BILINEAR INTEGRALS OF POLYHARMONIC FUNCTIONS AND OF ANALYTIC FUNCTIONS

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INTRODUCTION

In Part I of this paper we show that any two functions polyharmonic in open subsets of a euclidean space satisfy a certain identity which involves a bilinear integral. This identity is a generalization of a bilinear integral identity for harmonic functions which was derived by Gustin [4]. Gustin applied his identity to obtain new proofs of several fundamental theorems in the theory of harmonic functions. Similarly, we apply our identity to obtain some new proofs and new theorems in the theory of polyharmonic functions. In Part II we generalize our method to the case of arbitrary analytic functions of real variables.

PART I

1. For $i = 1, 2$, let $f_i(q_i)$ be a real-valued function which is n_i-harmonic in an open set D_i of the euclidean N-dimensional space E^N $(N \geq 2)$. We always denote by x a point on the unit sphere S. Suppose that q_i is a fixed point of D_i and that the whole sphere Q_i with center q_i and radius R_i is contained in D_i. We then consider the bilinear integral

$$(1) \qquad I(r_1, r_2) = \int_S f_1(q_1 + r_1 x) f_2(q_2 + r_2 x)\, dS_x,$$

defined for $0 \leq r_1 \leq R_1$, $0 \leq r_2 \leq R_2$. Using the Almansi development [1] for polyharmonic functions, we write f_i in the form

$$f_i(q_i + r_i x) = \sum_{j=0}^{n_i - 1} r_i^{2j} f_{ij}(q_i + r_i x) \qquad (i = 1, 2),$$

where the f_{ij} are functions harmonic in Q_i. Substituting this in (1), we get

$$(2) \qquad I(r_1, r_2) = \sum_{j=0}^{n_1 - 1} \sum_{k=0}^{n_2 - 1} r_1^{2j} r_2^{2k} \int_S f_{1j}(q_1 + r_1 x) f_{2k}(q_2 + r_2 x)\, dS_x.$$

Gustin [4] proved that for any two harmonic functions g_1 and g_2 the bilinear integral

$$\int_S g_1(q_1 + r_1 x)\, g_2(q_2 + r_2 x)\, dS_x$$

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depends only on q_1, q_2 and $\sqrt{r_1 r_2}$. Using this fact, we can write (2) in the form

$$I(r_1, r_2) = \sum_{j=0}^{n_1-1} \sum_{k=0}^{n_2-1} A_{jk}(q_1, q_2, \sqrt{r_1 r_2})\, r_1^{2j} r_2^{2k}, \tag{3}$$

where the A_{jk} are defined by

$$A_{jk}(q_1, q_2, \sqrt{r_1 r_2}) = \int_S f_{1j}(q_1 + r_1 x) f_{2k}(q_2 + r_2 x)\, dS_x. \tag{4}$$

2. We shall apply identity (3) to derive a nonlinear characterization for polyharmonic functions.

THEOREM 1. *A function* $f(q)$, *continuous in an open set* D *of* E^N, *is* n-*harmonic in* D *if and only if an identity of the form*

$$\int_S f(q + r_1 x) f(q + r_2 x)\, dS_x = \sum_{j,k=0}^{n-1} A_{jk}(q, r)\, r_1^{2j} r_2^{2k} \qquad (r = \sqrt{r_1 r_2}) \tag{5}$$

holds for all nonnegative numbers r_1, r_2 *for which the whole sphere of radius* $\max(r_1, r_2)$ *about* q *is contained in* D.

Proof. Taking $f_1 = f_2$, $q_1 = q_2$ in (3), we see that if $f(q)$ is n-harmonic in D, then (5) holds.

