SOME CONSEQUENCES OF A METHOD OF PROOF OF J. H. C. WHITEHEAD

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We give some results which follow from abstraction and extension of the method of proof used by J. H. C. Whitehead in [4]; specifically, the "converse" portion of the proof of Theorem B therein. We first note that the method cited yields a lemma which has other applications, two of which we give. We then extend the method of proof to obtain a lemma which, when applied to the local homology groups of a compact space S, reveals that certain conclusions of Alexandroff [1] regarding the local Betti numbers of a compact space S are incorrect.

1. THE FIRST LEMMA AND APPLICATIONS

The first lemma referred to above is the following.

LEMMA 1. *Let* $\{A_\mu, i_\mu^\lambda\}$ *and* $\{H_\mu, i_{\mu*}^\lambda\}$ *be dually paired inverse and direct systems* [3; p. 67] *of vector spaces (the* A_λ *being discrete) over a field* $\mathscr{F}$, *indexed over a well-ordered set of indices* $\mu = 0, 1, 2, \cdots$ *of order-type* Λ. *Then if* $\varinjlim H_\mu = 0$, *there exists an index* λ *such that* $i_0^\lambda A_\lambda = 0$.

The proof of this lemma is a paraphrase of the method of proof of Whitehead cited above, and it need not be repeated here. As an interesting application of it, consider Problem 2.1, p. 381, of my Colloquium book [5]: *If* A *and* B *are disjoint closed subsets of a compact space* S, *and every* r*-cycle on* A *bounds on a compact subset of* S - B, *does there exist an open set* U, *containing* B, *such that every* r*-cycle on* A *bounds on* S - U? In [5; p. 218, Theorem 4.5] I showed that the answer is affirmative if S is a metric space. Now metric spaces are a special subclass of spaces satisfying the following axiom:

AXIOM (A'). If B is a compact set, there is a basis $\{V_\mu\}$ for the open neighborhoods of B which is well-ordered by inclusion (that is, V_μ precedes V_λ if and only if $V_\mu \supset V_\lambda$).

We can then prove the following.

THEOREM 1. *If* S *is a compact space satisfying Axiom* (A'), *then the answer to the question above is affirmative.*

Proof. Applying Axiom (A') to the set B in the statement of the question, we may assume that $A \subset S - V_0$. Then the groups $H^r(S: A, 0; S - V_\mu, 0)$ [5; p. 166, Def. 18.28] form a direct system under the homomorphisms $i_{\mu*}^\lambda$ induced by the inclusions

$$i_\mu^\lambda : S - V_\lambda \to S - V_\mu, \ \lambda < \mu.$$

Under the given conditions, $\varinjlim \{H^r(S: A, 0; S - V_\mu, 0); i_{\mu*}^\lambda\} = 0$. By Lemma 1 there consequently exists, for the dual system $\{H_r(S: S, V_\mu; S, S - A); i_\mu^\lambda\}$, a μ,

Received October 2, 1956.

This research was supported in part by National Science Foundation Grant G2783.

say $\mu = \lambda$, such that $i_0^\lambda H_r(S\colon S, V_\lambda; S, S - A) = 0$. This implies that all r-cocycles mod V_λ are cobounding mod S - A, and hence that all r-cycles on A bound on $S - V_\lambda$.

Remark. Axiom (A') is more restrictive than Whitehead's Axiom (A). For it is easy to give an example of a space S having two points p and q, of characters $\aleph_0$ and $\aleph_1$, respectively, and satisfying Whitehead's axiom, but such that the set $B = p \cup q$ does not satisfy Axiom (A') (for example, the space of all ordinal numbers of the first and second classes together with the first ordinal ω_1 of the third class, with the order topology; let $p = \omega$ and $q = \omega_1$).

For another application of Lemma 1, we recall that in his thesis [2] S. Kaplan left unsettled the following question: Let B be an arbitrary subset of the n-sphere S^n, and let A be a compact subset of S^n - B such that, for every compact subset F of S^n - B containing A, some r-cycle of A fails to bound on F; then does there exist a compact cycle in S^n - B that fails to bound on a compact subset of S^n - B? [2; p. 264, relations between properties IV' and III']. We can show that the answer is affirmative if B is a G_δ set; that moreover we may replace S^n by any compact space such that the set B in question satisfies the following axiom:

AXIOM (A"). This is Axiom (A') with the assumption that B is compact deleted.

