THE CONSTRUCTION OF AUTOMORPHIC FORMS FROM THE DERIVATIVES OF GIVEN FORMS

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1. I have recently [4] given a method for constructing automorphic forms from the derivatives of a given form, and have found a basis from which all such forms can be obtained. I show here how this method can be applied when there is more than one given form.

Let T denote the bilinear transformation

$$w = \frac{az+b}{cz+d} = Tz,$$

and let $\mathfrak{H}$ be the half-plane $\operatorname{Im} z > 0$. Let Γ be a given horocyclic group (see [3]), and suppose that $f_0(z), f_1(z), \cdots, f_m(z)$ are $m+1$ given meromorphic automorphic forms of real or complex dimensions $-k_0, -k_1, \cdots, -k_m$, with multiplier systems $v_0, v_1, \cdots, v_m$, and belonging to Γ. We write this

$$f_i \in \{\Gamma, k_i, v_i\} \qquad (i = 0, 1, \cdots, m).$$

We then have

$$f_i(w) = f_i(Tz) = v_i(T)(cz+d)^{k_i} f_i(z),$$

for all $T \in \Gamma$ and $z \in \mathfrak{H}$. For nonintegral k_i, $(cz+d)^{k_i}$ denotes a certain uniquely determined root of $cz+d$. See, for example, [1]. We then have, as is shown in [4],

$$(1) \quad f_i^{(\nu)}(w) = v_i(T) S^{k_i+2\nu} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} (k_i+\nu-1)(k_i+\nu-2)\cdots(k_i+\nu-\mu)\lambda^{\mu} f_i^{(\nu-\mu)}(z),$$

where

$$(2) \qquad S = cz+d, \quad \lambda = c/S.$$

Our object is to find those polynomials $P(z)$ in the given functions f_i and their derivatives, which, for every Γ, k_i and v_i, are automorphic forms belonging to $\{\Gamma, k', v'\}$, for some dimension $-k'$ and multiplier system v' depending on the k_i and v_i respectively. Each term in $P(z)$ is of the form

$$g_i f_i^{\alpha_0} \left(f_i'\right)^{\alpha_1} \left(f_i''\right)^{\alpha_2} \cdots \left(f_i^{(\nu)}\right)^{\alpha_\nu},$$

where the coefficient g_i is a product of the functions f_j and their derivatives for $j \neq i$. We say that such a term has degree r_i and weight s_i in f_i, and total degree r and total weight s, if

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(3) $$\alpha_0 + \alpha_1 + \cdots + \alpha_\nu = r_i, \qquad \alpha_1 + 2\alpha_2 + \cdots + \nu\alpha_\nu = s_i,$$

and

(4) $$r = \sum_{i=0}^{m} r_i, \qquad s = \sum_{i=0}^{m} s_i.$$

If $P \in \{\Gamma, k', v'\}$, then $P(w)$ is transformed, by means of (1), into a polynomial in λ from which all terms with positive powers of λ cancel out. It follows that, for each term in $P(z)$, we must have

(5) $$\prod_{i=0}^{m} v_i^{r_i} = v', \qquad 2s + \sum_{i=0}^{m} k_i r_i = k'.$$

Such a polynomial $P(z)$ we call an *admissible* polynomial.

These conditions are satisfied, in particular, if $v_i = v$, $k_i = k$ for $i = 0, 1, \cdots, m$, and if each term in $P(z)$ has the same total degree r and total weight s. For we then have

(6) $$v' = v^r, \qquad k' = 2s + kr.$$

As in [4] we define

(7) $$h_{i\nu}(z) = h_{i\nu} = \frac{f_i^{(\nu)}}{\Gamma(k_i + \nu)\,\nu!}, \qquad c_{i\nu}(z) = c_{i\nu} = \psi_{i\nu}/h_{i0}^{\nu},$$

where

(8) $$\psi_{i\nu}(z) = \psi_{i\nu} = (-1)^{\nu-1} \begin{vmatrix} h_{i1} & 2h_{i2} & 3h_{i3} & \cdots & (\nu-h)h_{i,\nu-1} & \nu h_{i\nu} \\ h_{i0} & h_{i1} & h_{i2} & \cdots & h_{i,\nu-2} & h_{i,\nu-1} \\ 0 & h_{i0} & h_{i1} & \cdots & h_{i,\nu-3} & h_{i,\nu-2} \\ \cdot & \cdot & & \cdots & & \cdot \\ 0 & 0 & 0 & \cdots & h_{i1} & h_{i2} \\ 0 & 0 & 0 & \cdots & h_{i0} & h_{i1} \end{vmatrix}$$

and we consider only points z of $\mathfrak{H}$ at which the functions f_i and their reciprocals are holomorphic.

