NOTE ON THE CONDITION n-colc

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Let X be a locally compact space which satisfies the following condition, relative to a point $x \in X$;

(A) *There is a basis* (V) *for the (open) neighbourhoods of* x, *which is well-ordered by the relation* $\supset$ (that is, V *precedes* V' *if, and only if,* $V \supset V'$).

After adding a "point at ∞," if necessary, we assume that X is compact (this is justified by the excision axiom [2], in the study of local properties at x). If B is a subspace of any space Y, then* $H_q(Y, B)$ will denote $\underrightarrow{\text{Lim}}\{H_q(Y_\lambda, B_\lambda)\}$ for all compact pairs $(Y_\lambda, B_\lambda) \subset (Y, B)$, partially ordered by inclusion, where $H_q(Y_\lambda, B_\lambda)$ refers to the Čech theory with coefficients in a field F. Similarly

$$H^q(Y, B) = \underleftarrow{\text{Lim}}\{H^q(Y_\lambda, B_\lambda)\},$$

with F as coefficient group. If (Y, B) is itself a compact pair, there is a natural identification of $H_q(Y, B)$ with the vector space of linear maps $H^q(Y, B) \to F$.

I recall that X is n-colc at x [3] (see [1]) if, and only if, for every neighbourhood U of x, there is a neighbourhood U' such that $x \in U' \subset U$ and

$$i^*H^n(X, A') = 0 \in H^n(X, A),$$

where $A = X - U$, $A' = X' - U'$ and $i : (X, A) \subset (X, A')$.

THEOREM (B). *Let* X *satisfy* (A). *Then it is* n-colc *at* x *if, and only if,*

$$H_n(X, X - x) = 0. \qquad \text{(C)}$$

Proof. We identify $H_n(X, B)$ $(B = A, A')$ with the vector space of linear maps $H^n(X, B) \to F$. Then $i_* : H_n(X, A) \to H_n(X, A')$ is defined by $i_*u = u \circ i^*$, where $u : H^n(X, A) \to F$. Therefore $i_*H_n(X, A) = 0$ if $i^*H^n(X, A') = 0$, and (C) is satisfied if X is n-colc at x.

Conversely, let X satisfy (C), let ρ be the ordinal number of the sequence (V), let Λ be the set of ordinals less than ρ and let (V) be indexed to Λ in the obvious way. Let $U = X - A$ be a given neighbourhood of x and let us assume, as we obviously may, that $U = V_0$. Let $A_\lambda = X - V_\lambda$, and let

$$G_\lambda = i_\lambda^* H^n(X, A_\lambda) \subset H^n(X, A) = G,$$

where $i_\lambda : (X, A) \subset (X, A_\lambda)$. Then (B) will follow when we have proved that $G_\lambda = 0$ for some λ. Notice that if $\lambda \leq \mu$, then $V_\lambda \supset V_\mu$, $A_\lambda \subset A_\mu$ and $G_\lambda \supset G_\mu$.

Received June 1, 1956.

* H_q denotes an homology and H^q a cohomology group. If $A \subset X$ is closed, I write $H^q(X - A) = H^q(X, A)$, in conformity with [2], where $H^q(X - A)$ is defined as in [3].

Let $u : G \to F$ be given. Since $i_{\mu}*u = u \circ i^*_{\mu}$, it follows from (C) that there is a $\mu \in \Lambda$, and hence a $\mu \geq \lambda$ for a given λ, such that $uG_{\mu} = 0$. Therefore $uG_{\lambda} = 0$ for every u if $G_{\lambda} = G_{\mu}$ for every $\mu \geq \lambda$. Since every linear map $G_{\lambda} \to F$ has a (linear) extension $G \to F$, it follows that $G_{\lambda} = 0$ if $G_{\lambda} = G_{\mu}$ for every $\mu \geq \lambda$. Therefore, if $G_{\lambda} \neq 0$ for every λ, the sequence Λ has no last element and we may assume, after discarding repetitions, that $G_{\lambda} \neq G_{\lambda+1}$.

Assume that $G_{\lambda} \neq G_{\lambda+1}$, and let $g_{\lambda} \in G_{\lambda} - G_{\lambda+1}$ for every $\lambda \in \Lambda$. Then it is obvious that the vectors g_{λ} are linearly independent. Therefore they belong to a basis for G, and there is a linear map $u : G \to F$ such that $ug_{\lambda} = 1$, whence $uG_{\lambda} \neq 0$ for every λ. This contradicts the preceding paragraph, and (B) is proved.

REFERENCES

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