ON THE PROPERTIES OF A SINGULAR STURM-LIOUVILLE EQUATION DETERMINED BY ITS SPECTRAL FUNCTIONS

Avner Friedman

1. INTRODUCTION. Consider the equation

$$y'' + [\lambda - q(x)]y = 0 \qquad (0 \leq x < \infty) \tag{1}$$

with the boundary condition

$$y(0) \sin \alpha - y'(0) \cos \alpha = 0. \tag{2}$$

As is well known (see, for instance, [1]), if $q(x)$ is integrable on finite intervals, then the system (1), (2) determines spectral functions $\rho(\lambda)$ (in the sense of [3]).

Considering the inverse problem, Gelfand and Levitan [3] proved that if $\rho(\lambda)$ $(-\infty < \lambda < \infty)$ is a monotone increasing function such that

(i) $$\int_{-\infty}^{0} e^{\sqrt{|\lambda|}x}\, d\rho(\lambda) < \infty \text{ for all real } x, \text{ and}$$

(ii) one of the functions

$$F_i(x, y) \equiv \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}y}{\lambda}\, d\sigma_i(\lambda) \qquad (i = 1, 2) \tag{3}$$

belongs to C^{n+3},

then $\rho(\lambda)$ is a spectral function of a *uniquely* determined system (1), (2). Here,

$$\sigma_1(\lambda) = \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda} \qquad (\lambda > 0),$$

$$\sigma_2(\lambda) = \rho(\lambda) - \frac{2}{3\pi}\lambda^{3/2} \qquad (\lambda > 0),$$

$$\sigma_i(\lambda) = \rho(\lambda) \qquad (i = 1, 2;\ \lambda \leq 0).$$

Defining $f_i(x, y) = \dfrac{\partial^2 F_i(x, y)}{\partial x \partial y}$, they showed that there exists a unique solution $K_i(x, y)$ of the integral equation

$$f_i(x, y) + K_i(x, y) + \int_0^x f_i(y, t) K_i(x, t)\, dt = 0 \qquad (x \geq 0, y \geq 0), \tag{4}$$

Received January 16, 1957.
This paper was written under Office of Naval Research Contract N58304.

and that

(5) $$K_1(x, x) = \operatorname{tg}\alpha + \frac{1}{2}\int_0^x q(t)\,dt \qquad (\alpha \neq \pi/2)$$

if $F_1 \in C^{n+3}$, and

(6) $$K_2(x, x) = \frac{1}{2}\int_0^x q(t)\,dt \qquad (\alpha = \pi/2)$$

if $F_2 \in C^{n+3}$.

This paper deals mainly with the following problem. Given some properties of $\rho(\lambda)$ or of $f_i(x, y)$, to determine some properties of $q(x)$. Results of this kind were recently announced by Neigauz [4] for $\alpha = \pi/2$; but his assumptions on $\rho(\lambda)$ are very restrictive. In Section 2 we give a method for estimating $K_i(x, x)$ and its derivatives in terms of $f_i(x, y)$ and its derivatives, under a very mild assumption on $\rho(\lambda)$. In Section 3 it is proved that if $f_i(x, y)$ has a series development in x and y about the origin, then the same holds for $q(x)$. We also give a lower bound for the radius of convergence of the power series of $q(x)$. In Section 4 we give an application of the Gelfand-Levitan results to the problem of moments.

2. It will be sufficient to consider the case where $F_1 \in C^{n+3}$, so that (5) holds. For simplicity, we write $f_1 = f$, $\sigma_1 = \sigma$, and so forth.

