CONVERGENCE PROPERTIES OF SEQUENCES OF LINEAR FRACTIONAL TRANSFORMATIONS

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1. INTRODUCTION

We are concerned with two aspects of the convergence behavior of arbitrary sequences of transformations

$$(1) \qquad T_n = T_n(z) = \frac{A_n z + B}{C_n z + D} \qquad (n = 1, 2, \cdots).$$

One aspect is the nature of the point set where such a sequence converges; the other is the character of the limit function, $T(z)$, on that point set. While the two aspects of the problem are not quite independent, the connection between them is surprisingly slight, and they can be treated almost separately.

In order to make the further discussion precise, we mention here some conventions. No cases of interest will be lost through the assumption that all of the transformations T_n are nonsingular; we shall therefore assume, throughout, that $A_n D_n - B_n C_n \neq 0$. Also, the subject naturally demands that the transformations be regarded as mappings of the extended plane onto itself; therefore a sequence of points will be called divergent only if it has at least two limit points in the extended plane. A point set E will be called a set of convergence provided some sequence (1) converges everywhere on E and nowhere outside of E. The complement of a set of convergence will be called a set of divergence, or an SD, for short. In other words, the statement "E is an SD" shall have the meaning: "there exists a sequence (1) which diverges everywhere on E and converges everywhere outside of E."

It happens that the limit function of (1) is very simple, throughout the set of convergence, regardless of how the set of convergence is constituted. In fact, only finitely many essentially different situations can occur. Therefore we shall first treat the properties of the limit function (Section 2), and then we shall study the more difficult problem of finding the point sets which are sets of divergence (Sections 3 and 4). This latter problem is not yet completely solved. The continuity of the linear fractional transformations implies that every SD is a set of type $G_{\delta\sigma}$ (see [1], p. 273). On the other hand we shall prove, for example, that every set of type G_δ (but not every set of type F_σ) is an SD, and that not every SD is a set of type G_δ. We shall also show that it is not possible to characterize the SD's in purely topological terms. The essential reason for this state of affairs is the fact that the level curves of the function $1/(z - h)$ are concentric circles.

The present study originated from Thron's investigations on the convergence behavior of continued fractions. If $t_n(z) = a_n/(b_n + z)$ and

$$T_n(z) = t_1(t_2 \cdots (t_n(z)) \cdots),$$

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then the sequence $\{T_n(0)\}$ is the sequence of approximants of the continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots.$$

This fact has proved very fruitful in the study of continued fractions (for examples, see [2], [3], [4], [5], [6]), and it is hoped that the present paper will provide information useful in the theory of convergence behavior of continued fractions.

2. THE LIMIT FUNCTION

THEOREM 1. *On the set of convergence of a sequence of nonsingular linear fractional transformations, the limit function is either*

(a) *a nonsingular linear fractional function,*

(b) *a function taking on precisely two distinct values, or*

(c) *a constant.*

In case (a), *the sequence converges everywhere in the extended plane; in case* (b), *the sequence converges either everywhere, and to the same value everywhere except at one point, or it converges at only two points; case* (c) *can occur with every possible set of convergence.*

In the proof of the first part of this theorem, we can clearly assume that the sequence (1) converges at two distinct points z_1 and z_2 (the other cases are trivial). We can also assume that $z_1 = 0$ and $z_2 = \infty$; for if this is not the case, we need only replace the function $T_n(z)$ by the functions

$$T_n^*(z) = T_n\left(\frac{z_2 z + z_1}{z + 1}\right),$$

which are also nonsingular. Similarly, we may assume that $T(\infty) = \lim T_n(\infty) \neq \infty$. This assumption implies that $C_n \neq 0$, except for at most finitely many indices n; the exceptional elements T_n can be suppressed, since they do not affect the convergence behavior of the sequence. Our transformations then have the form

$$T_n(z) = \frac{A_n}{C_n} - \frac{A_n D_n - B_n C_n}{C_n^2 z + C_n D_n}.$$

Moreover, since the determinant of T_n is not 0, its value may be assumed to be 1, and the transformations take the form

$$(2) \qquad T_n(z) = \frac{A_n}{C_n} - \frac{1}{C_n^2 z + C_n D_n}.$$

Two alternatives arise: either the two limits $T(0)$ and $T(\infty)$ are different, or they are equal.

