ON MAPS OF THE THREE-SPHERE INTO THE PLANE

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1. INTRODUCTION

The following theorem deals with a special case of part of the Knaster conjecture [2].

THEOREM. *Let* $f\colon S^3 \to E^2$ *be continuous, and let* p, p_1, p_2 *be points of* S^3 *which are vertices of an equilateral triangle in* E^4. *Then there exists a rotation* $r \in SO(4)$ *such that* $f(rp) = f(rp_1) = f(rp_2)$.

This note consists of a proof of this theorem. Before giving the proof, let us fix the notation. E^n is Euclidean n-space, S^{n-1} the unit sphere of E^n; SO(n) is the group of proper rotations of E^n, considered here as operating on S^{n-1}; and P^n is real projective n-space. For $x \in S^3$, let G_x denote the subgroup of SO(4) consisting of rotations which leave x fixed. Let $f_1\colon S^3 \to E^1$ be the map obtained by following f by the projection of E^2 onto E^1 which is defined by the rule $(x_1, x_2) \to x_1$. Without loss of generality, suppose that p is a point of S^3 at which f_1 attains its maximum value, and that this maximum is positive. It is an elementary matter to show that there exists a rotation $a \in SO(4)$ satisfying

(1) $ap = p_1, \quad ap_1 = p_2, \quad a^3 = 1$ (where 1 denotes the identity element of SO(4)),

(2) a leaves some point, say $z \in S^3$, fixed.

Then G_p and G_z are conjugate subgroups of SO(4), and they carry homologous, nonbounding, integral 3-cycles of SO(4). The proof will now proceed as follows: we shall construct a map $\psi\colon SO(4) \to S^3$, under the assumption that the theorem is false. Then we shall see that $\psi | G_p$ and $\psi | G_z$ have different degrees. Since this is impossible, our proof by contradiction will then be complete.

2. CONSTRUCTION OF ψ

This construction is well known. Define the three maps

$$\phi\colon SO(4) \to E^6, \qquad T\colon SO(4) \to SO(4), \qquad T^1\colon E^6 \to E^6$$

by the conditions

$$\phi(r) = (f(rp), f(rp_1), f(rp_2)),$$

$$T(r) = r\cdot a,$$

$$T^1(x_1, \cdots, x_6) = (x_3, x_4, x_5, x_6, x_1, x_2).$$

Then $\phi \circ T = T^1 \circ \phi$, since $ap = p_1$, and so forth. Let

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$$\Delta = \{(x_1, \cdots, x_6) \in E^6 \mid x_1 = x_3 = x_5, x_2 = x_4 = x_6\}.$$

Our theorem is false if and only if $\phi(SO(4)) \cap \Delta$ is empty, for some f. Assume f to be such that $\phi(SO(4)) \cap \Delta$ is empty. Let ∇ be the 4-dimensional linear subspace orthogonal to Δ, that is, let ∇ be given by $x_1 + x_3 + x_5 = 0$, $x_2 + x_4 + x_6 = 0$. Let $\pi: E^6 \to \nabla$ be defined by the equation

$$\pi(x_1, \cdots, x_6) = (x_1 - \alpha(x), x_2 - \beta(x), x_3 - \alpha(x), x_4 - \beta(x), x_5 - \alpha(x), x_6 - \beta(x)),$$

where $\alpha(x) = (x_1 + x_3 + x_5)/3$, $\beta(x) = (x_2 + x_4 + x_6)/3$. Let ω be the origin of E^6. Now define $\rho: \nabla - \omega \to \mathfrak{S}^3$ (where $\mathfrak{S}^3$ denotes the unit sphere of ∇) by the relation

$$\rho(x_1, \cdots, x_6) = \left(\frac{x_1}{|x|}, \cdots, \frac{x_6}{|x|}\right) \qquad \left(|x| = (\Sigma_1^6 x_i^2)^{1/2}\right),$$

and observe that $\pi \circ T^1 = T^1 \circ \pi$, $\rho \circ T^1 = T^1 \circ \rho$. Then the mapping

$$\psi = \rho \circ \pi \circ \phi: SO(4) \to \mathfrak{S}^3$$

satisfies the condition $\psi \circ T = T^1 \circ \psi$.

3. THE DEGREE OF $\psi | G_Z$

First note that $T(G_Z) = G_Z$, since $a \in G_Z$. Hence $\psi | G_Z: G_Z \to \mathfrak{S}^3$ satisfies $(\psi | G_Z) \circ T = T^1 \circ (\psi | G_Z)$. All that is needed now, in order to conclude that the degree of $\psi | G_Z$ is not zero, is the following theorem of Eilenberg [1, p. 405]: If X is a topological space with a periodic transformation $\Lambda: X \to X$ of prime period p; if P is a simplicial polyhedron of dimension not greater than q, with a simplicial periodic transformation $\Lambda: P \to P$ of period p without fixed points; if $f: X \to P$ is a continuous mapping such that $f\Lambda(x) = \Lambda f(x)$ for each $x \in X$; and if X is acyclic in dimensions less than q over some ring J with a unit in which the equation $px = 1$ has no solution; then the homomorphisms

$$f_q: H_q(X, J_p) \to H_q(P, J_p), \quad f_q: H_q(X, J) \to H_q(P, J)$$

are not trivial. We now set

$$X = G_Z, \quad P = \mathfrak{S}^3, \quad J = Z_3 \text{ (the group of integers mod 3)},$$

$$p = 3, \quad \Lambda = T \text{ on } G_Z, \quad \Lambda = T^1 \text{ on } \mathfrak{S}^3.$$

4. THE DEGREE OF $\psi | G_p$

If $r \in G_p$, then with $\phi(r) = (x_1, \cdots, x_6)$, we have $x_1 \geq x_3$, $x_1 \geq x_5$, $x_1 > 0$, since f_1 has a positive maximum at p. Therefore, $x_1 - \alpha(x) \geq 0$, and since ρ does not change the sign of any coordinate, it follows that if $r \in G_p$, then $\psi(r)$ has a nonnegative first coordinate. Hence $\psi(G_p) \not\supset \mathfrak{S}^3$, and therefore the degree of $\psi | G_p$ is zero. As we remarked in the Introduction, this completes the proof.

REFERENCES

1. S. Eilenberg, *Homology of spaces with operators. I*, Trans. Amer. Math. Soc. 61 (1947), 378-417.

2. B. Knaster, *Problème 4*, Colloquium Math. 1 (1948), 30-31.

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