Assumption 1. If $\lambda \geq 0$, then

(7) $$d\sigma(\lambda) \geq -\theta\frac{2}{\pi}d\sqrt{\lambda},$$

where $\theta < 1$ is a constant. We note that the relation $d\sigma(\lambda) \geq -\frac{2}{\pi}d\sqrt{\lambda}$ always holds. We also remark that in the case where $F_2 \in C^{n+3}$, (7) is replaced by

$$d\sigma(\lambda) \geq -\theta\frac{2}{3\pi}d\lambda^{3/2}.$$

We shall now estimate $K(x, x)$ and its derivatives in terms of $f(x, y)$ and its derivatives, under Assumption 1. Consider first the integral

$$\int_0^x\int_0^x f(y, t)\,h(y)\,h(t)\,dy\,dt,$$

where $h(t)$ is any real continuous function. Since $F(x, y)$ has continuous derivatives of the second order,

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

(8) $$= \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon, y+\varepsilon) - F(x-\varepsilon, y+\varepsilon) - F(x+\varepsilon, y-\varepsilon) + F(x-\varepsilon, y-\varepsilon)}{4\varepsilon^2}$$

$$= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \left(\frac{\sin\sqrt{\lambda}\varepsilon}{\sqrt{\lambda}\varepsilon}\right)^2 \cos\sqrt{\lambda}x \cos\sqrt{\lambda}y \, d\sigma(\lambda),$$

and the limit is obtained uniformly in x and y in finite intervals. It follows that

$$\int_0^x \int_0^x f(y, t) h(y) h(t) \, dy \, dt =$$

(9) $$= \lim_{\varepsilon \to 0} \int_{-\infty}^{0} \left(\frac{\sin\sqrt{\lambda}\varepsilon}{\sqrt{\lambda}\varepsilon}\right)^2 \left(\int_0^x h(t) \cos\sqrt{\lambda}t \, dt\right)^2 d\sigma(\lambda)$$

$$+ \lim_{\varepsilon \to 0} \int_0^{\infty} \left(\frac{\sin\sqrt{\lambda}\varepsilon}{\sqrt{\lambda}\varepsilon}\right)^2 \left(\int_0^x h(t) \cos\sqrt{\lambda}t \, dt\right)^2 d\sigma(\lambda).$$

The first integral on the right side of (9) is nonnegative, since $d\sigma(\lambda) \geq 0$ if $\lambda \leq 0$. The second integral is greater than

$$-\theta \int_0^{\infty} \left(\int_0^x h(t) \cos\sqrt{\lambda}t \, dt\right)^2 d\left(\frac{2}{\pi}\sqrt{\lambda}\right) = -\theta \int_0^x [h(t)]^2 \, dt,$$

by Assumption 1 and Parseval's equation. Thus

(10) $$\int_0^x \int_0^x f(y, t) h(y) h(t) \, dy \, dt \geq -\theta \int_0^x [h(t)]^2 \, dt.$$

Multiplying (4) (with $f_i = f$, $K_i = K$) by $K(x, y)$, integrating with respect to y $(0 \leq y \leq x)$ and using (10), we get

$$\int_0^x f(x, y) K(x, y) \, dy + (1 - \theta) \int_0^x [K(x, y)]^2 \, dy \leq 0.$$

Using Schwarz's inequality, we conclude that

(11) $$\left(\int_0^x [K(x, y)]^2 \, dy\right)^{1/2} \leq (1 - \theta)^{-1} \left(\int_0^x [f(x, y)]^2 \, dy\right)^{1/2}.$$

Taking $y = x$ in (4) and using (11), we derive the following estimate for $K(x, x)$:

$$|K(x, x)| \leq |f(x, x)| + (1 - \theta)^{-1} \int_0^x [f(x, t)]^2\, dt. \tag{12}$$

