CONFORMAL MAPPINGS AND PEANO CURVES

by

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Salem and Zygmund [1945] have shown that if $\{n_k\}$ is a sequence of integers with the property $n_{k+1}/n_k \geq \lambda > \lambda_0$, where λ_0 is ome universal constant less than $1 + 10\sqrt{2}\,\pi$, then the function $\sum a_k z^{n_k}$ maps the unit circle into a Peano curve provided the series $\sum |a_k|$ converges so slowly that

$$\lambda^{1-p}|a_1| + \lambda^{2-p}|a_2| + \ldots + |a_p| < c \sum_{k>p} |a_k|$$

for $p = 1, 2, \ldots$; the constant c depends on λ; it is sufficient that it be less than

$$[\lambda(\lambda - 1) - 2^{3/2}\pi(5\lambda - 1)]/[\lambda(\lambda - 1) + 2^{3/2}\pi(5\lambda - 1)].$$

In the present note, we present a theorem which is contained in the result of Salem and Zygmund. Its publication is justified by the extreme simplicity of the proof.

Theorem. There exists a function $f(z)$ which is holomorphic in the open unit disc and continuous in the closed unit disc, and which has the property that the set of points $f(e^{i\theta})$ $(0 \leq \theta < 2\pi)$ fills a square.

The example by means of which the theorem will be proved is of the form

$$f(z) = \sum_{r=1} k_r[1 - (1 - z/t_r)^{\alpha_r}],$$

where $\{k_r\}$ is a complex null sequence, $\{t_r\}$ is a sequence of distinct points on the unit circle, and $\{\alpha_r\}$ is a sequence of positive constants approaching zero very rapidly; the function $(1 - r/t_r)^{\alpha_r}$ is understood to be real and positive on the line segment joining the origin to the point $z = t_r$. It should be observed that, with the notation

$$f_r(z) = k_r[1-(1-z/t_r)^{\alpha_r}],$$

$f_r(t_r) = k_r$. Moreover, if N_r is any neighborhood of the point t_r, the quantity $|f_r(z)/k_r|$ can be made arbitrarily small in the complement of N_r relative to the unit disc, simply by choosing the constant α_r small enough; and throughout the unit disc, $|f_r(z)/k_r| \leq 1$.

Let Q be the square whose sides lie on the four lines $x = \pm 1$, $y = \pm 1$. Let Q be divided into four equal squares Q_1, Q_2, Q_3, Q_4 (with the subscripts chosen so that they indicate the quadrants in which the squares lie); let each of these be subdivided into four equal squares, and let these be denoted by Q_{11}, Q_{12}, Q_{13}, Q_{14}, Q_{21}, ... , Q_{44}; and so forth. For r = 1, 2, 3, 4, let $t_r = k_r = e^{i\theta_r}$, where $\theta_r = (2r+1)\pi/4$; then, if the constants α_r are chosen small enough, the curve C_4 into which the function $F_4(z) = \sum_{r=1}^{4} f_r(z)$ maps the unit circle passes through the interior of each of the squares Q_1, Q_2, Q_3, Q_4. For r = 5, 6, 7, 8, let t_r be four points such that the points $F_4(t_r)$ lie in Q_1; let k_5 be chosen so that $F_5(t_5)$ lies in Q_{11}, and let α_5 be small enough so that the curve C_5 still passes through the interiors of Q_2, Q_3, and Q_4. Let k_6 be chosen so that $F_6(t_6)$ lies in Q_{12}, and let α_6 be small enough so that C_6 passes through Q_2, Q_3, Q_4, and Q_{11}. And let this process be continued, subject to the further precaution that at each stage the constant α_r is chosen small enough so that $|f_r(z)| < 1/r^2$ when z lies in the complement, relative to the unit disc, of some circular disc containing t_r and containing none of the points $t_1, t_2, \ldots, t_{r-1}$.

Since the sequence k_r tends to zero, and since the inequality

$$\sum_{r=1}^{\infty} |f_r(z)| \leq \sqrt{2} \sum_{j=1}^{\infty} 2^{2-j} + \sum_{r=1}^{\infty} 1/r^2$$

holds throughout the unit disc, the function f(z) is holomorphic in $|z| < 1$ and continuous in $|z| \leq 1$. Because each square obtained in the subdivision of Q contains one of the points $f(e^{i\theta})$, the theorem is proved.

Note 1. If the precaution is taken that each of the curves C_r lies in the interior of Q, then the set of points $f(e^{i\theta})$ $(0 \leq \theta < 2\pi)$ is identical with Q.

Note 2. If the sequence $\{\alpha_r\}$ approaches zero rapidly enough, the series $\sum f_r(z)$ converges uniformly in every bounded set in the plane. Because the construction does not require the set $\{t_r\}$ to be everywhere dense on the unit circle, it is possible to choose the constants in such a way that the function f(z) is holomorphic in R, where R is obtained from the complex plane by deleting an appropriate arc of the unit circle and a curve joining this arc to the point at infinity.

Note 3. The set of singular points of f(z) on the unit circle can be made to be a set of measure zero. This is not the case with the functions constructed by Salem and Zygmund; there the unit circle is necessarily a natural boundary.

Note 4. If the sequence $\{\alpha_r\}$ approaches zero rapidly enough, the Taylor series of our function is certain to have uniform, non-absolute convergence on the unit circle. As the description stands, the series does not have gaps. But since the Taylor series for each of the functions $f_r(z)$ converges absolutely on the unit circle, gaps can be introduced by replacing each of the functions $f_r(z)$ by the product of a partial sum of its Taylor series with an appropriate power of z/t_r.

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Bibliography

R. Salem and A. Zygmund, Lacunary power series and Peano curves, Duke Math. J. 12 (1945) 569-578.