REMARKS ON A PAPER BY A. FRIEDMAN

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In this note we shall give a slight generalization of Theorem 2 in the preceding paper by Friedman, and we shall remark on additional problems in the same direction. We adhere to Friedman's notation.

In the following generalization of Friedman's Theorem 2, we assume that n and p are fixed positive integers with $n > 2p$, that ϕ is a fixed real number, and that $[(\alpha, \beta), n, R, \phi]$ denotes the closed finite region bounded by a certain oriented polygon centered at (α, β).

THEOREM 2'. *Let* $u(x, y)$ *be a real-valued function, continuous in the domain* D. *If there exist real-valued functions* $A_k(x, y)$ $(0 \le k \le p - 1)$, *continuous in* D, *and with* $A_0 \equiv 1$, *such that*

$$\text{(i)} \qquad \frac{1}{sR^2} \iint\limits_{[(\alpha,\beta),n,R,\phi]} u(x, y)\, dx\, dy = \sum_{k=0}^{p-1} A_k(\alpha, \beta) R^{2k}$$

holds for all $[(\alpha, \beta), n, R, \phi]$ *in* D, *then* $u(x, y)$ *is a* p-*harmonic polynomial of degree at most* pn, *and its derivative in the* ϕ-*direction vanishes identically.*

Proof. As Friedman remarks, it is sufficient to consider the case $\phi = 0$, for the general case can then be obtained by a rotation. We note that Friedman's formula (2.1) can be modified to yield the following result:

$$\text{(ii)} \qquad \sum_{k=1}^{n} \left[\left(\cos \frac{2\pi k}{n}\right) \frac{\partial}{\partial x} + \left(\sin \frac{2\pi k}{n}\right) \frac{\partial}{\partial y} \right]^t u(\alpha, \beta) = \begin{cases} \frac{n}{4m} \binom{2m}{m} \triangle^{2m} u(\alpha, b) & (0 < t = 2m < n), \\ 0 & (0 < t = 2m - 1 < n). \end{cases}$$

First we assume that $u(x, y)$ has continuous partial derivatives of the first 2p orders in D. If we develop $u(x, y)$ in a finite Taylor expansion about an arbitrary point (α, β) in D,

$$u(x, y) = \sum_{k=0}^{2p} \frac{p^k}{k!} \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)^k u(\alpha, \beta) + o(p^{2p}),$$

and make use of (ii), we obtain

$$\text{(iii)} \qquad \frac{1}{sR^2} \iint\limits_{[(\alpha,\beta),n,R,0]} u(x, y)\, dx\, dy = \sum_{j=0}^{p} B_j \triangle^j u(\alpha, \beta) R^{2j} + o(R^{2p}),$$

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which must hold for all small R. From (i) and (iii) we conclude that $B_p \triangle^p u(\alpha, \beta) = 0$. A direct calculation shows that $B_p \neq 0$. Hence $\triangle^p u(\alpha, \beta) = 0$. Since (α, β) is an arbitrary point in D, we conclude that u(x, y) is p-harmonic in D.

Now we assume that u(x, y) is merely continuous in D. Then the mean-value function

$$M_1(u, x, y, r) \equiv \frac{1}{\pi r^2} \iint\limits_{\xi^2+\eta^2 \leq r^2} u(x + \xi, y + \eta)\, d\xi\, d\eta$$

has continuous partial derivatives of the first order in a subdomain of D, for all small r. Fubini's Theorem and (i) imply that

$$\frac{1}{sR^2} \int\limits_{[(\alpha,\beta),n,R,0]} M_1(u, x, y, r)\, dx\, dy = \sum_{k=0}^{p-1} M_1(A_k, \alpha, \beta, r) R^{2k}$$

holds, so that $M_1(u, x, y, r)$ satisfies an identity of the form (i). By using iterated averages

$$M_{k+1}(u, x, y, r) \equiv M_1(M_k, x, y, r) \qquad (1 \leq k \leq 2p - 1),$$

we can show in the same way that $M_{2p}(u, x, y, r)$ also satisfies an identity of the form (i) in a subdomain of D, for all small r. Moreover, $M_{2p}(u, x, y, r)$ has continuous partial derivatives of the first 2p orders in that same subdomain of D. It now follows from the first part of our proof that $M_{2p}(u, x, y, r)$ is p-harmonic in a subdomain of D, for all small r. Since $M_{2p}(u, x, y, r)$ approximates u(x, y) uniformly on compact subsets of D, as $r \to 0$, the function u(x, y) is also p-harmonic in D.

It remains to show that u(x, y) is a polynomial with certain properties. That follows from Friedman's own work. This completes our proof.

Additional Remarks. We note that the area averages in (i) can be replaced by peripheral averages and by discrete averages, just as in Friedman's paper.

We also note that the form of the functions $A_k(\alpha, \beta)$ in (i) was not specified. But a comparison with Friedman's formula (0.2) shows that

$$A_j(\alpha, \beta) = \frac{\gamma_{j,n} \triangle^j u(\alpha, \beta)}{(2^j j!)^2 (j + 1)} \qquad (1 \leq j \leq p - 1).$$

Finally we raise the question of the possibility of using the general results of Choquet and Deny, which we used in [3] (see Friedman's references), in order to obtain all the preceding results, and perhaps more.

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