To prove the converse, suppose that (5) is satisfied. Taking $r_2 = 0$, we obtain

$$f(q) \int_S f(q + r_1 x)\, dS_x = \sum_{j=0}^{n-1} A_{j0}(q, 0)\, r_1^{2j}. \tag{6}$$

Let D_1 be the set of points of D at which $f(q) \neq 0$, and let D_2 be the interior of the complement of D_1 relative to D. Clearly, every point q of D_2 has a neighborhood where $f = 0$. From (6) it follows that if $q \in D_2$, then the polynomial on the right side of (6) vanishes for all sufficiently small positive r_1, and consequently

$$A_{j0} = 0 \quad (0 \leq j \leq n - 1).$$

We conclude that the identity

$$\int_S f(q + r_1 x)\, dS_x = \sum_{j=0}^{n-1} B_j(q)\, r_1^{2j} \tag{7}$$

holds for all sufficiently small positive r_1, with $B_j(q) = 0$ $(0 \leq j \leq n - 1)$. If $q \in D_1$, then $f(q) \neq 0$, and from (6) we derive (7), with

$$B_j(q) = \frac{A_{j0}(q, 0)}{f(q)}.$$

By means of the identity (7), we can express the $B_j(q)$ in terms of

$$\int_S f(q + r_1^k x)\, dS_x \qquad (k = 1, 2, \cdots, n),$$

where the r_1^k are nonnegative numbers, arbitrary except for the requirement that $r_1^k \neq r_1^l$ if $k \neq l$. It follows that the $B_j(q)$ can be defined throughout $\overline{D}_1 \cap D$ by continuity, and we conclude that (7) holds for every $q \in \overline{D}_1 \cap D$. Thus we have proved (7) for every q in D.

We now apply a theorem due to Nicolesco [6], which states that if a function $f(q)$, summable in an open set D, satisfies the condition

$$(8) \qquad \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{N}{N+2} & \cdots & \frac{N}{N+2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \left(\frac{N}{N+2}\right)^{n-1} & \cdots & \left(\frac{N}{N+2n-2}\right)^{n-1} \end{vmatrix} f(q) = \begin{vmatrix} \mu_0 & 1 & \cdots & 1 \\ \mu_1 & \frac{N}{N+2} & \cdots & \frac{N}{N+2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \left(\frac{N}{N+2}\right)^{n-1} & \cdots & \left(\frac{N}{N+2n-2}\right)^{n-1} \end{vmatrix}$$

for all sufficiently small positive r, where

$$\mu_0 = \mu_0(q; r) = \int_S f(q + rx)\, dS_x$$

and

$$\mu_i = \mu_i(q; r) = \frac{N}{r^N} \int_0^r \rho^{N-1} \mu_{i-1}(q; \rho)\, d\rho \qquad (i = 1, 2, \cdots, n-1),$$

then $f(q)$ has $2n$ continuous derivatives and is n-harmonic in D. Now from (7) it follows easily that

$$(9) \qquad \mu_i(q; r) = B_0(q) + \sum_{j=1}^{n-1} B_j(q) \left(\frac{N}{N+2j}\right)^i r^{2j} \qquad (i = 0, 1, \cdots, n-1).$$

The continuity of $f(q)$ implies that $B_0(q) = f(q)$; solving (9) for $f(q)$, we obtain (8). By Nicolesco's theorem, f is n-harmonic in D.

3. *Definition.* A function $g(x)$ defined on the unit sphere is called surface n-harmonic of degree t if there exists an n-harmonic function $f(q)$, defined for all q of E^N, such that for each $x \in S$ and for every nonnegative number r,

$$f(rx) = r^t g(x).$$

It is easy to see that $f(q)$ must be a homogeneous polynomial of degree t.

THEOREM 2. *Let* $g_i(x)$ $(i = 1, 2)$ *be a surface* n_i*-harmonic function of degree* t_i. *Then* $g_1(x)$ *and* $g_2(x)$ *are orthogonal if one of the following conditions holds:*

1) $t_1 - t_2$ *is odd;*

2) $t_1 - t_2$ *is even and either* $t_1 - t_2 > 2n_1 - 2$ *or* $t_2 - t_1 > 2n_2 - 2$.

Proof. We have to show that the bilinear integral $A = \int_S g_1(x)\, g_2(x)\, dS_x$ is zero.

