AMBIGUOUS POINTS OF A FUNCTION HOMEOMORPHIC INSIDE A SPHERE

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This note answers a question raised by G. S. Young, who is directing the author's dissertation. For a definition of the term *ambiguous*, see [1].

THEOREM. *There exists a homeomorphism* h *of the cube*

$$S = \{(x, y, z) \mid 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\}$$

onto itself such that every point of the face

$$T = \{(x, y, 0) \mid 0 < x < 1,\ 0 < y < 1\}$$

is an ambiguous point of h.

The proof, like the proof in [1], is based on the construction of two trees, G_1 and G_2, such that each point of T can be reached from S along each of two Jordan arcs J_1 and J_2 lying on G_1 and G_2, respectively. The homeomorphism h will be defined so that it maps each horizontal plane onto itself, so that it reduces to the identity on the tree G_1, and so that it carries the tree G_2 into a tree $h(G_2)$ whose branches all lead to the point (1/2, 0, 0) on the boundary of T.

To describe G_1 (see Fig. 1), we denote by T_n ($n = 1, 2, \cdots$) the intersection of the plane $z = 2^{-n}$ with S. The tree G_1 begins at (1/2, 1/2, 1/2) in T_1, and at this point branches into two segments ending at the points (1/4, 1/2, 1/4) and (3/4, 1/2, 1/4) in T_2. The first branch divides into two segments which meet T_3 at (1/4, 1/4, 1/8) and (1/4, 3/4, 1/8); the second branch behaves analogously. In general, G_1 meets each plane T_{2k-1} ($k = 1, 2, \cdots$) in the set of all points of the form $(m/2^k, n/2^k, 1/2^{2k-1})$ ($m, n = 1, 3, 5, \cdots, 2^k - 1$); G_1 meets T_{2k} ($k = 1, 2, \cdots$) in the set of points of the form $(m/2^{k+1}, n/2^k, 1/2^{2k})$ ($m = 1, 3, 5, \cdots, 2^{k+1} - 1$; $n = 1, 3, \cdots, 2^k - 1$).

The tree G_2 is a homeomorphic image of G_1. It is defined by the condition that (x, y, z) lies on G_2 if and only if (x, y, 2z) lies on G_1. Figure 2 shows the intersection of T_n ($n = 1, 2, \cdots$) with the trees G_1 and G_2; points of G_1 are indicated by dots, points of G_2 by small crosses.

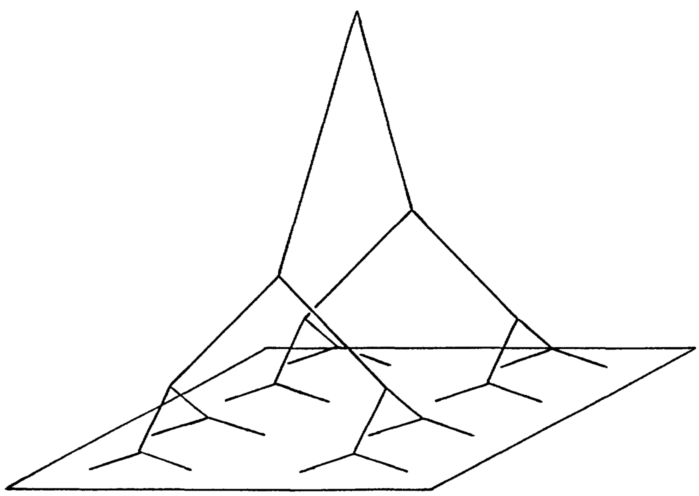

Figure 1

In each region T_n of S, we draw a simple curve R_n which joins the points $(0, 0, 1/2^n)$ and $(1, 0, 1/2^n)$ in such a way that it separates $G_1 \cap T_n$ from $G_2 \cap T_n$ (see Fig. 2). It follows from the

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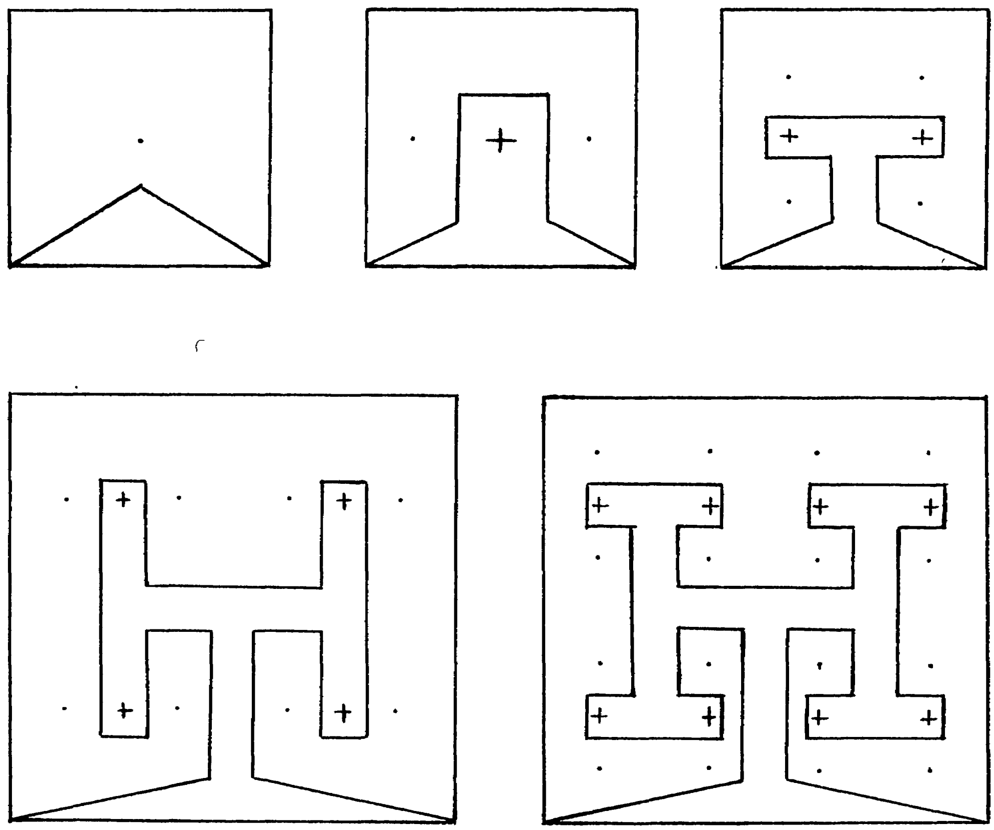

Figure 2

construction of G_1 and G_2 that the curves R_n can be embedded in a surface R, whose normal is nowhere vertical and which divides S into two components C_1 and C_2 such that C_1 contains G_1 and C_2 contains G_2. On C_1, we define h to be the identity. On C_2, we define h so that it carries the intersection of G_2 with each plane $z = z_0$ $(0 < z_0 < 1)$ onto a set of points of distance less than or equal to z_0 from the point $(1/2, 0, z_0)$.

Remark. A modification of this construction yields an infinitely differentiable homeomorphism of S onto S such that *every* point of the sphere is ambiguous.

REFERENCE

1. G. Piranian, *Ambiguous points of a function continuous inside a sphere*, Michigan Math. J. 4 (1957), 151-152.

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