To estimate $\frac{d}{dx}K(x, x)$, we take $y = x$ in (4) and differentiate with respect to x. We obtain

$$\frac{d}{dx}f(x, x) + \frac{d}{dx}K(x, x) + f(x, x)K(x, x) + \int_0^x f_x(x, t)K(x, t)\,dt + \int_0^x f(x, t)K_x(x, t)\,dt = 0, \tag{13}$$

and it is clearly sufficient to estimate

$$I = \left(\int_0^x [K_x(x, t)]^2\,dt\right)^{1/2}.$$

To do this, we differentiate (4) with respect to x, multiply the resulting equation by $K_x(x, y)$ and integrate with respect to y $(0 \leq y \leq x)$. Using (10) and Schwarz's inequality, we obtain

$$I \leq (1 - \theta)^{-1}\left(\int_0^x [f_x(x, t)]^2\,dy\right)^{1/2} + |K(x, x)|\left(\int_0^x [f(x, y)]^2\,dy\right)^{1/2}. \tag{14}$$

Bounds on the higher derivatives of $K(x, x)$ can be derived in a similar way. This will become clear, from the proof of Theorem 1 below.

Let $\{M_n\}$ be a sequence of nonnegative numbers. We denote by $C^p\{M_n; a\}$ $(a > 0)$ the class of all infinitely differentiable functions $g(x)$ $(x = (x_1, \cdots, x_p))$ defined in the domain $0 \leq x_i \leq a$ $(i = 1, \cdots, p)$ and possessing the following property: To every function $g(x)$ there correspond constants H_0 and H such that if $x_i \in (0, a)$ $(i = 1, \cdots, p)$, then

$$\left|\frac{\partial^n g(x)}{\partial x_1^{i_1} \cdots \partial x_p^{i_p}}\right| \leq H_0 H^n M_n \qquad (n = 1, 2, \cdots).$$

THEOREM 1. *If* $f(x, y) \in C^2\{M_n; a\}$ *then* $q(x) \in C^1\{M_{n+1}; a\}$, *provided that* $\sigma(\lambda)$ *satisfies Assumption* 1 *and that, for some* $A > 0$, *the* M_n *satisfy the monotonicity condition*

$$\binom{n}{i} M_i M_{n-i} \leq A M_n \qquad (i = 1, \cdots, n;\ n = 1, 2, \cdots). \tag{15}$$

Note that if $f(x, y)$ is infinitely differentiable, the same is true of $K(x, y)$ and $q(x)$. Theorem 1, with $M_n = n!$, shows that if $f(x, y)$ is analytic, the same is true of $q(x)$.

Proof. We shall prove by induction that if $x \in (0, a)$, then

$$\left(\int_0^a \left(\frac{\partial^m K(x, t)}{\partial x^m}\right)^2 dt\right)^{1/2} \leq H_0 H^m M_m, \tag{16$_1$}$$

$$(16_2)\qquad \left| \frac{d^j}{dx^j}\left(\left[\frac{\partial^{m-j}}{\partial x^{m-j}} K(x, t) \right]_{t=x} \right) \right| \leq H_0 H^m M_m \qquad (j = 0, 1, \cdots, m).$$

The theorem then follows from (5).

Assuming (16_i) $(i = 1, 2)$ to hold for $m = 1, 2, \cdots, n - 1$, we shall prove it for $m = n$. In the sequel, A_i will be used to denote appropriate constants independent of n. For simplicity, we write

$$\frac{\partial^i K(x, x)}{\partial x^i} = \frac{\partial^i K(x, t)}{\partial x^i}\bigg|_{t=x}, \qquad \frac{\partial^i f(x, x)}{\partial x^i} = \frac{\partial^i f(x, t)}{\partial x^i}\bigg|_{t=x}.$$

Differentiating (4) n times with respect to x, multiplying the resulting equation by $\frac{\partial^n K(x, y)}{\partial x^n}$, integrating with respect to y $(0 \leq y \leq x)$ and using (10), we obtain, after using Schwarz's inequality,

$$(17)\qquad \begin{aligned} &(1 - \theta) \left(\int_0^x \left[\frac{\partial^n K(x, y)}{\partial x^n} \right]^2 dy \right)^{1/2} \\ &\leq \left(\int_0^x \left[\frac{\partial^n f(x, y)}{\partial x^n} \right]^2 dy \right)^{1/2} + \sum_{i=0}^{n-1} \left(\int_0^x \left[\frac{\partial^{n-1-i}}{\partial x^{n-1-i}} \left(f(y, x) \frac{\partial^i K(x, x)}{\partial x^i} \right) \right]^2 dy \right)^{1/2}. \end{aligned}$$

