. respectively. Note that the exponents of the right-hand members of (5) and (6) are the expressions under the two summation signs occurring in the exponent of the number 3 in Davis' formula for asym (p). Combining the results (5) and (6), we obtain

$$f_1^{j_1} f_2^{j_2} \cdots f_p^{j_p} \rightarrow \left(\prod_{k=1}^{p} e_k^{t_k} \right) \left(\prod_{1 \leq r < s \leq p} e_{[r,s]}^{j_r j_s (r,s)} \right). \tag{7}$$

Substituting the right-hand member of (7) for the occurrence of the left-hand member in (3), we obtain $Z(Q_p)$, the cycle index of the configuration group. Applying Pólya's Theorem, we have

$$\theta_p(x) = Z(Q_p, 1 + 2x). \tag{8}$$

This formula may be illustrated by substituting $1 + 2x^k$ for e_k in $Z(Q_3)$ above to obtain the polynomial

$$\theta_3(x) = 1 + x + 3x^2 + 2x^3,$$

which is verified pictorially by Figure 1.

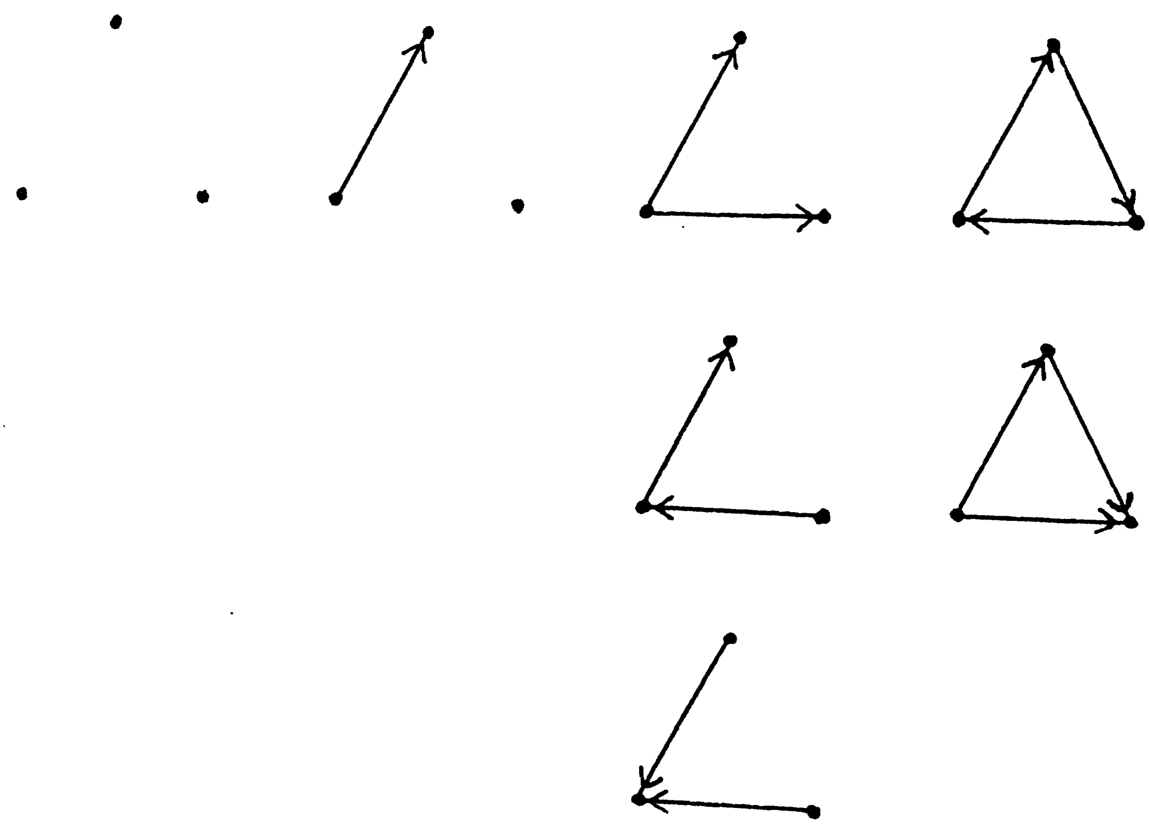

Figure 1

The counting polynomials for $\theta_p(x)$ through $p = 5$ are given by

$$
\begin{aligned}
\theta(x,y) &= \sum_{p=1}^{\infty} \theta_p(x)\, y^p \\
&= y + y^2(1 + x) \\
&\quad + y^3(1 + x + 3x^2 + 2x^3) \\
&\quad + y^4(1 + x + 4x^2 + 10x^3 + 12x^4 + 10x^5 + 4x^6) \\
&\quad + y^5(1 + x + 4x^2 + 13x^3 + 41x^4 + 78x^5 + 131x^6 + 144x^7 + 107x^8 + 50x^9 + 12x^{10}) \\
&\quad + y^6(1 + x + 4x^2 + 14x^3 + 55x^4 + 187x^5 + 539x^6 + 1292x^7 + 2500x^8 + 3817x^9 \\
&\qquad + 4512x^{10} + 4112x^{11} + 2740x^{12} + 1274x^{13} + 376x^{14} + 56x^{15}) \\
&\quad + \cdots .
\end{aligned} \tag{9}
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REFERENCES

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