We note first the equivalence of the following two propositions:

(P_1). If for every compact subset F of S - B containing A, some r-cycle of A fails to bound on F, then there exists an r-cycle on A which fails to bound on every compact subset of S - B.

(P_2). If every r-cycle on A bounds on a compact subset of S - B, then there exists an open set U containing B and such that every r-cycle of A bounds on S - U.

THEOREM 2. *Let* S *be a compact space, and* A *and* B *disjoint subsets of* S *such that* A *is compact and* B *satisfies Axiom* (A"). *Then proposition* (P_2) *holds, and consequently proposition* (P_1) *holds.*

The proof is like that of Theorem 1.

2. THE SECOND LEMMA

In [4], Whitehead considers a local group of the following kind: Let S be any compact space, $x \in S$, and $H_n(S, S - x) = \varinjlim H_n(S_\lambda, A_\lambda)$, where S_λ and A_λ are compact subsets of S and S - x respectively, ordered by inclusion (Whitehead uses the lower index n to denote homology). Since S is the maximal compact subset of S, and since there exists, for each A_λ, a neighborhood U_λ of x such that $A_\lambda \subset S - U_\lambda$, we may set $H_n(S, S - x) = \varinjlim H_n(S, S - U_\lambda)$.

Now Alexandroff [1; p. 20] defined a group $H_x^n(S)$ as follows: If Z^n is an n-cycle mod S - U(x) for some neighborhood U(x) of x, then Z^n is called a *cycle through* x. The cycles through x form a group $Z_x^n(S)$. For any such cycle, let $Z^n \sim 0$ if there exists a neighborhood V(x) such that $Z^n \sim 0$ mod S - V(x). The collection of all cycles Z^n such that $Z^n \sim 0$ forms a group $B_x^n(S)$, and $H_x^n(S) = Z_x^n(S)/B_x^n(S)$.

LEMMA 2. *The groups* $H_n(S, S - x)$ *and* $H_x^n(S)$ *are isomorphic.*

Proof. The isomorphism follows in a natural fashion from a mapping

$$i\colon H_x^n(S) \longrightarrow H_n(S, S - x)$$

defined as follows: As remarked above, $H_n(S, S - x) = \varinjlim H_n(S, S - U_\lambda)$; and if Z^n is a representative cycle of an element of $H_x{}^n(S)$ (hence a cycle Z^n mod $S - U_\lambda$, for some U_λ), it determines an element of $H_n(S, S - x)$ in obvious fashion. That the mapping thus defined is an isomorphism is easily shown.

Definition. By $p_x{}^n(S)$ we denote the dimension of the vector space $H_x{}^n(S)$ (and hence, by Lemma 2, the dimension of $H_n(S, S - x)$). In case $p_x{}^n(S)$ is infinite, but the dimension of $H_n(S, S - U_\lambda)$ is finite for a cofinal set of U_λ's, we agree to write $p_x{}^n(S) = \omega$.

By $p^n(S, x)$ we denote the local Betti number of S at x (see [5; pp. 190-191]). We also make the convention that symbols such as U(x), V(x), etc. denote (open) neighborhoods of x.

LEMMA 3. *For every* $x \in S$, $p^n(S, x) \geq p_x{}^n(S)$.

Proof (compare [1; p. 22, Remark]). Suppose k is a positive integer such that $p_x{}^n(S) \geq k$. Then there exist cycles $Z_i{}^n$ $(i = 1, \cdots, k)$ representative of independent elements of $H_x{}^n(S)$, which for some U(x) are all cycles mod S - U(x). Since, for every $V(x) \subset U(x)$, they are lirh mod S - V(x), it follows that $p^n(x; U(x)) \geq k$ (see [5; p. 191]). But $p^n(S, x) \geq p^n(x; U(x))$, and hence $p^n(S, x) \geq k$.

THEOREM 3 (Alexandroff). *If* S *has no* n-*dimensional condensation at* x, *then* $p^n(S, x) = p_x{}^n(S)$.

[The notion of n-dimensional condensation at a point was introduced by Alexandroff [1; p. 16]; see also the discussion in [5; p. 356]. Theorem 3 is given by Alexandroff (for metric space) [1; p. 23].]

