THE SETS OF LUSIN POINTS OF ANALYTIC FUNCTIONS

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I. INTRODUCTION

Let the function f be holomorphic in the unit disk D. A point $e^{i\theta}$ on the unit circle C is then called a Lusin point of f provided f maps each circular disk having $e^{i\theta}$ as a boundary point and lying in D onto a Riemann domain of infinite area. By using Taylor series with Fabry gaps, Lohwater and Piranian have shown [2] that a function can be continuous on the closure of D and yet possess a Lusin point at each point $e^{i\theta}$. By means of the direct construction of an appropriate Riemann surface, Jenkins has proved [1] that a function can be bounded in D, possess a radial limit of modulus 1 on almost every radius of D, and have almost all points of C as Lusin points. It is highly plausible that Jenkins' beautiful example actually has all points of C as Lusin points; but the proof of this seems to be difficult.

In the present paper we describe a new bounded function (see Sections 2 to 6) whose radial limit has modulus 1 almost everywhere and for which every point $e^{i\theta}$ is a Lusin point. In Section 7 we solve the problem of characterizing the sets E on C for which there exists a function f whose set of Lusin points is precisely the set E. In Section 8 we consider the areas of the images of subregions of D which are more general than circular disks.

2. THE EXTENSION OF JENKIN'S THEOREM

THEOREM 1. *There exists a function* f, *holomorphic and bounded in* D, *with* $|\lim_{r\to 1} f(re^{i\theta})| = 1$ *for almost all* θ, *and such that each point* $e^{i\theta}$ *is a Lusin point of* f.

This theorem will be proved by means of an example of the form $f = \lim f_n$, where $f_n = \Pi_{m=1}^{n} g_m$ and

$$g_m(z) = \exp \sum_k a_{mk}(z + z_{mk})/(z - z_{mk}) \qquad (m = 1, 2, \cdots). \tag{1}$$

In (1), the a_{mk} are certain positive constants with the property that

$$\sum_k a_{mk} < 2^{-m}, \tag{2}$$

and the z_{mk} are points on C.

That f is holomorphic in D follows from condition (2) and the relation

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$$f(z) = \exp \sum_{m,k} a_{mk}(z + z_{mk})/(z - z_{mk}).$$

The inequality $|f| < 1$ follows from the fact that the real part of $(z + z_{mk})/(z - z_{mk})$ is negative throughout D, for each z_{mk}. To see that the radial limit of f had modulus 1 almost everywhere, it is sufficient to write f in the form

$$f(z) = \exp \int_{-\pi}^{\pi} \frac{z + e^{is}}{z - e^{is}}\, d\mu(s);$$

since $\mu(s)$ is a pure jump function (with a saltus a_{mk} at the point s_{mk} if $z_{mk} \equiv \exp is_{mk}$), the derivative $\mu'(s)$ vanishes for almost all s, and therefore the real part of $\log f(z)$ has the radial limit 0, almost everywhere.

3. OUTLINE OF THE CONSTRUCTION

It remains to choose the constants a_{mk} and z_{mk} in such a way that each point of C is a Lusin point of f. We begin with certain preliminary considerations concerning the maps of disks which are internally tangent to C at points z_{mk}.

For a point t on C and a positive constant r $(0 < r < 1)$, let $D(r, t)$ denote the circular disk of radius r which is internally tangent to C at t. The function

$$z^*(z) = a(z + t)/(z - t)$$

maps the disk $D(r, t)$ onto the half-plane $\Re z^* < a(1 - r^{-1})$, and therefore the function $Z(z) = \exp z^*(z)$ maps $D(r, t)$ onto a Riemann domain which covers the punctured disk

$$0 < |Z| < \exp a(1 - r^{-1})$$

infinitely often; clearly, the area of this Riemann domain is infinite.