Now, by assumption,

$$(18_1)\qquad \begin{gathered} \left| \frac{\partial^m f(x, y)}{\partial x^m} \right| \leq A_1 A_2^m M_m, \qquad \left| \frac{d^j}{dx^j} \frac{\partial^{m-j} f(x, x)}{\partial x^{m-j}} \right| \leq A_1 A_2^m M_m \\ (x, y \in (0, a);\ m = 1, 2, \cdots), \end{gathered}$$

and therefore

$$(18_2)\qquad \left\{ \int_0^a \left[\frac{\partial^m f(x, y)}{\partial x^m} \right]^2 dy \right\}^{1/2} dy \leq A_3 A_2^m M_m.$$

If we take $H > 2A_2$, and make use of (15) and of the inductive assumption, we get

$$\begin{aligned} \left| \frac{\partial^{n-1-i}}{\partial x^{n-1-i}} \left(f(y, x) \frac{\partial^i K(x, x)}{\partial x^i} \right) \right| &\leq \sum_{j=0}^{n-1-i} \binom{n-1-i}{j} A_1 A_2^{n-1-i-j} M_{n-1-i-j} H_0 H^{i+j} M_{i+j} \\ &\leq A_4 H_0 H^{n-1} M_{n-1}, \end{aligned}$$

and from (17) it follows that

$$(19)\qquad \left\{ \int_0^a \left[\frac{\partial^n K(x, y)}{\partial x^n} \right]^2 dy \right\}^{1/2} \leq A_5 H_0 H^{n-1} M_n.$$

To prove (16_2) for $m = n$, we differentiate (4) $n - j$ times with respect to x, then substitute $y = x$ and finally differentiate the resulting equation j times with respect to x. We get

$$\frac{d^j}{dx^j}\frac{\partial^{n-j}f(x,x)}{\partial x^{n-j}} + \frac{d^j}{dx^j}\frac{\partial^{n-j}K(x,x)}{\partial x^{n-j}} + \sum_{i=0}^{n-j-1}\frac{d^j}{dx^j}\left\{\left[\frac{\partial^{n-1-j-i}}{\partial x^{n-1-j-i}}\left(f(y,x)\frac{\partial^i K(x,x)}{\partial x^i}\right)\right]_{y=x}\right\}$$

$$+\sum_{i=0}^{j-1}\frac{d^{j-1-i}}{dx^{j-1-i}}\left\{\left[\frac{\partial^i}{\partial x^i}\left(f(x,t)\frac{\partial^{n-j}K(x,t)}{\partial x^{n-j}}\right)\right]_{t=x}\right\} + \int_0^x \frac{\partial^j}{\partial x^j}\left(f(x,t)\frac{\partial^{n-j}K(x,t)}{\partial x^{n-j}}\right)dt = 0.$$

If $j = n$, the first sum does not appear, and if $j = 0$, the second sum does not appear. Using (15), (18_1), (18_2), (19) and the inductive assumption, we obtain, after some calculations,

$$\left|\frac{d^j}{dx^j}\frac{\partial^{n-j}K(x,x)}{\partial x^{n-j}}\right| \leq A_6 H_0 H^{n-1} M_n,$$

where A_6 does not depend on j. We take $H > \max(A_5, A_6)$, and the proof is complete.