Alternative I. We may assume that $T(\infty) = 1$ and $T(0) = 0$, so that

$$A_n/C_n \to 1, \qquad C_n D_n \to 1.$$

If C_n approaches a finite limit other than 0, we have case (a) of our theorem. If $C_n \to 0$ or $C_n \to \infty$, we have case (b), with convergence everywhere. Finally, if the sequence $\{C_n\}$ diverges, we have case (b) with convergence at the two points 0 and ∞ only.

Alternative II. We may assume that $T(\infty) = T(0) = 0$, so that

$$A_n/C_n \to 0, \qquad C_n D_n \to \infty.$$

For convenience, we replace the transformations (2) by the transformations

$$T_n(z) = \frac{1}{C_n^2 z + C_n D_n} = \frac{1/C_n D_n}{zC_n/D_n + 1}; \tag{3}$$

this is permissible, since the convergence behavior is not affected by the change.

Suppose first that the sequence $\{z_n\} = \{-D_n/C_n\}$ converges to some finite or infinite limit p. Then $T_n(z) \to 0$ for all z, except possibly for $z = p$ (also for $z = p$, by hypothesis, if $p = 0$ or $p = \infty$). If $p \neq 0$ and $p \neq \infty$, the sequence $\{T_n(p)\}$ may converge to a value other than 0 (case (b) of the theorem); or it may converge to 0, or it may diverge (case (c) of the theorem). The reader will have no difficulty in constructing relevant examples to show that each of these possibilities can actually be realized.

Suppose next that the sequence $\{z_n\}$ diverges. It then has at least two distinct limit points h_1 and h_2. To each of these points h_i $(i = 1, 2)$, there corresponds a subsequence of $\{T_n(z)\}$ which converges to 0 everywhere except possibly at h_i. It follows that at each point z of the plane the sequence $\{T_n(z)\}$ either converges to 0, or diverges. In other words, if $\{z_n\}$ diverges, we have case (c) of the theorem.

We have now shown that the limit function of each sequence (1) falls into one of the three cases described in the theorem. The assertion that case (c) can occur with every possible set of convergence will follow incidentally from the proof of Theorem 3 below.

3. GENERAL THEOREMS ON SETS OF DIVERGENCE

THEOREM 2. *Every* SD *is of type* $G_{\delta\sigma}$.

This theorem is an immediate consequence of the fact that the functions $T_n(z)$ are continuous on the extended plane.

THEOREM 3. *If* E *is a set of type* G_δ, *it is an* SD.

If E is the extended plane, it is the SD of the sequence $\{T_n\}$ with

$$T_{3n} = z, \qquad T_{3n+1} = z + 1, \qquad T_{3n+2} = 1/z \qquad (n = 1, 2, \cdots).$$

If E omits at least one point, we may assume that it omits the point $x = \infty$. We shall show that E is then the SD of a sequence of the form

$$T_n(z) = \frac{d_n}{z - z_n}. \tag{4}$$

Let E be the intersection of the open sets G_k^* ($G_k^* \supset G_{k+1}^*$, $\infty \notin G_k^*$, $k = 1, 2, \cdots$; we note that the set E can be represented as such an intersection even if it is empty). Let G_k denote the intersection of G_k^* with the disk $|z| < k$. Let H_{k1} denote the set of points which lie at a distance at least 1/2 from the complement of G_k. And for $j = 2, 3, \cdots$, let H_{kj} denote the set of all points whose distance from the complement of G_k is less than 2^{1-j} and at least 2^{-j}.

