RECTILINEAR LIMITS OF A FUNCTION DEFINED INSIDE A SPHERE

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Denote by x, y, z the Cartesian coordinates of a point in three-dimensional Euclidean space, and set

$$D = \{(x, y, 0): x^2 + y^2 < 1\}, \qquad K = \{(x, y, 0): x^2 + y^2 = 1\},$$

$$S = \{(x, y, z): x^2 + y^2 + z^2 < 1\}, \qquad T = \{(x, y, z): x^2 + y^2 + z^2 = 1\}.$$

Let $f(Q)$ be an arbitrary single-valued, real-valued function defined for every point $Q \in D$. It has been shown [1, p. 382] that there are at most enumerably many points $P \in K$ possessing the following property: There exist two Jordan arcs J_1 and J_2 lying wholly in D except for their common end point P, such that

$$\lim_{Q \to P, Q \in J_1} f(Q) \quad \text{and} \quad \lim_{Q \to P, Q \in J_2} f(Q)$$

exist and are distinct. It is very easy to see that $f(Q)$ can be defined for $Q \in S$ in such a manner that there are $2^{\aleph_0}$ points $P \in T$ possessing the foregoing property with D replaced by S. Gail S. Young, Jr. asked, in the course of a recent conversation with the author, if $f(Q)$ can be defined for $Q \in S$ so that this property is possessed by *every* $P \in T$. The purpose of this note is to show that this is indeed possible. It is not necessary for the Jordan arcs in question to be complicated: they can be taken to be rectilinear segments. Moreover, at every point $P \in T$ there can be $2^{\aleph_0}$ coplanar rectilinear segments along which the function tends to distinct limits.

If $P \in T$, we shall mean by a *segment at* P a rectilinear segment in S extending from a point $Q \in S$ to the point P; and by a *disk at* P, the intersection with S of a plane that passes through P but is not tangent to T.

THEOREM. *There exists a single-valued, real-valued function* $f(Q)$ *defined for every* $Q \in S$ *and possessing the following property: at every point* $P \in T$ *there is a disk containing* $2^{\aleph_0}$ *segments at* P *on each of which* f *is constant;* f *does not assume the same value on any two of these segments.*

Proof. Let ω_γ be the initial ordinal number of $Z(2^{\aleph_0})$. Arrange the set of real numbers in a transfinite sequence

$$r_0, r_1, \cdots, r_\xi, \cdots \qquad (\xi < \omega_\gamma).$$

Let the sequence

$$P_0, P_1, \cdots, P_\xi, \cdots \qquad (\xi < \omega_\gamma)$$

be such that, for every $\xi < \omega_\gamma$, $P_\xi \in T$, and, for every $P \in T$, there are $2^{\aleph_0}$ ordinal numbers $\xi < \omega_\gamma$ such that $P_\xi = P$. We shall define, by transfinite induction, for every $\xi < \omega_\gamma$, a disk d_ξ at P_ξ, a segment s_ξ at P_ξ that lies in d_ξ, and the value of f at every point of s_ξ.

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Take any disk d_0 at P_0 and any segment s_0 at P_0 that lies in d_0, and define the value of f at every point of s_0 to be r_0.

Suppose that $0 < \alpha < \omega_\gamma$, and that there have been defined, for every $\mu < \alpha$, a disk d_μ at P_μ and a segment s_μ at P_μ that lies in d_μ, satisfying the following condition (C_α):

If a point $P \in T$ *is a term of the sequence* $\{P_\mu\}_{\mu<\alpha}$, *then there is one (and hence, precisely one) disk* d *at* P *that is a term of the sequence* $\{d_\mu\}_{\mu<\alpha}$ *and contains a segment at* P *that is a term of the sequence* $\{s_\mu\}_{\mu<\alpha}$; *and* d *contains no term of the sequence* $\{s_\mu\}_{\mu<\alpha}$ *that is not a segment at* P. *The value of* f *at every point of* s_μ $(\mu < \alpha)$ *is* r_μ, *and* f *is undefined at every point of* $S - \bigcup_{\mu<\alpha} s_\mu$.

Condition (C_α) is obviously satisfied if $\alpha = 1$.

Consider P_α; this point of T either is or is not a term of the sequence $\{P_\mu\}_{\mu<\alpha}$.

(I) If the first alternative holds, let τ be the smallest ordinal number such that $P_\tau = P_\alpha$ (so that $\tau < \alpha$), and set $d_\alpha = d_\tau$. According to the induction hypothesis, s_τ is a segment at P_τ that lies in d_τ, and hence, if, in condition (C_α), we identify P_α with P, then d_α plays the role of d, and consequently each term of the sequence $\{s_\mu\}_{\mu<\alpha}$ is either a segment at P_α that lies in d_α, or else intersects d_α in at most one point. Thus, since $|\alpha| < 2^{\aleph_0}$, there exists a segment—call it s_α—at P_α that lies in d_α, does not lie in any disk at P_α that both differs from d_α and is a term of the sequence $\{d_\mu\}_{\mu<\alpha}$, and does not intersect $\bigcup_{\mu<\alpha} s_\mu$. Define f to have the value r_α at every point of s_α.