More generally, for each fixed pair of indices n and k, suppose that the point z_{nk} is not a limit point of the set

$$\{z_{mh}\} \qquad (m = 1, 2, \cdots, n;\ h = 1, 2, \cdots).$$

(This is a condition which will be satisfied by the point set $\{z_{mk}\}$ of our construction.) Let

$$w_{nk}(z) = \exp a_{nk}(z + z_{nk})/(z - z_{nk}),$$

and let the function W_{nk} be defined by the formula $f_n = w_{nk} W_{nk}$. Then

$$|f_n'| \geq |w_{nk}' W_{nk}| - |w_{nk} W_{nk}'|.$$

By (1), the function W_{nk} is holomorphic at z_{nk}, and $|W_{nk}(z_{nk})| = 1$; therefore the inequalities

$$|W_{nk}| > 1/2, \qquad |W_{nk}'| < A_{nk} < \infty$$

are satisfied throughout some neighborhood of z_{nk}. Also, $|w_{nk}(z)| < 1$ in D, and

$$\iint_{D(r,z_{nk})} |w'_{nk}|^2 \, d\sigma = \infty$$

when $0 < r < 1$. It follows that

$$\iint_{D(r,z_{nk})} |f'_n|^2 \, d\sigma = \infty \qquad (0 < r < 1);$$

that is, z_{nk} is a Lusin point of f_n. In order to ensure that z_{nk} is a Lusin point of the function f as well, it will be sufficient to choose the constants a_{mh} and z_{mh} $(m > n)$ in such a way that the ratio f'/f'_n is bounded away from 0, throughout the disk $D(1/2, z_{nk})$. This detail will receive our attention in Sections 5 and 6.

Next we describe a device which will ensure that every point in $C - \{z_{mk}\}$ is a Lusin point of f. For the sake of a later theorem, we replace the consideration of disks tangent to C by the consideration of regions whose boundaries have contact of order lower than 1 with C.

Let the finite region R be bounded by a curve $\theta = \pm\lambda(1 - r)$ $(0 \leq r \leq 1)$, where the monotonic function λ is subject to the two restrictions that

$$\lambda(x)/x \to \infty, \qquad \lambda(x)/x^{1/2} \to 0$$

as $x \to 0$. The first condition is equivalent to this, that the boundary of R possesses a tangent at $z = 1$; the second condition makes the region R so narrow near $z = 1$ that, for every sufficiently small ρ, the disk $D(\rho, 1)$ contains the part of R which lies outside of the circle $|z| = 1 - \rho$. By $R(\theta)$ we denote the image of R under the rotation of the z-plane through an angle θ about the origin.

With each pair of indices m and k we associate a circular disk D_{mk} of center $z = z_{mk}(1 - 2a_{mk})$ and radius a_{mk}. The constants z_{mk} and a_{mk} will be chosen in such a manner that no two disks D_{mk} overlap, that for each index m and each point $e^{i\theta}$ in $C - \{z_{mk}\}$ the region $R(\theta)$ contains at least one of the disks D_{mk}, and that the function f maps each of the disks D_{mk} onto a Riemann domain whose area exceeds a certain positive universal constant. Clearly, each point in $C = \{z_{mk}\}$ will then be a Lusin point of the function f.

4. THE FIRST ROUND

For $m = 1$, the index k takes on only a finite number of values. The points z_{1k} are evenly spaced on C, in counterclockwise order; and the constants a_{1k} are all equal. The common value $a = a_{1k}$ is chosen small enough so that $2\pi a/\lambda(a) < 1/2$; and the number of points z_{1k} is the integral part of the number $2\pi/\lambda(a)$. The angular distance between adjacent points z_{1k} is then of the order of magnitude $\lambda(a)$; that is, it has the value $\lambda(a)(1 + \varepsilon)$, where $\varepsilon \to 0$ as $a \to 0$. And since $a = o(\lambda(a))$ and the angular measure of the intersection of the region R with the circle $|z| = 1 - a$ is $2\lambda(a)$, it follows that if the constant a is small enough, each region $R(\theta)$ contains one of the disks D_{1k}. Also,

$$\sum a_{1k} = \left[\frac{2\pi}{\lambda(a)}\right] a < 1/2 .$$

To obtain an estimate of the value of $|f_1'|$ in the disks D_{1k}, we write f_1' in the form

$$f_1'(z) = -2f_1(z)\,a \sum z_{1k}(z - z_{1k})^{-2}.$$

Now

$$\log |f_1(z)| = \Re\, a \sum (z + z_{1k})/(z - z_{1k}).$$

In the disk D_{1h}, the term of index h under the last summation sign has modulus less than $2a^{-1}$. Since

$$\frac{1 - r^2}{1 + r^2 - 2r\cos\theta} = \frac{1 - r^2}{(1 - r)^2 + 4r\sin^2\theta/2},$$

the real part of the kth term under the summation sign, for z in D_{1h} and $k \neq h$, is not greater than $A_1\theta^{-2}$, where A_1 is a universal constant and θ is the angular distance between z_{1h} and z_{1k}. It follows that throughout D_{1h} an inequality of the form

$$|f_1(z)| > \exp\left\{-2 - 2a^2A_1(\lambda(a))^{-2}\sum_1^\infty p^{-2}\right\} = \exp\{-2 + o(1)\}$$

is satisfied.