Example. In addition to Assumption 1, assume also that $d\sigma(\lambda) = 0$ if $\lambda < 0$ and $\int_0^\infty |d\sigma(\lambda)| = |\sigma| < \infty$. Using (12), we obtain

$$\limsup_{x\to\infty}\left|\frac{1}{x}\int_0^x q(t)\,dt\right| \leq (1-\theta)^{-1}|\sigma|^2.$$

Similarly, if

$$\int_0^\infty \lambda^k |d\sigma(\lambda)| < \infty \qquad (k = 1, 2, \cdots),$$

then the following inequalities hold:

$$\limsup_{x\to\infty} x^{-(n+2)}|q^{(n)}(x)| < \infty \qquad (n = 0, 1, \cdots).$$

3. Consider again the integral equation

$$f(x,y) + K(x,y) + \int_0^x f(y,t)K(x,t)\,dt = 0, \tag{20}$$

where $f(x, y) = f_1(x, y)$, and assume that $f(x, y)$ has a series development about the origin, namely,

$$f(x,y) = \sum_{m,n=0}^{\infty} f_{mn}x^m y^n \qquad (|x| \leq A,\ |y| \leq A).$$

Substituting formally $K(x, y) = \Sigma K_{mn}x^m y^n$ into (20), we find that the following system of equations must be satisfied:

$$K_{0n} = -f_{0n},$$

(21)
$$K_{mn} = -f_{mn} + \sum_{i+j+k=m-1} \frac{-f_{nk}}{j+k+1} K_{ij} \qquad (m \geq 1).$$

Note that (21) can be used to define K_{mn} recursively, since K_{mn} is given in terms of K_{ij} and $i + j \leq m - 1 < m$. It follows that if $\{L_{mn}\}$ is an infinite matrix of numbers which satisfy (21) with K_{mn} replaced by L_{mn} and $-f_{mn}$ replaced by F_{mn}, where $F_{mn} \geq |f_{mn}|$, then $L_{mn} \geq |K_{mn}|$, and consequently if $\Sigma L_{mn} x^m y^n$ converges, then also $\Sigma K_{mn} x^m y^m$ converges.

To define F_{mn}, assume that the series $\Sigma f_{mn} x^m y^n$ is majorized by the product $\Sigma a_n x^n \Sigma a_n y^n$, for $|x| \leq B$, $|y| \leq B$ $(B \leq A)$; and write $f(x) = \Sigma_{n=0}^{\infty} a_n x^n$ $(|x| \leq B)$. It is clear that if the (majorant) integral equation

(22)
$$-f(x) f(y) + L(x, y) - \int_0^x f(y) f(t) L(x, t)\, dt = 0$$

has a solution $L(x, y) = \Sigma L_{mn} x^m y^n$, and if the series converges for $|x| < B$, $|y| < B$, then also $K(x, y)$ has a series development for $|x| < B$, $|y| < B$, and consequently $q(x) = \Sigma q_n x^n$ $(|x| < B)$. Since the solution of (22) is

$$L(x, y) \equiv \frac{f(x) f(y)}{1 - \int_0^x [f(t)]^2\, dt},$$

the following result is proved.

THEOREM 2. *If* $f(x, y) = \Sigma f_{mn} x^m y^n$ *is majorized by* $f(x) f(y) \equiv \Sigma a_n x^n \Sigma a_n y^n$ *for* $|x| \leq B$, $|y| \leq B$, *and if* $\int_0^B [f(x)]^2\, dx \leq 1$, *then*

$$q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

and the power series converges for $|x| < B$.

Theorem 2 gives a sharp lower bound on the radius of convergence R of $q(x)$, in the sense that there are cases in which $R = B$. As an example, define

$$\rho(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{\lambda} & (\lambda \geq 0) \\ -a & (\lambda < 0) \qquad (a > 0). \end{cases}$$

Then $f(x, y) = a$ and $K(x, y) = -a/(ax + 1)$, so that $R = 1/a$. On the other hand, for the majorant of $f(x, y)$ determined by $f(x) = \sqrt{a}$, $\int_0^B [f(x)]^2\, dx = 1$ becomes $B = 1/a = R$.

