UPPER AND LOWER BOUNDS OF ORDER TYPES

by

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1. In 1940, Fraissé [1] defined the relation

$$\alpha \leqq \beta$$

to mean that an ordered set A of type α is similar to a subset of an ordered set B of type β. If, at the same time, B is not similar to any subset of A, then we shall write $\alpha < \beta$. It is obvious that this definition depends only on the order types α and β, and is independent of the special sets A and B. If $\alpha \leqq \beta$ and $\beta \leqq \alpha$ both hold, we shall write $\alpha \equiv \beta$ and say that α and β are equivalent (even though α and β may be distinct). If neither $\alpha \leqq \beta$ nor $\beta \leqq \alpha$ holds, then α and β will be said to be non-comparable.

In terms of these relations it is natural to discuss the notions of upper and lower bounds of two order types, or their least upper and greatest lower bounds. Thus, γ would be called a least upper bound for α and β if $\alpha \leqq \gamma$, $\beta \leqq \gamma$, while for any δ such that $\alpha \leqq \delta$ and $\beta \leqq \delta$ it would follow that either $\gamma < \delta$ or that γ and δ are non-comparable.

2. Throughout this note we shall assume as known the usual terminology and symbolism for order-types and ordinals.

The purpose of this note is to give a method for demonstrating the following theorem:

If $\alpha = \omega \cdot r + m$, $\beta = \omega \cdot s + n$, where r and s are natural numbers and m and n are integers $\geqq 0$, then α and β^* have only a finite number of distinct least upper bounds, namely, all types of the form

$$\text{(I)} \qquad n + \omega^* \cdot b_1 + \omega a_1 + \dots + \omega^* \cdot b_t + \omega \cdot a_t + m$$

where t and the coefficients $a_1, \dots, a_t, b_1, \dots, b_t$ are natural numbers except that b_1 or a_t may be 0, and

$$\sum_{i=1}^{t} a_i = r, \qquad \sum_{i=1}^{t} b_i = s.$$

We do not actually prove this theorem; however its proof would be only a slight modification of the proof of Theorem VI in section 4.

Hereafter, we shall call the types (I), with m = n = 0, the mixed sums of α and β^*; similarly, an order-type γ will be called a mixed sum if γ can be represented in the form (I) for some ordinals α and β.

3. We first prove a number of auxiliary theorems about mixed sums and their relation to general order types.

Theorem I: Let $\alpha = \omega \cdot r$ and $\beta = \omega \cdot s$, where r and s are natural numbers, and let γ, and δ be two mixed sums of α and β *. Then $\gamma \leqq \delta$ if and only if $\gamma = \delta$.

Proof: Let C and D be ordered sets such that $\overline{C} = \gamma$, $\overline{D} = \delta$. Then C can be represented as an ordered sum of disjoint ordered sets, each of these summands being of type ω or ω *; say,

$$C = C_1 + C_2 + \ldots + C_j + \ldots + C_m$$

where $m = r + s$. In exactly the same way, with the same conditions, we may write

$$D = D_1 + D_2 + \ldots + D_j + \ldots + D_m .$$

Since $\gamma \leqq \delta$, let f be a similarity transformation which carries C into a subset of D:

$$f(C) \subset D .$$

For any natural j between i and m inclusive, we have

$$f(C_j) = f(C_j) \cap D_1 + f(C_j) \cap D_2 + \ldots + f(C_j) \cap D_m ,$$

and therefore

$$\overline{f(C_j)} = \overline{f(C_j) \cap D_1} + \ldots + \overline{f(C_j) \cap D_m} .$$

But

$$\overline{f(C_j)} = \overline{C_j} = \omega \text{ or } \omega^*$$

Since in any representation of ω or ω^* as a finite sum, there is exactly one infinite summand, it follows that only one of the summands of $f(C_j)$ is an infinite set, while all the others are finite (or empty). Hence, by discarding a finite number of the elements of C -- which does not change the type of either C or of any of its summands ---, we will have that f will carry any summand of C into just one summand of D. We suppose that this is already so; it is clear, further, that f will carry different summands of C into different summands of D. Since the number of summands in both C and D is the same finite number m, we must have