Condition $(C_{\alpha+1})$ is now satisfied. For let $P \in T$. If P is a term of the sequence $\{P_\mu\}_{\mu<\alpha+1}$, then it is also a term of the sequence $\{P_\mu\}_{\mu<\alpha}$, because $P_\alpha = P_\tau$ and $\tau < \alpha$. According to (C_α), then, there is only one disk d at P that is a term of the sequence $\{d_\mu\}_{\mu<\alpha}$—and hence, of the sequence $\{d_\mu\}_{\mu<\alpha+1}$, since $d_\alpha = d_\tau$ and $\tau < \alpha$—and contains a segment at P that is a term of the sequence $\{s_\mu\}_{\mu<\alpha}$. Suppose that d' is a disk at P that is a term of the sequence $\{d_\mu\}_{\mu<\alpha+1}$ (and hence, of the sequence $\{d_\mu\}_{\mu<\alpha}$), and that s_α is a segment at P that lies in d'. By the definition of s_α, we have $P = P_\alpha$ and $d' = d_\alpha$. But $d_\alpha = d_\tau$, where $\tau < \alpha$, and hence d' contains the segment s_τ at P, so that, according to (C_α), d' = d. We have now shown that there is only one disk d at P that is a term of the sequence $\{d_\mu\}_{\mu<\alpha+1}$ and contains a segment at P that is a term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$. By (C_α), d contains no term of the sequence $\{s_\mu\}_{\mu<\alpha}$ that is not a segment at P. Suppose that d contains s_α and that s_α is not a segment at P. It follows from the definition of s_α that $d = d_\alpha = d_\tau$, and since $P \neq P_\alpha = P_\tau$, d contains s_τ that is not a segment at P, which contradicts (C_α). Thus, d contains no term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$ that is not a segment at P. According to (C_α), the value of f at every point of s_μ $(\mu < \alpha)$ is r_μ; we have defined f to have the value r_α at every point of s_α; and f is undefined at every point of $S - \bigcup_{\mu<\alpha+1} s_\mu$. Hence, $(C_{\alpha+1})$ is satisfied.

(II) If, however, the second alternative holds, then, since $|\alpha| < 2^{\aleph_0}$, there exists a disk—call it d_α—at P_α whose frontier contains no point that is a term of the sequence $\{P_\mu\}_{\mu<\alpha}$. Consequently, each term of the sequence $\{s_\mu\}_{\mu<\alpha}$, since it is a segment at a point that is a term of the sequence $\{P_\mu\}_{\mu<\alpha}$, intersects d_α in at most one point, and so there exists a segment—call it s_α—at P_α that lies in d_α, does not lie in any one of the fewer than $2^{\aleph_0}$ disks that are terms of the sequence

$\{d_\mu\}_{\mu<\alpha}$, and does not intersect $\bigcup_{\mu<\alpha} s_\mu$. Define f to have the value r_α at every point of s_α.

Condition $(C_{\alpha+1})$ is again satisfied. For let P be a term of the sequence $\{P_\mu\}_{\mu<\alpha+1}$; then either (a) P is a term of the sequence $\{P_\mu\}_{\mu<\alpha}$ or (b) $P = P_\alpha$. If (a) holds, then, according to (C_α) and the facts that s_α is not a segment at P and d_α is not a disk at any point of T that is a term of the sequence $\{P_\mu\}_{\mu<\alpha}$, there is only one disk d at P that is a term of the sequence $\{d_\mu\}_{\mu<\alpha+1}$ and contains a segment at P that is a term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$; and, according to (C_α) and the fact that s_α does not lie in any disk that is a term of the sequence $\{d_\mu\}_{\mu<\alpha}$, d contains no term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$ that is not a segment at P. If (b) holds, then, for every $\mu < \alpha$, s_μ is a segment at $P_\mu \neq P_\alpha = P$, and s_α is not contained in any term of the sequence $\{d_\mu\}_{\mu<\alpha}$; hence, no term of the sequence $\{d_\mu\}_{\mu<\alpha}$ contains a segment at P that is a term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$, and so d_α is the only disk at P that is a term of the sequence $\{d_\mu\}_{\mu<\alpha+1}$ and contains a segment (namely, s_α) at P that is a term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$. The disk d_α was chosen so as to contain no term of the sequence $\{s_\mu\}_{\mu<\alpha}$, and hence d_α contains no term of the sequence $\{s_\mu\}_{\mu<\alpha+1}$ that is not a segment at P. The rest of condition $(C_{\alpha+1})$ is obviously satisfied.