Denoting by $f_i(rx)$ $(i = 1, 2)$ the n_i-harmonic function associated with $g_i(x)$, we have

$$(10) \qquad \int_S f_1(r_1x)\, f_2(r_2x)\, dS_x = r_1^{t_1} r_2^{t_2} \int_S g_1(x)\, g_2(x)\, dS_x = A\, r_1^{t_1} r_2^{t_2}.$$

By the identity (3), the left side of (10) is equal to

$$\sum_{j=0}^{n_1-1} \sum_{k=0}^{n_2-1} A_{jk}(\sqrt{r_1 r_2})\, r_1^{2j}\, r_2^{2k}.$$

Comparing this expression with the right side of (10) and taking $r_1 = r$, $r_2 = r^{-1}$, we get the identity

$$A\, r^{t_1 - t_2} = \sum_{j=0}^{n_1-1} \sum_{k=0}^{n_2-1} A_{jk}(1)\, r^{2(j-k)},$$

from which the assertion of the theorem follows.

It should be remarked that Theorem 2 is sharp in the following sense: If $t_1 - t_2$ is even and $0 \leq t_1 - t_2 = 2k \leq 2n_1 - 2$, then there exist surface n_i-harmonic functions $g_i(x)$, of degrees t_i, such that $\int_S g_1(x)\, g_2(x)\, dS_x \neq 0$. To show this, let $g(x)$ be an arbitrary surface harmonic function of degree t_2; then it is also surface n_2-harmonic of the same degree t_2. Since $r^{t_2} g(x)$ is harmonic, $r^{2k} r^{t_2} g(x)$ is $(k+1)$-harmonic (see [1]), and since $k \leq n_1 - 1$, it is also n_1-harmonic. Therefore $g(x)$ is surface n_1-harmonic of degree $2k + t_2 = t_1$; but it is also surface n_2-harmonic of degree t_2, and $\int_S g(x)\, g(x)\, dS_x \neq 0$.

4. LEMMA. *Let* f *be an* n-*harmonic function in a sphere* Q *with center* q *and radius* R_2, *and suppose that* f *vanishes in a concentric sphere of radius* $R_1 < R_2$. *Then* f *vanishes identically in* Q.

Proof. If the assertion did not hold, then we could assume that there does not exist a sphere of radius greater than R_1 in which $f \equiv 0$. We shall derive a contradiction.

Substituting $r_1 = sr$, $r_2 = s^{-1}r$ in (5) (so that $r = \sqrt{r_1 r_2}$, $s = \sqrt{r_1 r_2^{-1}}$), we obtain

$$(11) \qquad \int_S f(q + r_1x)\, f(q + r_2x)\, dS_x = \sum s^{2k} \left(\sum r^{2j} A(q,r) \right) \equiv \sum{}' c_k(r)\, s^{2k}.$$

The left side of (11) vanishes if $0 < r_1 < R_1$, $0 < r_2 < R_2$, so that $\Sigma c_k(r)\, s^{2k} = 0$ for $s_1 < s < s_2$ and $s_i = s_i(r)$ $(i = 1, 2)$. We write $\Sigma c_k(r)\, s^{2k}$ in the form

$$s^{-1} \sum_{k=0}^{m} d_k(r)\, s^k;$$

then the polynomial $\Sigma d_k(r)\, s^k$ vanishes throughout the interval $s_1 < s < s_2$; consequently, it vanishes identically, hence $\Sigma c_k(r)\, s^{2k} = 0$ for $0 < s < \infty$, and therefore $c_k(r) = 0$ for every k. Note that in this argument we have not used the usual method of series expansion.

In particular, it follows that $c_k(r) = 0$ for $R_1 < r < \sqrt{R_1 R_2}$, and so the left side of (11) vanishes for $r_1 = r_2 = r$; that is,

$$\int_S f(q + rx)^2\, dS_x = 0 \qquad (R_1 < r < \sqrt{R_1 R_2}).$$

This implies that f vanishes in a sphere of radius $\sqrt{R_1 R_2}$, which is a contradiction. Using the lemma above, we obtain by a well-known argument (see, for instance, [5] p. 250) a new proof of the following result.

