ON FABER SERIES

1. A PROBLEM OF TRANSFER

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1. INTRODUCTION. In Sections 2 and 3 a method for the treatment of Faber series [2] is developed. The method is applied, in Section 4, to give a new proof of a recent result of Iliev [4], and to establish one new theorem. Further applications are indicated in Section 5.

1.1. *Notation.* The letter C will denote the same simple closed analytic curve throughout, and I(C) will denote its interior. The symbol F(z) will represent a function analytic in I(C), although not necessarily the same one in different usages. The symbol Σ will indicate a summation in which the index of the summand ranges from 0 to ∞. A sequence will be represented by placing braces about a general element. Again, the index will be understood to range from 0 to ∞.

1.2. *The Problem of Transfer.* The following proposition constitutes the basic result in the theory of Faber series.

LEMMA 1 (Faber). *There exists a sequence of polynomials* $\{F_n(z)\}$ *which can be associated with* C, *such that each function analytic in* I(C) *can be represented by a unique series*

$$\Sigma\, a_n F_n(z) \tag{1}$$

converging uniformly in each closed subset of I(C).

These polynomials are now called Faber polynomials. A series of type (1), whether it converges for any z, or not, is called a Faber series. When the series converges uniformly in closed subsets of I(C), it converges to an analytic function and is called the Faber expansion of the function.

Recently, Iliev investigated the nature of the analytic function represented by a Faber series in the case where the number of different values assumed by the coefficients is finite. The corresponding problem for power series was solved by Szegö [6]. He proved that the function represented by the power series is either a rational function, or is analytic inside the circle $|z| = 1$ and has each point of this circle as a singularity.

Iliev's result has a similar character, and his proof follows the pattern of the proof for power series. This suggests the *problem of developing a general method for transferring a theorem on power series to Faber series.* As indicated, Iliev's proof is of a special nature. In Faber's work, however, certain findings are related to this problem. It will be indicated in what respect they are inadequate as a method of transfer. A new lemma is then added, which, together with Faber's results, constitutes the proposed method.

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2. OUTLINE OF THE METHOD. Faber obtained results on the relationship between the function $F(z)$ represented by $\Sigma a_n F_n(z)$ and the associated function $G(t)$ represented by the power series $\Sigma a_n t^n$ formed with the same coefficients. After preliminaries, the pertinent facts due to Faber are stated, and an extension is given.

2.1. *Some Conformal Mapping Theory.* There exists a unique analytic function $w = f(z)$ which (a) maps the exterior of C in a one-to-one manner onto the exterior of a circle K centered at $w = 0$, and (b) is represented for large z by a series of the form

$$f(z) = z + b_0 + \frac{b_1}{z} + \cdots . \tag{2}$$

The function $w = f(z)$ is known as the normalized exterior mapping function of C. The radius k of K is determined by the conditions (a) and (b), and is called the exterior mapping radius of C. The formal expansion of $[f(z)]^n$ by (2) begins with a polynomial of degree n. This is the Faber polynomial of degree n associated with the curve C.

Since C is an analytic curve, $f(z)$ is analytic and one-to-one in an unbounded, simply connected domain D containing C and its exterior. Thus the function $w = f(z)$ maps D onto a domain L containing K. Let the inverse of this mapping be denoted by $z = g(w)$. There exists a number k_0 ($k_0 < k$) such that the circle $K_0 : |w| = k_0$, and its exterior lie in L. Then if $k_i \geq k_0$, the image of $K_i : |w| = k_i$, by $z = g(w)$ is a simple analytic curve. This is called the level curve of C of height k_i.

2.2. *Faber's Result.* In Lemma 1 a function $F(z)$ is given, and the assertion concerns the possibility of its representation by a series of polynomials. In the following result, a series is given, and the assertion concerns the function represented by the series.

LEMMA 2 (Faber). *Let C have exterior mapping radius k, and let $\{F_n(z)\}$ be the associated sequence of Faber polynomials.*

(a) *If a sequence $\{c_n\}$ satisfies the condition*

$$\limsup_{n \to \infty} |c_n|^{1/n} = 1/k_1, \tag{3}$$

where $k_1 > k$, then $\Sigma c_n F_n(z)$ converges inside of C_1, the level curve of height k_1, to an analytic function $F(z)$. The curve C_1 is called the curve of convergence.

(b) *The image K_1 of C_1 under the mapping $w = f(z)$ is the circle of convergence of $\Sigma c_n t^n$. It will also be referred to as a curve of convergence. Let $G(t)$ be the function defined by this series. Then a point z_0 on C_1 is a regular point of $F(z)$ if and only if $t_0 = f(z_0)$ is a regular point of $G(t)$.*

Note that this lemma describes a method of transfer. An assertion is made about a function defined by a Faber series, based on information concerning a function defined by a power series. However, it should also be noted that this transfer of information is limited to the behavior on the curves of convergence, C_1 and K_1.