Proof. If $p^n(S, x) \geq k$, then there exists a U(x) such that $p^n(x; U(x)) \geq k$; and since S has no n-dimensional condensation at x, there exists a $V(x) \subset U(x)$ such that if Z^n is a cycle mod S - U(x), with $Z^n \not\sim 0$ mod S - V(x), then for all $W(x) \subset V(x)$, $Z^n \not\sim 0$ mod S - W(x). Since $p^n(x; U(x)) \geq k$, there exist a W(x) and cycles $Z_i{}^n$ mod S - U(x) $(i = 1, \cdots, k)$ that are lirh mod S - W(x), and *a fortiori* lirh mod S - U(x). Then the $Z_i{}^n$ are lirh mod S - V'(x) for all $V'(x) \subset V(x)$, and hence $p_x{}^n(S) \geq k$. Consequently $p_x{}^n(S) \geq p^n(S, x)$, and the theorem follows from this relation and that of Lemma 3.

LEMMA 4. *If* $p^n(S, x) \leq \omega$, *then* S *has no* n-*dimensional condensation at* x.

[See Alexandroff [1; p. 18, Corollary I]; also [5; p. 358, Corollary 1.12]. Although in [5] the proof of the corollary depends on a theorem which assumes S to be of countable character at x, there is no difficulty in giving a proof without this assumption.]

As a corollary of Lemma 4 and Theorem 3 we have:

THEOREM 4. *If* $p^n(S, x) \leq \omega$, *then* $p^n(S, x) = p_x{}^n(x)$.

We now prove the second lemma mentioned in our introductory remarks:

LEMMA 5. *Let* $\{A_\lambda; i_\mu^\lambda\}$ *and* $\{H_\lambda; i_{\mu *}^\lambda\}$ *be inverse and direct systems of vector spaces, in which the* A_λ *are discrete, dually paired to a field* $\mathcal{F}$, *and indexed over a well-ordered set of order type* $\Lambda \geq \omega$. *For each* $\lambda \in \Lambda$, *let* $G_\lambda = i_0^\lambda A_\lambda$, *and suppose that there does not exist a* λ *such that* $G_\lambda = G_{\lambda+\nu}$ *for all* ν $(\nu = 0, 1, \cdots;$ $\lambda + \nu < \Lambda)$. *Then the dimension of* $\varinjlim \{H_\lambda; i_\mu^\lambda{}_*\}$ *is infinite.*

Proof. We may consider H_0 to be the vector space of all linear mappings of A_0 into $\mathcal{F}$. And after discarding repetitions, we may assume that $G_\lambda \neq G_{\lambda+1}$ for all λ (at least infinitely many!). Let $g_\lambda \in G_\lambda - G_{\lambda+1}$. The collection $\{g_\lambda\}$ forms an independent set of elements of $G_0 = A_0$, and it may be considered a subset of a basis of A_0.

For each natural number n, let $u_n \in H_0$ be defined as follows: Consider each ω-sequence $\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots$ of subscripts λ of the G_λ's, where λ_1 is either 0 or a limiting ordinal, and where $\lambda_{k+1} = \lambda_1 + k$; then $u_n g_{\lambda_k} = 1$ or 0 according as k is a power of the nth prime or not. We let u_n map base elements of A_0 not in $\{g_\lambda\}$ into 0, and we extend u_n linearly to non-base elements of A_0. To show that the mappings u_n determine a linearly independent set of elements of $\varinjlim \{H_\lambda; i^\lambda_{\mu *}\}$; consider any finite linear combination $u = \Sigma a^i u_i$ in which $a^i \neq 0$ for $i = n_1, n_2, \cdots, n_k$, say. If $i^\lambda_{0*} u = 0$ for some λ, then $u \circ i^\lambda_0 A_\lambda = 0 = uG_\lambda$. But there exists a $\mu > \lambda$ such that $u_{n_1} g_\mu = 1$ and hence, because of the way in which the u_n were defined, such that $ug_\mu = a^{n_1} \neq 0$ and hence $uG_\lambda \neq 0$.

LEMMA 6. *If* S *satisfies Axiom* (A) *at* x, *and* S *has* n*-dimensional condensation at* x, *then* $p_x^n(S) = \infty$.

