INVARIANCE OF NORMAL DISTRIBUTIONS

J. G. Wendel

It is a well-known and elementary fact that an affine transform of a multivariate normal distribution is again normal; but, so far as we are aware, no one has pointed out that there is a partial converse to this. After the introduction of the relevant notation, such a theorem is stated and proved below.

Let g be a one-to-one transformation of Euclidean n-space onto itself, with the property that the set gE is Borel measurable if and only if E is Borel measurable; and let $N(u, U)$ denote the normal distribution with mean u and covariance operator U. We say that g transforms $N(u, U)$ into $N(v, V)$ provided gX has the distribution $N(v, V)$ whenever X is distributed according to $N(u, U)$. Let u_0 denote the zero-vector, and let the vectors $u_1, u_2, \cdots, u_n$ span E^n; let U denote an arbitrary non-singular covariance operator.

THEOREM. *If* g *transforms* $N(u_j, U)$ *into normal distributions* $N(v_j, V_j)$ $(j = 0, 1, \cdots, n)$, *then* g *is affine, up to almost-everywhere equivalence.*

Proof. Let $f_j(x)$ denote the densities corresponding to $N(u_j, U)$, and $h_j(x)$ those corresponding to $N(v_j, V_j)$; the latter densities exist, since the V_j must be non-singular. By hypothesis we have, for any Borel set E,

$$(1) \qquad \int_E h_j(x)\, dm(x) = \Pr\{gX \in E\} = \Pr\{X \in g^{-1}E\} = \int_{g^{-1}E} f_j(x)\, dm(x),$$

where m denotes Lebesgue measure. Since the densities are everywhere positive, the formula shows that $m(gE)$ and $m(g^{-1}E)$ are absolutely continuous with respect to $m(E)$. Therefore we may write $dm(gx) = r(x)\, dm(x)$, where r is the appropriate Radon-Nikodym derivative. From (1) we deduce that

$$\int_E h_j(x)\, dm(x) = \int_E f_j(gx)\, r(x)\, dm(x),$$

and hence that, almost everywhere,

$$(2) \qquad h_j(x) = f_j(gx)\, r(x) \qquad (j = 0, 1, \cdots, n).$$

In virtue of the two-way absolute continuity, $r(x)$ is positive almost everywhere. Therefore $h_j(x)/h_0(x) = f_j(gx)/f_0(gx)$ almost everywhere $(j = 1, 2, \cdots, n)$. Appealing to the explicit forms for the densities f_j and h_j, we find that, up to null-sets,

$$(gx, U^{-1}u_j) = Q_j(x) \qquad (j = 1, 2, \cdots, n),$$

where we use the inner-product notation on the left, and where the Q_j are inhomogeneous quadradic forms (of unknown signature). Hence, relative to any basis, the components of the vector gx are quadratic forms in the components of x, and by continuity we can dispense with the qualification "almost everywhere."

Received March 30, 1957.

It follows that $r(x)$ is a polynomial in the components of x, namely, the Jacobian of gx with respect to x. But (2) has the form

$$\exp\{\text{quadratic form in } x\} = r(x)\exp\{\text{quadratic form in } gx\},$$

and the forms in braces are positive-definite. Therefore r must be constant, and, finally, gx must be linear, since otherwise terms of fourth degree would appear. This completes the proof.

In conclusion we remark that the theorem can not be expected to hold under assumptions about fewer than $n + 1$ normal distributions. For example, in the case $n = 1$, the mapping

$$gx = x \text{ if } |x| < 3, \qquad gx = -x \text{ if } |x| \geq 3,$$

is nonaffine, but sends $N(0, 1)$ into $N(0, 1)$; this phenomenon can easily be carried up to spaces of any dimension.

University of Michigan.