Then, for $\nu \geqslant 2$, $c_{i\nu} \in \{\Gamma, 2\nu, 1\}$ and $\psi_{i\nu} \in \{\Gamma, \nu(k_i + 2), v_i^{\nu}\}$, and Theorem 1 of [4] states that, if k_i is not a nonpositive integer, a polynomial $P_i(z)$ in f_i and its derivatives (for fixed i) belong to $\{\Gamma, r_i k_i + 2s_i, v_i^{r_i}\}$ if and only if P_i is of the form $f_i^{r_i - s_i} Q_i$, where Q_i is a polynomial of weight s_i in the functions $\psi_{i2}, \psi_{i3}, \cdots$. This theorem we now generalize. To do so we need to introduce m new functions

(9) $$w_i(z) = w_i = k_0 f_0 f_i' - k_i f_0' f_i, \qquad t_i(z) = t_i = w_i/(f_0 f_i),$$

for $i = 1, 2, \cdots, m$. These are particular cases of the Wronskian functions defined in formula (23) of [4]. It is easily verified that

$$w_i \in \{\Gamma, k_0 + k_i + 2, v_0 v_i\}, \qquad t_i \in \{\Gamma, 2, 1\}.$$

For example, if we take $m = 1$, $f_0 = G_4$ and $f_1 = G_6$, where G_4 and G_6 are the Eisenstein series of dimensions -4 and -6 belonging to the full modular group and having constant terms unity in their Fourier expansions, we obtain

$$w_1 = 4G_4G_6' - 6G_6G_4' = -6912\pi i \triangle = 4\pi i(G_6^2 - G_4^3).$$

Here $\triangle$ is the modular discriminant.

2. We prove the following theorem.

THEOREM 1. *Suppose that, for* $i = 0, 1, \cdots, m$, $f_i(z) \in \{\Gamma, k_i, v_i\}$, *where the* k_i *are not zero and are not positive integers, and that* $P(z)$ *is an admissible polynomial in the functions* f_i *and their derivatives. Then* $P(z) \in \{\Gamma, k', v'\}$, *where* k' *and* v' *are defined by* (5), *if and only if* P *is a polynomial in the functions* f_i, $c_{i\mu}$ $(\mu \geq 2)$ *and* t_i. *This polynomial can be written as a polynomial in the functions* f_i, $1/f_i$, $\psi_{i\mu}$ $(\mu \geq 2)$ *and* w_i.

Proof. Suppose that $P \in \{\Gamma, k', v'\}$ and that τ is a term of $P(z)$ whose degrees r_i, weights s_i, and total weight s satisfy (3), (4) and (5). Then

$$\tau = f_0^{r_0} f_1^{r_1} \cdots f_m^{r_m} \tau^*,$$

where τ^* is a product of powers of the quotients $h_{i\mu}/h_{i0}$ times a constant factor. For $\mu \geq 2$, by (7) and (8),

$$\frac{h_{i\mu}}{h_{i0}} = \frac{c_{i\mu}}{\mu} + c_{i\mu}^*,$$

where $c_{i\mu}^*$ is a polynomial in the $h_{i\kappa}/h_{i0}$ for $\kappa < \mu$. By eliminating the $h_{i\mu}/h_{i0}$ successively in this way, we express τ^* as a polynomial in the functions $c_{i\mu}$ and f_i'/f_i $(\mu \geq 2;\ i = 0, 1, \cdots, m)$, with constant coefficients. We now put

$$\frac{f_i'}{f_i} = \frac{1}{k_0}\left\{t_i + k_i\frac{f_0'}{f_0}\right\} \qquad (i = 1, 2, \cdots, m),$$

and in this way we express τ^* as a polynomial of the form

$$\sum_{q=0}^{n} p_q^* \left\{\frac{f_0'}{f_0}\right\}^q,$$

where p_q^* is a polynomial in the $c_{i\mu}$ $(\mu \geq 2;\ i - 0, 1, \cdots, m)$ and t_i $(i = 1, 2, \cdots, m)$, of total weight $s - q$. Here terms $c_{i\mu}^{\alpha_{i\mu}}$, t^{β_i} are counted as having degrees zero and weights $\mu\alpha_{i\mu}$, β_i respectively. Hence $p_q^* \in \{\Gamma, 2s - 2q, 1\}$, and so

$$P(z) = \sum_{q=0}^{n} p_q \left\{\frac{f_0'}{f_0}\right\}^q,$$

where p_q is a polynomial in the f_i, $c_{i\mu}$ ($\mu \geq 2$, $0 \leq i \leq m$) and t_i ($1 \leq i \leq m$), and where $p_q \in \{\Gamma, k' - 2q, v'\}$. Each term of p_q is of degree r_i in the functions f_i and of total weight $s - q$, where the r_i and s satisfy (5).