(II) $$f(C_1) \subset D_1,\ f(C_2) \subset D_2,\ \ldots,\ f(C_m) \subset D_m .$$

Finally, since the type of any C_j or D_j is ω or ω^* ,

$$\overline{C_1} = \overline{f(C_1)} = \overline{D_1},\ \ldots,\ \overline{C_m} = \overline{f(C_m)} = \overline{D_m} ,$$

and so $\gamma = \delta$.

Obviously, if $\gamma = \delta$, then $\gamma \leqq \delta$. This completes the proof.

The above proof, particularly the inclusions in (II), indicate the truth of the following theorem:

Theorem II: If two mixed sums of α and β^* differ in the magnitude and/or order of their coefficients, then they represent distinct order-types.

Theorem III: If α is mixed sum and $\beta \leqq \alpha$, then either $\beta < \alpha$ or $\beta = \alpha$.

Proof: Let A and B be ordered sets such that $\overline{A} = \alpha$ and $\overline{B} = \beta$, and let f be a similarity transformation which carries B into an ordered subset of A. By hypothesis

$$A = A_1 + A_2 + \dots + A_j + \dots + A_m$$

where $A_i \cap A_j = 0$, $i = j$, and $\overline{A_j} = \omega$ or ω^*, $i, j = 1, \dots, m$. Since $f(B) \subset A$, we have

$$\beta = \overline{B} = \overline{f(B)} = \overline{A'_1} + \dots \overline{A'_j} + \dots \overline{A'_m},$$

where $A'_j = f(B) \cap A_j$, $1 \leqq j \leqq m$. If each A'_j is infinite, then $\overline{A'_j} = \overline{A_j}$ and

$$\beta = \overline{A_1} \dots + \overline{A_j} + \dots \overline{A_m} = \overline{A} = \alpha .$$

If even one of the A'_j is finite (or empty) then β is a sum of m summands, with less than m summands of type ω and ω^*, the other summands being finite. Since α is the sum of precisely m ω or ω^* summands, A cannot be similar to an ordered subset of B (if A'_j is finite), and so $\alpha \leqq \beta$ is false in this case. This completes the proof.

Theorem IV: If α is a mixed sum and $\beta \equiv \alpha$, then $\beta = \alpha$.

Proof: If $\beta \neq \alpha$, then since $\beta \equiv \alpha \rightarrow \beta \leqq \alpha$, we conclude by the preceding theorem that $\beta < \alpha$, i.e., that $\alpha \leqq \beta$ is false. But $\beta \equiv \alpha$ does imply $\alpha \leqq \beta$.

Theorem V: If $\alpha = \omega \cdot r$ and $\beta = \omega \cdot s$, where r and s are natural numbers, and if δ is an upper bound for α and β^*, then there exists a mixed sum γ of α and β^* such that $\gamma \leqq \delta$.

Proof: Let D be an ordered set of type δ, and let

$$A = A_1 + A_2 + \dots A_r$$

and

$$B = B_1 + B_2 + \dots + B_s$$

be ordered subsets of D such that $\overline{A} = \alpha$, $\overline{B} = \beta^*$, where $\overline{A_i} = \overline{A_j} = \omega$, $A_i \quad A_j = 0$ for $i \neq j$, $i \leqq r$, $j \leqq r$, and similarly for the set B. Since $\overline{A_i} = \omega$ and $\overline{B_j} = \omega^*$, then $A_i \quad B_j$ is finite or empty for any $i \leqq r$ and $j \leqq s$. Hence by discarding a finite number of elements from D, we will have