4. Consider the following problem of moments:

$$\text{(a)} \quad \begin{cases} \int_{-\infty}^{\infty} \lambda^n d\sigma(\lambda) = c_n & (n = 0, 1, \cdots; \ c_n = 0 \text{ if } n > N \text{ and } c_N \neq 0), \\ \int_{-\infty}^{\infty} e^{\sqrt{|\lambda|}\, x} |d\sigma(\lambda)| & \text{is finite for all positive } x, \\ d\sigma(\lambda) \geq 0 \text{ if } \lambda < 0; & d[\sigma(\lambda) + \frac{2}{\pi}\sqrt{\lambda}] \geq 0 \text{ if } \lambda \geq 0. \end{cases}$$

Suppose (a) has a solution $\sigma = \sigma(\lambda)$. Since the function

$$\rho(\lambda) = \begin{cases} \sigma(\lambda) & (\lambda < 0), \\ \sigma(\lambda) + \frac{2}{\pi}\sqrt{\lambda} & (\lambda \geq 0) \end{cases}$$

satisfies the conditions of Gelfand and Levitan (with $\alpha \neq \pi/2$), the function $K_1(x, y)$ which satisfies (4) exists for all nonnegative values of x and y. Since the function

$$f_1(x, y) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}y \, d\sigma(\lambda)$$

is analytic by our assumptions, we conclude from the relation

$$\left.\frac{\partial^{i+j} f_1(x, y)}{\partial x^i \partial y^j}\right|_{(0,0)} = \begin{cases} 0 & \text{if either } i \text{ or } j \text{ is odd}, \\ (-1)^n c_n & \text{if both } i \text{ and } j \text{ are even and } i + j = 2n, \end{cases}$$

that the function $f_1(x, y)$ must be of the form

$$\sum_{i+j \leq N} a_{ij} x^{2i} y^{2j} \quad \text{or} \quad \sum_{i=0}^{N} a_i(x^2) y^{2i},$$

where $a_i(z)$ is a polynomial in z of *exact* degree $N - i$ and with coefficients that are uniquely determined from the c_n. It follows that the integral equation

$$\text{(23)} \qquad \sum_{i=0}^{N} a_i(x^2) y^{2i} + K(x, y) + \sum_{i=0}^{N} y^{2i} \int_0^x a_i(t^2) K(x, t) \, dt = 0 \qquad (0 \leq y < \infty)$$

has a unique solution $K(x, y)$ for every fixed positive x.

From the theory of integral equations with degenerate kernels [2], it follows that

$$\text{(24)} \qquad A(x) \equiv \det\left(\int_0^x a_i(s^2) s^{2j} ds + \delta_{ij}\right)$$

must be either positive or negative for all x. We conclude that *if* (a) *has a solution, then necessarily* $A(x) \neq 0$ *for* $0 \leq x < \infty$.

In the case $c_n = 0$ $(n = 0, 1, \cdots)$ we have $f_1(x, y) \equiv K_1(x, y) \equiv 0$. Consequently, $q(x) \equiv 0$ and $\operatorname{tg} \alpha = 0$. Since the function $\rho(\lambda)$ corresponding to $\sigma(\lambda)$ is a spectral function, and since in the limit-point case there is only one spectral function (see [1; p. 259]), we conclude that $\rho(\lambda) = (2/\pi)\sqrt{\lambda}$, so that $\sigma(\lambda) \equiv 0$. Thus we have proved that *if all the* c_n *vanish, then* (a) *has no nontrivial solutions* $\sigma(\lambda)$.

REFERENCES

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, New York, 1955.
2. R. Courant and D. Hilbert, *Methods of mathematical physics*, *I*, New York, 1953.
3. I. M. Gelfand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Izvestiya Akad. Nauk SSSR. Ser. Mat. 15 (1951), 309-360.
4. M. G. Neigauz, *On determination of the asymptotic behavior of a function* $q(x)$ *by properties of the spectral function of the operator* $-y'' + q(x)y$. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 25-28.

University of Kansas