For $T \in \Gamma$ we have

$$v' S^{k'} P(z) = P(w) = \sum_{q=0}^{n} \{p_q v' S^{k'-2q}\}\{S^2(k_0\lambda + f_0'/f_0)\}^q ,$$

from which it follows that

$$\sum_{q=1}^{n} p_q\{(k_0\lambda + f_0'/f_0)^q - (f_0'/f_0)^q\} = 0 .$$

This holds for all $\lambda = c/(cz + d)$. For fixed $z \in \mathfrak{H}$ there are infinitely many different values of λ corresponding to different $T \in \Gamma$, and hence this equation is an identity in λ. Taking $\lambda = -f_0'/(k_0 f_0)$, we see that

$$\sum_{q=1}^{n} p_q \left\{\frac{f_0'}{f_0}\right\}^q = 0,$$

and so $P(z) = p_0$; that is, $P(z)$ is a polynomial in the functions f_i, $c_{i\mu}$ ($\mu \geq 2$) and t_i. When expressed in terms of the functions f_i, $\psi_{i\mu}$ and w_i, $P(z)$ becomes a polynomial in the functions f_i, $1/f_i$, $\psi_{i\mu}$ and w_i.

This completes the proof of the necessity of the conditions stated in the theorem; that the conditions are sufficient is obvious since the functions f_i, $c_{i\mu}$ and t_i are automorphic forms.

A similar but more complicated result holds when some or all of the k_i are allowed to be nonpositive integers. The last sentence of Theorem 1 remains true if, for those values of i for which k_i is a nonpositive integer, the functions $\psi_{i\mu}$ ($\mu \geq 2$) are replaced by $f_i^{(N_i)}$, $1/f_i^{(N_i)}$ and functions analogous to the functions δ_2, ϕ_μ, χ_ν defined in [4] (formulae (15), (17), (18) and Theorem 3), where $2 \leq \mu \leq -k_i = N_i - 1$, $\nu \geq 2$. This generalizes Theorem 3 of [4]. It may be noted that Theorem 3 of [4] arises as a particular case of Theorem 1 of the present paper, when this theorem is applied to the pair of automorphic forms $f(z)$ and $f^{(N)}(z)$.

3. In this section we drop the suffix i from the symbols f_i and $\psi_{i\nu}$, since we restrict the discussion to the case where there is only one given form. In [4], the homogeneous differential equation

$$13(\Delta')^4 + 10\Delta^2\Delta'\Delta''' - 24\Delta(\Delta')^2\Delta'' - 2\Delta^3\Delta^{(iv)} + 3\Delta^2(\Delta'')^2 = 0 \tag{10}$$

for the modular discriminant $\Delta = \Delta(z)$ was obtained by applying Theorem 1 of [4] to $f = \Delta$ and using the fact that, in this case, ψ_4 is a constant multiple of ψ_2^2. I am indebted to the referee for pointing out that the equation (10) was given by van der Pol (see formula (57) on p. 282 of [2]).

Suppose now, more generally, that Γ is a zonal (see [3]) horocyclic group and that $f \in \{\Gamma, k, v\}$, where f is an integral automorphic form having a zero of order m at infinity ($m > 0$). Then we see from (8) that ψ_ν is an integral form of $\{\Gamma, \nu(k + 2), v^\nu\}$, with a zero of order at least νm at infinity ($\nu \geq 2$). Now let

$N_\nu = N_\nu(\Gamma, k, v, m)$ be the number of linearly independent integral forms of $\{\Gamma, \nu(k+2), v^\nu\}$ having zeros of order at least νm at infinity, and let $\hat{p}(\nu)$ denote the number of partitions of ν in which each part is at least 2. Then

$$\hat{p}(\nu) = p(\nu) - p(\nu - 1),$$

where $p(\nu)$ is the number of unrestricted partitions of ν.

If ν is sufficiently large so that

$$\hat{p}(\nu) > N_\nu, \tag{11}$$

then some linear combination of the $\hat{p}(\nu)$ different forms of $\{\Gamma, \nu(k+2), v^\nu\}$, which can be formed by multiplying powers of the ψ_ν $(\nu \geq 2)$, must vanish; this yields a homogeneous differential equation for f of degree n_k, where n_k is the least integer ν such that (11) holds.

For example, if Γ is the full modular group with $v = 1$, and

$$f(z) = \Delta(z)\, G_{k-12}(z)$$

(with $G_0 = 1$), then we have $n_{12} = 4$, which yields (10). Similarly

$$n_k = 8 \text{ for } k = 16, 18;$$

$$n_k = 10 \text{ for } k = 20, 22;$$

$$n_k = 11 \text{ for } k = 24;$$

$$n_k = 12 \text{ for } k = 26, 28;$$

$$n_k = 13 \text{ for } k = 30;$$

$$n_k = 14 \text{ for } k = 32, 34, 36, 38.$$

For in these cases $m = 1$ and

$$N_\nu = \left[\frac{\nu(k-10)}{12}\right] + 1 \text{ if } \nu(k-10) \not\equiv 2 \pmod{12},$$

$$N_\nu = \left[\frac{\nu(k-10)}{12}\right] \text{ if } \nu(k-10) \equiv 2 \pmod{12}.$$

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