$$A \cap B = 0,$$

and for convenience we also suppose that, as unordered sets

$$A \cup B = D .$$

Suppose now that a_1 is the first element of A in D, and that a_2 is the first element of A after a_1 which is separated (in D) from a_1 by an element of B. In general, if a_k is already defined, let a_{k+1} be the first following element of A which is separated (in D) from a_k by an element of B. Let C_{2k-1} be the part of A between a_k and a_{k+1}, including a_k and excluding a_{k+1}, and let C_{2k} be the part of B between the same two elements. Clearly, every element of C_{2k-1} precedes any element of C_{2k}, and C_{2k}, as a subset of an inversely well-ordered set, must possess a last element, which we denote by b_k. It is further clear that b_k is the immediate predecessor of a_{k+1} in D, and that

$$b_1 < b_2 < \dots$$

This last sequence, being an increasing sequence of elements of B, must be finite; in other words, there will exist a natural number $n(\geqq 1)$ such that

$$a_1 < a_2 < \dots < a_n$$

and such that a_{n+1} is not defined. Finally, if C_1 is the subset of B which precedes a_1 in D, we can present D as an ordered sum of the ordered sets C:

$$D = C_1 + C_2 + \dots + C_{2n} + C_{2n+1} .$$

Here, C_1 and C_{2n+1} may be empty, while all the others are not, and they are alternately well-ordered and inversely well-ordered. If now a summand of A, say A_j, has elements in common with more than one of the C's, then the separation of A_j induced by this can only be of the form

$$A_j = A_j^1 + A_j^2 + A_j^m + A_j^{m+1} ,$$

where $m \geqq 1$ and $\overline{A_j^{m+1}} \doteq \omega$, the first m summands being finite. Similar remarks apply to the summands of B. Once again, therefore, by discarding a certain finite number of elements of D, we will have that α and β^* are still bounded above by $\overline{D}$, that any summand of A or B belongs to just one of the summands of D, and that, finally, any C is the ordered sum of a group of consecutive summands of either A alone or B alone. But this last means that the type of D is a mixed sum of α and β, and our proof is complete.

4. We are now ready to prove the main result of this note (see section 2):

Theorem VI: If α and β are limit ordinals $< \omega^2$, then the mixed sums of α and β^* are the only least upper bounds of α and β^*.

Proof: It is obvious that these mixed sums are upper bounds for the pair α and β^*. Suppose now that δ is any order type which is not a mixed sum and which is an upper bound for α and β^*. Then, by Theorem V, there exists a mixed sum γ for which $\gamma \leqq \delta$, and therefore, by Theorem IV, $\gamma < \delta$ (since $\gamma = \delta$ is excluded). This shows that δ cannot be a least upper bound. If δ is a type which is non-comparable with any of the mixed sums then Theorem V shows that δ cannot be an upper bound to both α and β^*.

Finally, suppose that γ_1 is a mixed sum, that $\delta < \gamma_1$, and we do have

$$\alpha \leqq \delta \ , \ \beta^* \leqq \delta \ .$$

Let now γ_2 be any mixed sum for which by Theorem V

$$\gamma_2 \leqq \delta$$

Then $\gamma_2 \leqq \delta$ and $\delta < \gamma_1$ would give

$$\gamma_2 < \gamma_1$$

which, by Theorem 1, is impossible. This means that γ_1 is a least upper bound for α and β^*. Our proof is complete.

5. Concerning lower bounds, we have the following theorem:

Theorem VII: If α and β are transfinite ordinals, then α and β^* have no greatest lower bound.

Proof: Any ordered set C_1 whose order type γ is a lower bound for the pair α , β^*, would be simultaneously similar to a subset of a well-ordered set and to a subset of an inversely well-ordered set; thus C itself would at the same time be well-ordered and inversely well-ordered, and therefore finite. Hence γ is a finite ordinal. But it is obvious that any finite ordinal is a lower bound for α and β^*. Hence no greatest lower bound can exist in this case.

Bibliography

[1] A note in the Comptes Rendus, 226, 1948, p. 1330.

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September, 1953