FABER SERIES AND THE LAURENT DECOMPOSITION

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1. INTRODUCTION

This paper deals with the problem of transfer described by J. L. Ullman [5]. Roughly speaking, we should like to have a method for deciding what statements about a Faber series

$$\sum_{0}^{\infty} a_n F_n(z) \tag{1}$$

are equivalent to analogous statements about the associated power series

$$\sum_{0}^{\infty} a_n z^n \tag{2}$$

with the same coefficients.

Ullman partially solved the problem by means of a lemma concerning the rationality of functions: if one of the series (1) and (2) represents a rational function, then the same is true of the other. The lemma leads to an immediate proof, for example, of Iliev's analogue [2] of Szëgo's theorem on power series whose coefficients assume only a finite number of different values.

We shall establish a result (Theorem 2) which is at the same time more elementary and more general than Ullman's lemma. It asserts that if $f(z)$ denotes the mapping function, normalized at $z = \infty$, which is associated with the analytic curve C giving rise to the Faber sequence $\{F_n(z)\}$, then the difference between the series (1) and the series

$$\sum_{0}^{\infty} a_n [f(z)]^n \tag{3}$$

can be continued so as to be holomorphic everywhere on C and outside of C. The proof of this theorem is based on a very simple tool, the *Laurent decomposition.* This device (developed by the author in connection with the extension of the Faber theory to Riemann surfaces; see [3], [4]) is described in Section 2. In Section 3, we give a brief development of the Faber theory, and we prove our fundamental result. The final section is devoted to applications of the theorem.

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2. THE LAURENT DECOMPOSITION

2.1. Let C be a rectifiable Jordan curve in the z-plane, with the interior I(C) and the exterior A(C). With every (single-valued) function ϕ holomorphic on C we associate the two functions

$$L\phi(z) = \frac{1}{2\pi i}\int_C \phi(\zeta)\frac{d\zeta}{\zeta - z} \quad \text{for } z \in I(C), \tag{4}$$

$$L^*\phi(z) = \frac{-1}{2\pi i}\int_C \phi(\zeta)\frac{d\zeta}{\zeta - z} \quad \text{for } z \in A(C). \tag{5}$$

Since the path of integration may be moved slightly, in the neighborhood of the curve C, we see that $L\phi$ is holomorphic in $I(C) \cup C$, and that $L^*\phi$ is holomorphic in $A(C) \cup C$, with $L^*\phi(\infty) = 0$. Therefore $L\phi$ and $L^*\phi$ are holomorphic on C, and

$$\phi = L\phi + L^*\phi \tag{6}$$

on C. In the special case where C is a circle, this relation reduces to the well-known Laurent decomposition of a function which is holomorphic in an annulus containing C. We therefore call L and L* the operators of the Laurent decomposition with respect to C.

2.2. From (4) it follows immediately that the operator L is linear and continuous; that is,

$$L(a\phi + b\psi) = aL\phi + bL\psi, \tag{7}$$

and $\phi_n \xrightarrow[C]{u} \phi$ implies $L\phi_n \xrightarrow[I(C)]{u} L\phi$, where the symbol $\xrightarrow[E]{u}$ denotes uniform convergence on every compact subset of E. In particular, if $\phi = F$ is holomorphic in $I(C) \cup C$, then $LF = F$, and therefore

$$\phi_n \xrightarrow[C]{u} F \tag{8}$$

implies immediately

$$L\phi_n \xrightarrow[I(C)]{u} F. \tag{9}$$

3. FABER THEORY

3.1. Henceforth, we assume that the curve C is analytic. Let $t = f(z)$ be a conformal mapping of A(C) onto the exterior of the circle K: $\{|t| = k\}$, with $f(\infty) = \infty$. Clearly, f is holomorphic and schlicht also on C.

Now let F be a function which is holomorphic in $I(C) \cup C$; then the mapping f carries F into a function G which is holomorphic on K and which has a Laurent expansion

$$G(t) = \sum_{-\infty}^{\infty} a_n t^n$$

on K. Interpreting this series in terms of values z on C and in the neighborhood of C, we deduce that

$$F(z) = \sum_{-\infty}^{\infty} a_n [f(z)]^n \tag{10}$$

on C; we shall call (10) a Laurent series. By (7), (8) and (9), it follows from (10) that

$$F(z) = \sum_{-\infty}^{\infty} a_n L[f(z)]^n \qquad (z \in I(C)). \tag{11}$$

3.2. By (6), the relation $f^n = Lf^n + L^*f^n$ holds throughout the neighborhood of C. Now the functions f^n and L^*f^n are defined throughout A(C); therefore the function Lf^n, which is holomorphic in I(C), can be continued into the entire plane, and

$$Lf^n = f^n - L^*f^n$$

throughout A(C). The second term on the right is holomorphic in A(C) and vanishes at $z = \infty$. Since the function f^n also vanishes at $z = \infty$, when $n < 0$, it follows that

$$Lf^n \equiv 0 \qquad (n < 0). \tag{12}$$

On the other hand, for $n \geq 0$ the function f^n is holomorphic in A(C) except for a pole of multiplicity n at $z = \infty$. It follows that Lf^n is a polynomial of degree n; this polynomial is called the *Faber polynomial* F_n (relative to the curve C):

$$Lf^n \equiv F_n \qquad (n \geq 0). \tag{13}$$

Returning to the function F discussed in Section 3.1, we note that because the curve C is analytic, the hypothesis that F is holomorphic on C can be dropped: if F is holomorphic in I(C), we can replace the curve C by a curve C' which lies in I(C) and which is sufficiently near to C so that f is also holomorphic throughout $A(C') \cup C'$. Nothing is changed, then, when we refer the operators L and L^* to the curve C' instead of to C. From (11), (12) and (13) we now obtain the following classical result:

THEOREM 1. *If the function* F *is holomorphic in* I(C), *it can be represented by a Laurent expansion* $F = \sum_{-\infty}^{\infty} a_n f^n$ *near* C *and by a Faber expansion* $F = \sum_0^{\infty} a_n F_n$ *in* I(C).