To complete the task of obtaining an estimate of $|f_1'|$ in the disk D_{1h}, we need an estimate of the quantity $|a\Sigma z_{1k}(z - z_{1k})^{-2}|$. For $h \neq k$, an inequality of the form

$$|z - z_{1k}|^{-2} < A_2(\arg z_{1h}/z_{1k})^{-2}$$

holds, and therefore it is obvious that

$$\left|\sum_{k\neq h} z_{1k}(z - z_{1k})^{-2}\right| < A_3(\lambda(a))^{-2}.$$

On the other hand,

$$|z_{1h}(z - z_{1h})^{-2}| > A_4 a^{-2}$$

in D_{1h}, and from the relation $(\lambda(a))^{-2} = o(a^{-2})$ it follows that

$$\left|a \sum z_{1k}(z - z_{1k})^{-2}\right| > A_5 a^{-2},$$

throughout D_{1h}. We conclude that

$$|f_1'(z)| > A_6 a^{-1} \qquad (3)$$

in D_{1h}. And we note that since the area of D_{1h} is $a^2\pi/4$, the function f_1 maps D_{1h} onto a Riemann domain of area greater than $A_6^2\pi/4$.

5. THE SECOND ROUND

We cover the open set $C - \{z_{1k}\}$ with closed intervals I_{2j} $(j = 1, 2, \cdots)$ which are disjoint (except for possible common end points) and which have the property that each lies at a positive distance d_{2j} from the set $\{z_{1k}\}$. For each index j we choose a number c_{2j} which will serve as the common value of the constants a_{2k} associated with the points z_{2k} in I_{2j}. The interval I_{2j} (of length $|I_{2j}|$) is divided into $[|I_{2j}|/\lambda(c_{2j})]$ equal subintervals, and the midpoint of each of these subintervals serves as one of the points z_{2k}. Each of the values c_{2j} is chosen small enough so that the following three conditions are satisfied.

First, $\Sigma a_{2k} < 1/4$; this requirement can be met by choosing each c_{2j} small enough so that $2\pi c_{2j}/\lambda(c_{2j}) < 2^{-2-j}$.

Second, throughout each of the disks $D(1/2, z_{1k})$ and D_{1k} the inequality

$$|f'|^2 > 2^{-1/2} |f_1'|^2$$

must hold. That this is possible follows from the relations

$$\begin{aligned} f_2(z) &= f_1(z) g_2(z), \\ f_2'(z) &= f_1'(z) g_2(z) \left\{ 1 + \frac{f_1(z) g_2'(z)}{f_1'(z) g_2(z)} \right\} \\ &= f_1'(z) g_2(z) \left\{ 1 + \frac{\Sigma a_{2k} z_{2k} (z - z_{2k})^{-2}}{\Sigma a_{1k} z_{1k} (z - z_{1k})^{-2}} \right\}. \end{aligned}$$

For since each of the intervals I_{2j} lies at a positive distance from the union of the disks $D(1/2, z_{1k})$ and D_{1k}, the quantity $g_2(z) = 1$ can be made arbitrarily small throughout these disks, by choosing each constant c_{2j} small enough; and the same applies to the fraction in the braces.