We now assert that, for every $\xi < \omega_\gamma$, a disk d_ξ and a segment s_ξ at P_ξ that lies in d_ξ are defined so that $(C_{\xi+1})$ is satisfied. For if this is not so, there exists a smallest ordinal number $\xi_0 < \omega_\gamma$ for which the assertion is false; by the remark immediately following the statement of (C_α) above, $\xi_0 > 0$. For every $\nu < \xi_0$, then, a disk d_ν at P_ν and a segment s_ν at P_ν that lies in d_ν are defined so that $(C_{\nu+1})$ is satisfied. We shall show that (C_{ξ_0}) must also be satisfied. This is obvious if ξ_0 is isolated (take $\nu = \xi_0 - 1$ in $(C_{\nu+1})$), and we may therefore assume that ξ_0 is a limit number. Suppose that the point P is a term of the sequence $\{P_\mu\}_{\mu<\xi_0}$, say $P = P_\eta$, and that at P there are two disks, say d and d', that are terms of the sequence $\{d_\mu\}_{\mu<\xi_0}$ and contain segments at P that are terms of the sequence $\{s_\mu\}_{\mu<\xi_0}$; say that $d = d_\alpha$ and $d' = d_\beta$, where $\alpha < \xi_0$ and $\beta < \xi_0$, and that d contains the segment s_ι at P and d' contains the segment s_κ at P, where $\iota < \xi_0$ and $\kappa < \xi_0$. Setting $\zeta = \max(\eta, \alpha, \beta, \iota, \kappa)$, we see that $\zeta < \xi_0$ and $(C_{\zeta+1})$ is not satisfied, which is a contradiction. Hence, there is only one disk, say d, at P that is a term, say d_α, of the sequence $\{d_\mu\}_{\mu<\xi_0}$ and contains a segment at P that is a term of the sequence $\{s_\mu\}_{\mu<\xi_0}$. Suppose that d contains a term, say s_ϕ, of the sequence $\{s_\mu\}_{\mu<\xi_0}$ that is not a segment at P; setting $\psi = \max(\eta, \alpha, \phi)$, we have $\psi < \xi_0$ and a contradiction of $(C_{\psi+1})$. Consequently, d contains no term of the sequence $\{s_\mu\}_{\mu<\xi_0}$ that is not a segment at P. If $\nu < \xi_0$, then, according to $(C_{\nu+1})$, the value of f at every point of s_ν is r_ν. If λ is a limit number and $\{\mu_\delta\}_{\delta<\lambda}$ is an increasing sequence of ordinal numbers less than ξ_0 such that $\lim_{\delta<\lambda} \mu_\delta = \xi_0$, then, for every $\delta < \lambda$, according to $(C_{\mu_\delta+1})$, f is undefined at every point of $S - \bigcup_{\mu<\mu_\delta+1} s_\mu$, and hence f is undefined at every point of

$$\bigcap_{\delta<\lambda} \Big(S - \bigcup_{\mu<\mu_\delta+1} s_\mu\Big) = S - \bigcup_{\delta<\lambda} \bigcup_{\mu<\mu_\delta+1} s_\mu = S - \bigcup_{\mu<\xi_0} s_\mu .$$

Thus, (C_{ξ_0}) is satisfied. But, by the argument following the statement of condition (C_α) above, this implies that a disk d_{ξ_0} at P_{ξ_0} and a segment s_{ξ_0} at P_{ξ_0} that lies in d_{ξ_0} are defined so that (C_{ξ_0+1}) is satisfied, which contradicts the definition of ξ_0. Hence, the assertion made at the beginning of this paragraph is true.

An argument analogous to one just given shows that f is undefined at every point of $S - \bigcup_{\xi<\omega_\gamma} s_\xi$; define f to have the value 0 at every point of this set. Then it is evident that f is a single-valued, real-valued function defined for every point of S.

Let $P \in T$. Then, according to the definition of the sequence $\{P_\xi\}_{\xi<\omega_\gamma}$, there are $2^{\aleph_0}$ ordinal numbers $\rho_0 < \rho_1 < \cdots < \rho_\xi < \cdots < \omega_\gamma$ $(\xi < \omega_\gamma)$ such that $P_{\rho_\xi} = P$ for every $\xi < \omega_\gamma$. The disk d_{ρ_0} contains $2^{\aleph_0}$ distinct segments s_{ρ_ξ} $(\xi < \omega_\gamma)$ at P, on each of which f is constant (namely, $f = r_{\rho_\xi}$ on s_{ρ_ξ}); and f does not assume the same value on any two segments s_ξ, $s_{\xi'}$ $(\xi < \xi' < \omega_\gamma)$.

This completes the proof of the theorem.

Remark 1. The proof can evidently be modified so as to yield even more complicated behavior for f at every point of T. Thus, for example, f can be made to have the following property at every $P \in T$: for every real number r, f is identically equal to r along each of $2^{\aleph_0}$ of the segments associated with P.

Remark 2. Is it possible for a function f to be defined and *continuous* at every point of S, and to possess the property that, for every point $P \in T$, there exist two Jordan arcs J_1 and J_2 lying wholly in S except for their common end point P, such that

$$\lim_{Q \to P, Q \in J_1} f(Q) \quad \text{and} \quad \lim_{Q \to P, Q \in J_2} f(Q)$$

exist and are distinct?

REFERENCE

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