

THE SETS OF AMBIGUOUS POINTS OF FUNCTIONS OF BOUNDED CHARACTERISTIC

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If $f(z)$ is a complex-valued function defined in the unit disk D , a point $P = e^{i\theta}$ is called an *ambiguous point* of $f(z)$ provided there exists a simple closed Jordan curve Γ , interior to D except for the point P , such that $f(z)$ approaches two different limits as $z \rightarrow P$ along Γ from opposite sides. Bagemihl [1, Theorem 2] has shown that even if the function $f(z)$ is not required to be continuous in D , its set of ambiguous points is at most denumerable. On the other hand, every denumerable set on the unit circle C is contained in the set of ambiguous points of some function which is regular and of bounded characteristic in D (Bagemihl and Seidel [2]). It is our purpose to prove that in the last statement the words "is contained in" can be replaced by "coincides with."

THEOREM. *Let E be a denumerable set on C . Then there exists a function $f(z)$ which is regular and of bounded characteristic in D and whose set of ambiguous points coincides with E .*

Let $E = \{z_n\}$, let $\{a_n\}$ be a decreasing sequence of positive constants ($\sum a_n < 1$), and let D_n denote the disk of diameter a_n which is internally tangent to C at z_n . Clearly the a_n can be chosen in such a way that no two of the disks D_n intersect. We consider the function

$$f(z) = g(z)/h(z) = g(z)k(z),$$

where

$$g(z) = \exp \left(- \sum a_n^2 / \sqrt{1 - z/z_n} \right),$$

$$k(z) = \exp \sum a_n^2 / (1 - z/z_n).$$

In the formula for $g(z)$, the symbol $\sqrt{1 - z/z_n}$ represents that branch of the corresponding function which has the value 1 at $z = 0$. That $f(z)$ is regular in D is seen directly from the formulas. Also, since $\Re(1 - z/z_n) > 0$ throughout D , the two quantities $\Re \log g(z)$ and $\Re \log h(z)$ are negative throughout D , and therefore $f(z)$ is of bounded characteristic.

Next we observe that the function $w = u + iv = a_n^2 / (1 - z/z_n)$ maps the disk D_n on the half-plane $u > a_n$, and the complementary set $D - D_n$ onto the strip $a_n^2/2 < u \leq \varepsilon$. From this it follows that, for each m , the inequality

$$\Re \log k(z) < \sum a_n < 1$$

is satisfied on the boundary C_m of D_m (taken relative to D), and, more generally, that $|k(z)| < e$ in the set $D^* = D - \bigcup D_n$. Since $\Re \log g(z) \rightarrow -\infty$ as $z \rightarrow z_m$ along

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C_m , it follows further that $f(z) \rightarrow 0$ as $z \rightarrow z_m$ along C_m . On the other hand, as $z \rightarrow z_m$ along the radius R_m of D ,

$$a_m^2 [1/(1 - z/z_m) - 1/\sqrt{1 - z/z_m}] \rightarrow \infty.$$

Also, the inequality $|1 - z/z_n| > a_n/2$ holds everywhere on R_m , for $n > m$, whence

$$\left| \sum_{n \neq m} a_n^2 / \sqrt{1 - z/z_n} \right| < 2 \sum a_n^{3/2} < 2,$$

everywhere on R_m . We conclude that $f(z) \rightarrow \infty$ as $z \rightarrow z_m$ along R_m . Since $f(z)$ has two different asymptotic values on C_m and R_m , the point z_m is ambiguous.

It remains to prove that if P is a point in $C - E$, it is not an ambiguous point of $f(z)$. Now let Γ be a simple closed curve through P , lying in D except for the point P , and suppose that $f(z)$ approaches the respective (finite or infinite) values b_1 and b_2 , as $z \rightarrow P$ along Γ in the clockwise or the counterclockwise direction relative to Γ . Then $|b_1| \leq e$ and $|b_2| \leq e$, since each arc of Γ which terminates at P passes through points of the set D^* . Therefore $|f(z)|$ has a finite bound $B = B(\Gamma)$, on Γ . It follows that $|f(z)| < B + e$, at every point which lies both in the interior of Γ and in one of the disks D_k . Since $|f(z)| < e$ in D^* , we conclude that $|f(z)| < B + e$ throughout the interior of Γ , and hence that $b_1 = b_2$. This concludes the proof.

REFERENCES

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