Third, the function f_2 must map each disk D_{2k} onto a Riemann domain whose area is greater than the universal constant $A_6^2 2^{-1/2}$ (see the end of Section 4). To show that this is possible, we note that, in each sector of D which is bounded by an arc I_{2j} and the two corresponding radii, the relation

$$\lim_{r \to 1} |f_1(re^{i\theta})| = 1$$

holds uniformly and the quantity $f_1'(z)$ is bounded. Since

$$\left| \frac{d}{dz} \left\{ \exp c_{21} \sum_{z_{2k} \in I_{21}} (z + z_{2k})/(z - z_{2k}) \right\} \right| > A_6 c_{21}^{-1}$$

(compare the inequality (3) in Section 4), the inequality

$$\left| \frac{d}{dz} \left\{ f_1(z) \exp c_{21} \sum_{z_{2k} \in I_{21}} (z + z_{2k})/(z - z_{2k}) \right\} \right| > A_7 c_{21}^{-1}$$

can be made to hold throughout the disks D_{2k} associated with I_{21}, for any A_7 less than A_6, simply by choosing c_{21} sufficiently small. Once c_{21} is chosen, the constant c_{22} can be taken small enough so that the contributions to f(z) and to f'(z) which come from the various points z_{2k} in I_{21} and I_{22} do not cancel each other beyond a tolerable degree, in any of the relevant disks. More generally, the constants c_{2j} $(j = 1, 2, \cdots)$ can be chosen in such a way that f_2 maps each of the disks D_{1k} and D_{2k} onto a Riemann domain of area greater than $A_6^2 2^{-1/2}$.

6. CONCLUSION OF THE PROOF

The rest is easy. Once the constants z_{mk} and a_{mk} have been chosen for $m = 1, 2, \cdots, n-1$, we cover the open set $C - \{z_{mk}\}$ $(m = 1, 2, \cdots, n-1;$ $k = 1, 2, \cdots)$ with closed arcs I_{nj}. For each j, we choose c_{nj}, then divide I_{nj} into $[|I_{nj}|/\lambda(c_{nj})]$ equal subintervals, choose the midpoint of each subinterval as one of the points z_{nk}, and associate with it the constant a_{nk} equal to c_{nj}. With the proper choice of the constants c_{nj}, the inequality (2) is satisfied. Also, if the c_{nj} are small enough, the condition

$$|f'_n|^2 > 2^{-(1-1/n)}|f'_m|^2$$

is satisfied throughout each of the disks $D(1/2, z_{mh})$ and D_{mh} $(m = 1, 2, \cdots, n-1)$; and the image under f of each of the disks D_{nk} has area greater than $A_6^2 2^{-1/2}$.

Since $f'_m \to f'$ uniformly in every closed subset of D, it follows that f maps every disk $D(1/2, z_{nk})$ onto a Riemann domain of infinite area, and every disk D_{nk} onto a Riemann domain of area at least $A_6^2 2^{-1}$. This completes the proof.

7. THE SET OF LUSIN POINTS OF A HOLOMORPHIC FUNCTION

To determine the structure of the set of Lusin points of a holomorphic function f, we consider a disk $R(r, \theta, \rho)$, of fixed radius ρ $(\rho < 1)$ and movable center $re^{i\theta}$ $(0 \leq r \leq 1 - \rho)$, and we denote by $A(r, \theta, \rho)$ the area of the Riemann domain onto which f maps this disk.

We assert that the quantity

$$A(\theta, \rho) = \lim_{r \to 1-\rho} A(r, \theta, \rho)$$

exists, for each θ, either as a finite number or as infinity. To see this, it is sufficient to note that the area of the image of the intersection of $R(r, \theta, \rho)$ with the disk $|z| < 1 - \rho$ is a continuous function of r $(r < 1 - \rho;$ θ fixed), while the area of the image of the remainder of $R(r, \theta, \rho)$ is a nondecreasing function of r.

Since the function $A(r, \theta, \rho)$ is continuous for $0 \leq \theta \leq 2\pi$ and $0 \leq r < 1 - \rho$ (ρ fixed), the set $E(\rho)$ of values θ for which $A(\theta, \rho) = \infty$ is of type G_δ. Clearly the set of Lusin points of f is the intersection of the sets $E(\rho)$ $(\rho = 1/2, 1/3, \cdots)$, and it is therefore also of type G_δ. By means of a few slight modifications in the proof of Theorem 1, we shall show that this result is in a certain sense the best possible.

THEOREM 2. *In order that a set* E *on* C *be the set of Lusin points of some function holomorphic in* D, *it is necessary and sufficient that* E *be of type* G_δ.

The necessity of the condition has already been established. To show the sufficiency, we suppose that $E = \bigcap G_n$ $(n = 1, 2, \cdots)$, where each set G_n is open, and where $G_n \supset G_{n+1}$.