2.3. *An Extension.* For an effective method of transfer, relationships between the functions $F(z)$ and $G(t)$ outside of their curves of convergence are required. The following is a result of this nature.

LEMMA 3. *A function* $F(z)$ *analytic in* $I(C)$ *and with a Faber expansion* $\Sigma a_n F_n(z)$ *is a rational function if and only if the function* $G(t)$ *defined by the associated power series* $\Sigma a_n t^n$ *is a rational function.*

3. PROOF OF LEMMA 3. It will be shown that $F(z)$ and $G(t)$ are related by a pair of complex inversion formulas. These form the analytical basis for the proof of the lemma.

3.1. *Cauchy's Integral Representation.* A function $F(z)$ analytic in $I(C)$ has the Cauchy integral representation

$$F(z) = (2\pi i)^{-1} \int_{C_2} F(\zeta)(\zeta - z)^{-1}\, d\zeta. \tag{4}$$

In (4), z is taken as a fixed point in $I(C)$, and C_2 is a level curve of C containing z and of height k_2 $(k_0 \leq k_2 < k)$. The sense of integration in this and other integrals is counterclockwise unless otherwise indicated. This choice for the path of integration is made in order to avoid possible singularities of $F(z)$ on C. The change of variable $\zeta = g(\omega)$ in (4) yields

$$F(z) = (2\pi i)^{-1} \int_{K_2} F[g(\omega)] g'(\omega) [g(\omega) - z]^{-1}\, d\omega, \tag{5}$$

where K_2 is the circle $|\omega| = k_2$. The next step is based on a fundamental contribution due to Faber. He showed [2, p. 391] that the coefficient of $F[g(\omega)]$ in the integrand in (5) is the generating function of the Faber polynomials, namely

$$g'(\omega)/[g(\omega) - z] = \Sigma F_n(z)/\omega^{n+1}. \tag{6}$$

This series converges uniformly on K_2. It can therefore be substituted in (5) and integrated term-by-term to give the relation

$$F(z) = \Sigma a_n F_n(z), \tag{7}$$

where

$$a_n = (2\pi i)^{-1} \int_{K_2} F[g(\omega)] \omega^{-n-1}\, d\omega. \tag{8}$$

This is the Faber expansion for $F(z)$.

3.2. *First Inversion Formula.* If $g'(\omega)/[g(\omega) - z]$ is replaced by $1/(\omega - t)$ in (5), for t fixed inside K_2, the following expressions are successively obtained:

$$(2\pi i)^{-1} \int_{K_2} F[g(\omega)](\omega - t)^{-1}\, d\omega \tag{9}$$

$$= (2\pi i)^{-1} \int_{K_2} F[g(\omega)]\, \Sigma t^n \omega^{-n-1}\, d\omega \tag{10}$$

$$= \Sigma a_n t^n = G(t). \tag{11}$$

The change of variable $\omega = f(\zeta)$ in (9) yields

$$G(t) = (2\pi i)^{-1} \int_{C_2} F(\zeta) f'(\zeta)[f(\zeta) - t]^{-1} \, d\zeta. \tag{12}$$

This is the desired formula.

3.3. *Second Inversion Formula.* If the path K_2 in (9) is replaced by any larger concentric circle lying inside K, the value of the integral is not changed. The representation then becomes valid for t inside the larger circle, and thus yields an analytic continuation of G(t). Therefore G(t) is analytic inside K, and has the Cauchy integral representation

$$G(t) = (2\pi i)^{-1} \int_{K_2} G(\omega)(\omega - t)^{-1} \, d\omega, \tag{13}$$

for all t inside K_2. It also follows that

$$a_n = (2\pi i)^{-1} \int_{K_2} G(\omega)\omega^{-n-1} \, d\omega. \tag{14}$$

Together with (6), the replacement of $1/(\omega - t)$ by $g'(\omega)/[g(\omega) - z]$ in (13), with z taken inside C_2, yields

$$(2\pi i)^{-1} \int_{K_2} G(\omega) g'(\omega)[g(\omega) - z]^{-1} \, d\omega \tag{15}$$

$$= (2\pi i)^{-1} \int_{K_2} G(\omega) \, \Sigma F_n(z) \omega^{-n-1} \, d\omega \tag{16}$$

$$= \Sigma \, a_n F_n(z) = F(z). \tag{17}$$

To summarize,

$$F(z) = (2\pi i)^{-1} \int_{K_2} G(\omega) g'(\omega)[g(\omega) - z]^{-1} \, d\omega \tag{18}$$

for z inside C_2. This is the second inversion formula.