THEOREM 3. *If a function* f *is* n-*harmonic in a domain* D *of* E^N *and vanishes identically over some nonempty open set, then it vanishes identically in* D.

It should be remarked that our proof, which is a generalization of Gustin's proof to the harmonic case, is of interest in that it makes no use of series expansion.

5. We shall give one more application of the identity (5), by proving a generalization of Liouville's theorem for polyharmonic functions.

THEOREM 4. *If a function* f *is* n-*harmonic throughout* E^N, *and if*

$$\int_S |f(r_k x)|\, dS_x \leq H \qquad (k = 1, 2, \cdots) \tag{12}$$

for some sequence $\{r_k\}$ $(r_k \nearrow \infty)$, *then* $f \equiv \text{const.}$

Proof. It is clear that we may assume that $f(0) = 0$. Using (5), we get the formulas

$$\int_S [f(x)]^2 dS_x = \sum_{j,k=0}^{n-1} A_{jk}(1), \tag{13}$$

$$\int_S f(r^{-1}x)\, f(rx)\, dS_x = \sum_{j,k=0}^{n-1} r^{-2j} r^{2k} A_{jk}(1) \equiv \Sigma\, r^{2m}\Big(\sum_{k-j=m} A_{jk}(1)\Big). \tag{14}$$

By (12) and the assumption that $f(0) = 0$, we have

$$\lim_{k\to\infty} \left| \int_S f(r_k^{-1}x)\, f(r_k x)\, dS_x \right| \leq H \limsup_{k\to\infty\, x\in S} |f(r_k^{-1}x)| = 0.$$

We conclude from (14) that the coefficients of r^{2m} vanish, for $m \geq 0$. Hence $\Sigma_{k-j=m} A_{jk}(1) = 0$ if $m \geq 0$. If in (14) we replace r by r^{-1} and r^{-1} by r, the left side remains unchanged, while the right side takes the form

$$\sum_{j,k=0}^{n-1} r^{2j} r^{-2k} A_{jk}(1) \equiv \sum r^{2m}\Big(\sum_{j-k=m} A_{jk}(1) \Big) .$$

We conclude, as before, that $\Sigma_{j-k=m} A_{jk}(1) = 0$ if $m \geq 0$. Using these remarks and (13), we have

$$\int_S [f(x)]^2 \, dS_x = \sum_{j,k=0}^{n-1} A_{jk}(1) = \sum_m \Big(\sum_{j-k=m} A_{jk}(1) \Big) = 0,$$

so that $f(x) \equiv 0$.

Let r be an arbitrary positive number; then the function $g(q) = f(rq)$ satisfies all the assumptions of Theorem 4; consequently, by the previous argument, we have $f(rx) = g(x) \equiv 0$, which proves the theorem.

PART II

1. Let $f_i(q_i)$ $(i = 1, 2)$ be real-valued functions, analytic in an open set D_i of the euclidean N-dimensional space E^N $(N \geq 2)$. As in Part I, we consider the bilinear integral

$$(15) \qquad I(r_1, r_2) = \int_S f_1(q_1 + r_1 x) f_2(q_2 + r_2 x) \, dS_x ,$$

defined for $0 \leq r_1 \leq R$, $0 \leq r_2 \leq R$. We shall use the following development, which is due to Cioranesco [2]:

$$(16) \qquad f_i(q_i + r_i x) = \sum_{j=0}^{\infty} r_i^{2j} f_{ij}(q_i + r_i x) \qquad (i = 1, 2),$$

where the f_{ij} are harmonic functions and the series converges absolutely and uniformly in a sufficiently small neighborhood of q_i. Substituting (16) in (15), we get the identity

$$(17) \qquad I(r_1, r_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} r_1^{2j} r_2^{2k} A_{jk}(q_1, q_2, \sqrt{r_1 r_2}),$$

where

$$A_{jk}(q_1, q_2, \sqrt{r_1 r_2}) = \int_S f_{1j}(q_1 + r_1 x) f_{2k}(q_2 + r_2 x) \, dS_x .$$