The first modification which we introduce into the proof of Theorem 1 is this, that the set of intervals I_{nj} $(j = 1, 2, \cdots)$ is made to cover not the set

$C - \{z_{mk}\}$ ($m = 1, 2, \cdots, n - 1$), but only the open intersection of this set with the open set G_n. At each point of the complement F_n of G_n we establish a tangent circular disk of radius 1/2. Like the disks $D(1/2, z_{mk})$, the circular disks just established must be protected, at all later stages, from excessive disturbances of the derivative f_m': but while in the one case we had to take care to keep the ratio f'/f_m' bounded away from 0, the new protected region requires a finite upper bound on the ratio f'/f_m'. That this finite bound can be obtained follows from the fact that each interval I_{mj} ($m > n$) lies at a positive distance from the set F_n.

Our second modification is designed to meet the requirement that every point of the set $\{z_{mk}\}$ must be a point of the set E. To this end, we increase to $[4\pi/\lambda(c_{nj})]$ the number of subintervals into which an interval I_{nj} is divided, and we examine the subintervals $I_{nj1}, I_{nj2}, \cdots, I_{njp}$ (assumed to lie in I_{nj} in the order of increasing indices p) for points of E.

If I_{nj1} contains points of E, we choose any one of them to serve as one of the base points z_{nk}; if I_{nj1} contains no points of E, then no points z_{nk} are needed in I_{nj1}, and therefore we pass over this interval. Similar treatment is accorded to $I_{nj2}, I_{nj3}, \cdots$, with this proviso: if at some stage no base point can be chosen which does not lie at a distance greater than $\lambda(c_{nj})/2$ from any of the base points previously chosen, then no point shall be chosen at that stage. A rule of this sort is necessary in order that the argument which leads to the inequality (3) should remain valid. We also rule that the common endpoint of two adjacent intervals I_{ni} and I_{nj} shall be considered as belonging only to that interval whose second index is larger. This rule ensures that each interval I_{nj} lies at a positive distance from the set of points z_{mk} which have been chosen prior to the treatment of the interval I_{nj}.

With these provisions in force, the discussion of Sections 2 to 6 can be carried through, *mutatis mutandis*, to prove Theorem 2.

8. MORE GENERAL BOUNDARY DOMAINS

Our proof of Theorem 2 supplies each point $e^{i\theta}$ of the set C - E with a "protected disk" of radius 1/2 whose image under f is a Riemann domain of finite area. There is no difficulty in replacing the protected disk by a more general protected region $R^*(\theta)$, provided the boundary of $R^*(\theta)$ is interior to D except for the point $e^{i\theta}$. In other words, the protected regions can have contact of arbitrarily high order with the unit circle C.

Similarly, the disks which are mapped onto Riemann domains of infinite area can be replaced by regions with contact of arbitrarily low order with C. The only reservation to this statement concerns the disks tangent at the points z_{mk}. The following theorem gives a detailed description of the state of affairs.

THEOREM 3. *Let* E *be a set of type* G_δ *on* C; *let* R^* *and* R *be two regions in* D, *with boundaries which lie in* D *except for one point of tangency at* $z = 1$; *and let* $R^*(\theta)$ *and* $R(\theta)$ *denote the images of* R^* *and* R, *under a rotation through an angle* θ *about the origin. Then there exists a function* f *with the following properties:*

i) f *is holomorphic and bounded in* D, *and on almost all radii of* D *the radial limit of* f *has modulus* 1;

ii) *if* $e^{i\theta}$ *lies in* C - E, *the function* f *maps the region* $R^*(\theta)$ *onto a Riemann domain of finite area*;

iii) *for all* $e^{i\theta}$ *in* E, *except possibly a denumerable set,* f *maps the region* $R(\theta)$ *onto a Riemann domain of infinite area; and if* $e^{i\theta}$ *belongs to the exceptional set, then* $e^{i\theta}$ *is a Lusin point of* f.

We omit the proof of this theorem, since it is implicit in the proof of Theorems 1 and 2. The statement in part (iii), concerning the denumerable exceptional set in E, may well be superfluous. But it seems that in order to eliminate the statement, it would be necessary to replace the function $\mu(s)$ (see Section 2) in the proof by a continuous function. Such a change would undoubtedly make the computations enormously more difficult.

REFERENCES

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