3.4. *Proof of the Lemma.* Because of the symmetry of the inversion formulae (12) and (18), it is sufficient to prove that if G(t) is rational, then F(z) is also rational. Since G(t) can be decomposed into partial fractions, it remains to show that the function

$$\phi(z) = (2\pi i)^{-1} \int_{K_2} (\omega - a)^k g'(\omega)[g(\omega) - z]^{-1} \, d\omega \tag{19}$$

is rational. In (19), the point a is exterior to K_2, since G(t) is regular inside K. Now

(20) $$\phi(z) = \frac{1}{(k-1)!} \frac{d^{k-1}}{d^{k-1}w} \left(\frac{g'(w)}{g(w) - z} \right) \Bigg|_{w = a}$$

(21) $$= \frac{A_1}{g(a) - z} + \cdots + \frac{A_k}{[g(a) - z]^k},$$

where the A_i depend on a and $A_k \neq 0$. This is a rational expression in z, and the proof is complete.

4. APPLICATIONS. Lemmas 2 and 3 are first used to derive Iliev's result. It is then shown that with the preparations made the analogue for Faber expansions of the Carlson-Pólya theorem [1], [5] for power series follows immediately.

4.1. THEOREM 1 (Iliev). *Let* C *be a simple, closed, analytic curve with exterior mapping radius* $k < 1$. *Let* $\{F_n(z)\}$ *be the Faber polynomials associated with* C. *Furthermore, let the sequence* $\{d_n\}$ *assume only a finite number of different values. The Faber series* $\Sigma d_n F_n(z)$ *converges inside* C_1, *the level curve of height one, to a function* F(z). *This function either has each point of* C_1 *as a singular point, or it is a rational function.*

4.2. *Proof.* By Szegö's theorem, $G(t) = \Sigma d_n t^n$ is either regular inside the unit circle K_1 and has each point of K_1 as a singularity, or it is a rational function. This result is transferred to Faber series as follows. First, $\limsup_{n \to \infty} |d_n|^{1/n} = 1$, so that by Lemma 2 the level curve C_1 of height one is the curve of convergence of $\Sigma d_n F_n(z)$. Next, if G(t) is singular at each point of K_1, then by Lemma 2 each point of C_1 is a singular point of F(z). The other alternative is that G(t) is rational, and in this case F(z) is rational by Lemma 3. This completes the proof.

4.3. THEOREM 2. *Let* C *be a simple, closed analytic curve with exterior mapping radius* $k < 1$. *Let* $\{F_n(z)\}$ *be the Faber polynomials associated with* C. *Furthermore, let* $\{e_n\}$ *consist of integers, with* $\limsup_{n \to \infty} |e_n|^{1/n} = 1$. *The function* F(z), *represented by the series* $\Sigma e_n F_n(z)$ *inside the level curve* C_1 *of height one, either has each point of* C_1 *as a singularity, or it is a rational function.*

4.4. *Proof.* The theorem that for the series $\Sigma e_n t^n$ the alternatives of this proposition hold, with the curve $K_1 : |t| = 1$, replacing C_1, is due to Carlson and Pólya. The transfer to Faber series then follows exactly along the lines of 4.2.

5. CONCLUSION. Lemma 3 complements the results of Faber so as to yield a method for transferring theorems about power series to Faber series. From the applications it is seen, however, that the theorems transferred involve the notion of rationality in an essential way. It thus appears that the problem of transfer is capable of a more thorough solution. The author intends to return to the consideration of this possibility. The following remarks concern the further exploitation of the method as it stands, and an indication of further deductions from formula (18).

5.1. *Location of Poles.* Hadamard [3] studied the problem of locating the poles of a function by means of the coefficients of one Taylor series expansion of the function. It is seen from (21) that if a is a pole of G(t), then g(a) is a pole of the same order of F(z). This fact permits Hadamard's techniques to be applied directly to the coefficients of the Faber expansions of F(z) to locate its polar singularities.

5.2. *Region of Single-valuedness.* Lemma 3 determines the rational character of the applications. Deductions of a different nature from (18) are possible which permit new directions for application.

Suppose F(z), analytic in I(C), has the Faber expansion $\Sigma a_n F_n(z)$. A region of single-valuedness for F(z) is a simply connected domain which contains C and in which F(z) is regular. By the monodromy theorem, F(z) is single-valued in such a region. Information concerning the regions of single-valuedness of F(z) can be obtained from the function G(t) defined by the associated power series $\Sigma a_n t^n$. Indeed, let M be a region of single-valuedness of G(t). Then the contour K_2 in (18) can be replaced by a simple curve N lying in M and having preassigned closeness to the boundary of M, With this change, (18) yields an analytic continuation of F(z) into the region bounded by F, the image of N by $z = g(w)$. Because of the permitted freedom in chosing N, it follows that F(z) is single-valued in the domain bounded by the image, under $z = g(w)$, of the boundary of M. This is the desired result.

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