3.3. If a series (1) converges uniformly in each compact subset of I(C), we call it a *Faber series*. Let F denote the holomorphic function to which such a series converges. If we replace each of the functions F_n by its Laurent series in f and note that this "series" consists of precisely two terms, we obtain the relation

$$F = \sum_{-\infty}^{\infty} b_n f^n,$$

where $b_n = a_n$ for $n \geq 0$. Since the Laurent coefficients are uniquely determined by F, the Faber series of the function is also uniquely determined. And since the function $\Sigma_{-\infty}^{-1} b_n f^n$ is holomorphic in $A(C) \cup C$ and vanishes at $z = \infty$, we have established the following conclusion.

THEOREM 2. *Let* $\Sigma_0^\infty a_n F_n$ *be a Faber series, let* $F(z)$ *denote its sum in* $I(C)$, *and let* $g(z) = \Sigma_0^\infty a_n f^n$. *Then the function* $F(z) - g(z)$ *can be continued analytically throughout* $A(C) \cup C$; *it is represented there by the series* $\Sigma_0^\infty a_n(F_n - f^n)$; *and it vanishes at* $z = \infty$.

COROLLARY. *Let* $\Sigma a_n F_n$ *be a Faber series, and let* R_1 *and* R_2 *denote the Riemann surfaces on which the two functions* $\Sigma_0^\infty a_n F_n$ *and* $\Sigma_0^\infty a_n f^n$ *are holomorphic. Then the portions of* R_1 *and* R_2 *which lie above the set* $A(C) \cup C$ *are identical.*

In particular, Ullman's lemma becomes immediate if we interpret the "power series" $\Sigma_0^\infty a_n f^n$ as the series $\Sigma_0^\infty a_n t^n$.

4. FABER SERIES

4.1. It follows from the proof of Theorem 2 that a Faber series $\Sigma_0^\infty a_n F_n$ and its associated power series $\Sigma_0^\infty a_n f^n$ have a common "ring" of convergence in I(C). Since the operator L^*, like L, is linear and continuous, it follows that in $A(C) \cup C$ the partial sums $s_n^* = \Sigma_0^n a_k L^* f^k$ converge uniformly to a holomorphic function. By (6) and (13), the corresponding partial sums

$$\sigma_n = \sum_0^n a_k F_k \quad \text{and} \quad s_n = \sum_0^n a_k f^k$$

satisfy the condition $\sigma_n = s_n - s_n^*$, on $A(C) \cup C$, and it follows that if one of two sequences $\{\sigma_{n_i}\}$ and $\{s_{n_i}\}$ converges (uniformly) on some subset of $A(C) \cup C$, then the other does likewise, and the difference of the two limits can be extended so that it is holomorphic in $A(C) \cup C$. We summarize:

THEOREM 3. *The series* $\Sigma a_n F_n$ *and* $\Sigma a_n f^n$ *have the same sets of convergence and of uniform convergence, in* $A(C) \cup C$. *The same applies to sets of overconvergence, natural boundaries, and sets of continuity at the boundary.*

More can be said; but it is obviously not feasible to compile a catalogue of theorems on Taylor series which can be restated in terms of Faber series.

4.2. Finally, we deduce a result of P. Heuser [1], by the method of the Laurent decomposition. Let C_i $(i = 1, 2)$ be an analytic simple closed curve in the z_i-plane, and let the functions $t = f_i(z_i)$ map the domains $A(C_i)$ conformally onto the exterior of the same circle K in the t-plane. Let $F_n^{(i)}(z_i)$ and L_i denote the corresponding Faber polynomials and Laurent operators. We wish to find a relation between functions

$$F^{(1)}(z_1) = \sum_0^\infty a_n F_n^{(1)}(z_1) \quad \text{and} \quad F^{(2)}(z_2) = \sum_0^\infty a_n F_n^{(2)}(z_2)$$

with the same set of Faber coefficients.

Let

$$F^{(1)}(z_1) = \sum_{-\infty}^{\infty} a_n[f_1(z_1)]^n$$

be the Laurent expansion of $F^{(1)}(z_1)$ near C_1. Since the relation $t = f_1(z_1) = f_2(z_2)$ defines a holomorphic mapping $z_1 = \gamma(z_2)$ of $A(C_2) \cup C_2$ onto $A(C_1) \cup C_1$, the relation

$$F^{(1)}(\gamma(z_2)) = \sum_{-\infty}^{\infty} a_n[f_2(z_2)]^n$$

holds near C_2. On the other hand,

$$L_2\left(\sum_{-\infty}^{\infty} a_n[f_2(z_2)]^n\right) = \sum_{0}^{\infty} a_n F_n^{(2)}(z_2) = F^{(2)}(z_2).$$

From this follows Heuser's result:

$$F^{(2)}(z_2) = L_2\left(F^{(1)}[\gamma(z_2)]\right) = \frac{1}{2\pi i}\int_{C_2} F^{(1)}[\gamma(\zeta)]\frac{d\zeta}{\zeta - z_2} \qquad (z_2 \in I(C_2)).$$

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