CONFORMAL MAPPING OF A JORDAN REGION WHOSE BOUNDARY HAS POSITIVE TWO-DIMENSIONAL MEASURE

by

A. J. Lohwater and G. Piranian

Jordan curves which pass through plane sets of positive two-dimensional measure have been constructed by Osgood [4] and Kline [1]. The present note is concerned with the relation between such Jordan curves and the following general problem. Let R be the finite region in the w-plane bounded by a Jordan curve C, and let E_w be a set of points on C. Let the function f(z) be continuous and univalent in $|z| \leq 1$ and holomorphic in $|z| < 1$, and let it map the set $|z| \leq 1$ upon the closure of R. The general problem (still unsolved) is that of finding necessary and sufficient conditions on E_w in order that the image E_z on the circle $|z| = 1$ have linear measure zero. A sufficient condition is that no points of E_w be accessible from R by rectifiable arcs ([2], [5], [6]). By means of this sufficient condition, the following result will be established.

THEOREM. There exists a Jordan region R such that, under a conformal mapping of R onto the region $|z| < 1$, a certain set lying on the boundary of of R and having positive two-dimensional measure is mapped into a set of linear measure zero on $|z| = 1$.

Let A be an isosceles right triangle in the w-plane. Let B_0 denote the intersection of A with a closed strip bounded by two parallel lines, one of which passes through the hypotenuse of A, while the other intersects the interior of A; and let A_0 denote the complement of B_0 relative to A. Let $B_{1,1}$ denote the intersection of A_0 with a closed strip which is symmetric with respect to the altitude upon the hypotenuse of A. If B_0 and $B_{1,1}$ are narrow enough, the complement relative to A of their union consists of two isosceles right triangles $A_{1,1}$ and $A_{1,2}$. From these two triangles we remove strips $B_{2,1}$ and $B_{2,2}$, each of which is symmetric with respect to the altitude upon the hypotenuse of the corresponding triangle; and we denote by $A_{2,r}$ $(r = 1, 2, 3, 4)$ the four triangles that remain. The process continues: at the n-th stage, strips $B_{n,r}$ $(r = 1, 2, \ldots, 2^{n-1})$ are removed from the triangles $A_{n-1,r}$, and this leaves 2^n triangles $A_{n,r}$.

Let M denote the union of B_0 and all the strips $B_{2n+1,r}$ $(n = 0, 1, \ldots; r = 1, 2, \ldots, 2^{2n})$, i.e. of all the strips in the construction that are parallel or perpendicular to the hypotenuse of A. Since every point of A which does not lie in one of the strips B_0 or $B_{n,r}$ is a boundary point of M, it may be assumed that the boundary of M is a set of positive two-dimensional measure (for this, it is only necessary that the strips constructed at the various stages be sufficiently narrow).

Let n and r be two integers $(n=1, 2, \ldots;$ $1 \leq r \leq 2^{2n-2})$. Let $b_{2n-1,r}$ denote that one of the strips which was constructed prior to $B_{2n-1,r}$ and has common boundary points with it. And let $B_{2n,s}$ and $B_{2n,t}$ be the two strips whose boundaries meet both those of $B_{2n-1,r}$ and $b_{2n-1,r}$. Then the closures of $B_{2n-1,r}$ and $B_{2n,s}$ meet in a line seqment on a side of $B_{2n-1,r}$; the same is true of the closures of $B_{2n-1,r}$ and $B_{2n,t}$. The two line segments just mentioned lie near the juncture of $b_{2n-1,r}$ and $B_{2n-1,r}$, and they shall therefore be called the guarding segments of $B_{2n-1,r}$. In the strip $B_{2n-1,r}$ we now erect a finite number of narrow rectangles; each of these rectangles has one of its short sides on one of the guarding segments of $B_{2n-1,r}$, and its long sides reach almost all the way across $B_{2n-1,r}$. Also, the rectangles erected in $B_{2n-1,r}$ are sufficiently numerous, and are arranged in such a way, that any path in M which leads from $b_{2n-1,r}$ to a point in $B_{2n-1,r}$ beyond the guarding segments, and which does not meet any of the rectangles, is necessarily of length greater than one.

Let $C_{2n-1,r}$ be the set of those points in $B_{2n-1,r}$ which do not lie in any of the rectangles described above. Let R be the interior of the union of B_0 and all the sets $C_{2n-1,r}$, and let C be the boundary of R. Since the boundary of M has positive two-dimensional measure, the same is true of the curve C. The proof that C is a Jordan curve is an obvious modification of the proof on p. 142 of [3], and is omitted here. Finally, any rectifiable path which lies in R except for one end point terminates on one of denumerably many rectilinear segments on C; that is, the set of points on C which are finitely accessible from R is a set of two-dimensional measure zero, and therefore the set E of points on C which are not finitely accessible from R is a set of positive two-dimensional measure. Under any conformal mapping of the unit circle upon R, E is the image of a set of linear measure zero, and the proof of our theorem is complete.

Bibliography

1. J. R. Kline, What is the Jordan curve theorem?, Amer. Math. Monthly 49 (1942) 281-286.
2. M. Lavrentiev, Sur la représentation comforme, C.R. Acad. Sci. Paris 184 (1927) 1407-1409.
3. A. J. Lohwater and W. Seidel, An example in conformal mapping, Duke Math. J. 15(1948) 137-143.
4. W. F. Osgood, A Jordan curve of positive area, Trans. Amer. Math. Soc. 4 (1903) 107-112.
5. W. Seidel and J. L. Walsh, On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence, Trans. Amer. Math. Soc. 52 (1942) 128-216.
6. M. Tsuji, On the theorems of Caratheodory and Lindelöf in the theory of conformal representation. Jap. J. Math. 7 (1930) 91-99.

The University of Michigan March, 1951