Proof. Since S has n-dimensional condensation at x, there exists a U(x) such that, for every $V(x) \subset U(x)$, there exists a cycle Z^n mod S - U(x) on a compact subset of S - x such that $Z^n \not\sim 0$ mod S - V(x) (see [5; p. 356, Lemma 1.6]). And since S satisfies Axiom (A) at x, there exists a well-ordered basis $\{V_\lambda\}$ of open neighborhoods of x; we may suppose that $V_0 = U(x)$.

The systems $\{H_n(V_\lambda); i^\lambda_\mu\}$, $\{H^n(S, S - V_\lambda); i^\lambda_{\mu *}\}$ form dually paired inverse and direct systems. Let $G_\lambda = i^\lambda_0 H_n(V_\lambda)$. We assert that there does not exist a $\lambda = \mu$ such that $G_\mu = G_{\mu+\nu}$ for all ν ($\nu = 0, 1, \cdots$). For consider any V_λ. There exists a $Z_\lambda{}^n$ mod S - V_0, on a compact subset K of S - x, such that $Z_\lambda{}^n \not\sim 0$ mod S - V_λ. Hence, by the duality between homology and cohomology, there exists a cocycle g_λ in V_0 such that $Z_\lambda{}^n \cdot g_\lambda = 1$. And since $x \notin K$, we may assume that g_λ is in V_0 - x, indeed, that g_λ is carried by an open subset of $V_\lambda - V_\mu$, where $\overline{V}_\mu \cap K = 0$. Hence the element $[g_\lambda]$ of G_λ is not in G_μ; for if it were, there would exist a cocycle class $\overline{g}_\mu$ of $H_n(V_\mu)$ such that $i^\mu_0 \overline{g}_\mu = [g_\lambda]$. But this would imply that $g_\lambda \sim g_\mu$ in V_0, where $g_\mu \in \overline{g}_\mu$, and hence that $Z_\lambda{}^n \cdot g_\mu = 1$, and this is impossible, since $K \cap V_\mu = 0$.

Applying Lemma 5, we conclude that $p_x{}^n(S) = \infty$.

THEOREM 5. *If* S *satisfies Axiom* (A) *at* x, *then* $p^n(S, x) = p_x{}^n(S)$.

Proof. If $p^n(S, x) \leq \omega$, the desired equality holds by Theorem 4. If $p^n(S, x) = \infty$, and S has no n-dimensional condensation at x, then $p_x{}^n(S) = \infty$ by Theorem 3; and if S does have n-dimensional condensation at x, then $p_x{}^n(S) = \infty$ by Lemma 6.

Remark. Theorem 5 shows that the values assigned to $p_a{}^1(F)$ in Examples 1, 2, 3 and 5 of Alexandroff [1; p. 23], should agree with the values of $p^1(a, F)$.

It is also of interest to note that Whitehead's theorem can be obtained as a corollary of Theorem 5:

COROLLARY (Whitehead). *A necessary and sufficient condition that* S *be* n-colc *at a point* x *satisfying Axiom* (A) *is that* $H_n(S, S - x) = 0$.

Proof. By Lemma 2, $H_n(S, S - x) \cong H_x{}^n(S)$. Hence if $H_n(S, S - x) = 0$, then $p_x{}^n(S) = 0$, and by Theorem 5, $p^n(S, x) = 0 = p_n(S, x)$. Thus S is n-colc at x by definition [5; p. 189, Def. 6.1].

Conversely, if S is n-colc at x, then $p_n(S, x) = 0 = p^n(S, x) = p_x{}^n(S)$. Therefore dimension $H_x{}^n(S)$ = dimension $H_n(S, S - x) = 0$, and consequently $H_n(S, S - x) = 0$.

REFERENCES

1. P. Alexandroff, *On local properties of closed sets*, Ann. of Math. (2) 36 (1935), 1-35.
2. S. Kaplan, *Homology properties of arbitrary subsets of euclidean spaces*, Trans. Amer. Math. Soc. 62 (1947), 248-271.
3. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications 27 (1942).
4. J. H. C. Whitehead, *Note on the condition* n-colc, Michigan Math. J. 4 (1957), 25-26.
5. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32 (1949).

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