This identity is a generalization of the identity (3). If $f_1(q_1)$ is n-harmonic, then the identity (17) takes the form:

$$I(r_1, r_2) = \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} r_1^{2j} r_2^{2k} A_{jk}(q_1, q_2, \sqrt{r_1 r_2}). \tag{18}$$

2. We shall discuss briefly some generalizations of the results of Part I. The method we used in the proof of Theorem 3 cannot be generalized without using the method of series expansion for the case of a power series in one variable. As to Theorem 4, the use of bilinear integrals seems to involve complications. Using a linear integral, namely, the well known Pizzetti formula

$$\frac{1}{|S|} \int_S f(rx)\, dS_x = \Gamma(N/2) \sum_{k=0}^{\infty} \frac{\Delta^k f(0)\, r^{2k}}{4^k k!\, \Gamma(k + N/2)}, \tag{19}$$

where $|S|$ is the area of S, we can prove a generalization of Liouville's theorem. This was already done by the author [3]. It remains to discuss the generalization of Theorem 2.

Definition. A function $g(x)$ defined on the unit sphere is called surface analytic of degree t if there exists an analytic function $f(q)$, defined for all q in E^N, such that, for each $x \in S$ and for every nonnegative number r,

$$f(rx) = r^t g(x).$$

THEOREM 5. *Let* $g_1(x)$ *be a surface* n*-harmonic function of degree* t_1, *and let* $g_2(x)$ *be a surface analytic function of degree* t_2. *Then* $g_1(x)$ *and* $g_2(x)$ *are orthogonal if one of the following conditions hold:*

1) $t_1 - t_2$ *is odd;*

2) $t_1 - t_2$ *is even and* $t_1 - t_2 > 2n - 2$.

Proof. Using the definition of surface analytic and surface n-harmonic functions and applying (18), we get the identity

$$\sum_{j=0}^{n-1} \sum_{k=0}^{\infty} A_{jk}(\sqrt{r_1 r_2})\, r_1^{2j} r_2^{2k} = I(r_1, r_2) = r_1^{t_1} r_2^{t_2} \int_S g_1(x) g_2(x)\, dS_x,$$

valid for $0 < r_1 < R'$, $0 < r_2 < R''$. Take $r_1 = s^{-1}r$, $r_2 = sr$; then

$$\sum_{j=0}^{n-1} \sum_{k=0}^{\infty} A_{jk}(r)\, r^{2(k+j)} s^{2(k-j)} = r^{t_1+t_2} s^{t_2-t_1} \int_S g_1(x) g_2(x)\, dS_x. \tag{20}$$

For each r, (20) holds for $s_1 < s < s_2$, where $s_i = s_i(r)$ $(i = 1, 2)$. If we take r to be a fixed number and compare the coefficients of $s^{t_2-t_1}$ on both sides of (20), the theorem follows.

Theorem 5 is sharp in the following sense: If $0 \leq t_1 - t_2 = 2k \leq 2n - 2$, then there exist surface analytic and surface n-harmonic functions, of degrees t_2 and t_1, respectively, which are not orthogonal. This follows from the analogous remark to Theorem 2, if we observe that polyharmonic functions are analytic.

The generalization of Theorem 5 to the case of two analytic functions can be proved by using (14), or even by using (19), but it follows almost immediately if we observe that a surface analytic function of degree t must be a homogeneous polynomial of degree t. We have the following result, which overlaps with Theorems 2 and 5.

THEOREM 6. *If* $g_1(x)$ *and* $g_2(x)$ *are surface analytic functions of degrees* t_1 *and* t_2, *and if* $t_1 - t_2$ *is odd, then* $g_1(x)$ *and* $g_2(x)$ *are orthogonal.*

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