In each of the sets H_j, we select a finite set of points z_{kjp} ($p = 1, 2, \cdots, p_{kj}$) such that the distance between each point of H_{kj} and the nearest of these points z_{kjp} is less than 2^{-k-2j}. We construct the functions

$$T_{kjp}(z) = \frac{2^{-k-2j}}{z - z_{kjp}};$$

and we arrange the family of functions T_{kjp} into a simple sequence $\{T_n\}$, according to any scheme.

If the point z lies in E, it lies in infinitely many of the sets G_k, hence in infinitely many sets H_{kj} ($k = k_z, k_z + 1, \cdots$; $j = j(k, z)$). Corresponding to each set H_{kj} which contains the point z, there exists a point z_{kjp} such that

$$|z - z_{kjp}| < 2^{-k-2j}.$$

Since the corresponding function T_{kjp} has modulus greater than 1 at z, the sequence $\{T_n\}$ does not converge to 0 at z. Therefore $\{T_n\}$ certainly diverges at z if the set $\{z_{kjp}\}$ has a limit point other than z. In case the set $\{z_{kjp}\}$ has only one limit point, we adjoin to $\{T_n\}$ the family of functions

$$T_h^* = \frac{1/h^2}{z - 1/h} \qquad (h = 1, 2, \cdots).$$

The augmented sequence $\{T_n\}$ then has a subsequence which converges to 0 everywhere, and therefore it diverges everywhere in E.

Suppose on the other hand that z is not a point of E. Then it lies in at most finitely many of the sets G_k. If $z \notin G_k$, then $|T_{kjp}(z)| < 2^{-k-2j}/2^{-j} = 2^{-k-j}$; and if $z \in G_k$, it is an interior point of G_k, and therefore $|T_{kjp}(z)| < \varepsilon$ except for a finite number n_ε of index sets (k, j, p). It follows that the sequence $\{T_n\}$ converges at z. Therefore E is the SD of a sequence (4), and the proof is complete.

We note that every SD which can arise under Alternative I in the proof of Theorem 1 is a set of type G_δ, and that under Alternative II the sequence $\{T_n\}$ is of the form (4). It follows now that every SD which does not contain the point $z = \infty$ is the SD of a sequence of the form (4). We also note that if an SD omits at least one finite point other than 0, then every point of the SD is a limit point of the set $\{z_n\}$ associated with a corresponding sequence (4).

THEOREM 4. *If each of the two sets* E_1 *and* E_2 *is an* SD, *then the union* $E_1 \cup E_2$ *is an* SD.

In the case where $E_1 \cup E_2$ omits only two points, the proposition is trivial. Otherwise, we may assume that each of the sets E_1 and E_2 is the SD of a corresponding sequence (4). By combining the two sequences into a single sequence, we obtain a sequence with the required SD.

4. DENUMERABLE SETS OF DIVERGENCE

THEOREM 5. *Every denumerable set on a line is an* SD.

COROLLARY. *Not every* SD *is of type* G_δ.

The corollary follows from the theorem because a set of type G_δ can not be both denumerable and dense on a line ([1], p. 138).

To prove the theorem, we suppose that E is a set of points x_k on the real axis. With each point x_k we associate a set of functions

$$T_{kj}(z) = \frac{1}{k^2 j(z - x_k - i/kj)} \qquad (j = 1, 2, \cdots),$$

and we arrange the functions T_{kj} into a sequence $\{T_n\}$. If z is not real, the distance $|z - x_k - i/kj|$ is bounded away from 0, except that it may vanish for one index pair (k, j); therefore $T_n(z) \to 0$. If z is real, we have two cases: If z is one of the points of E, say $z = x_h$, then

$$T_{hj}(z) = \frac{i}{h^2 j/hj} = i/h,$$

and therefore $\{T_n(z)\}$ diverges. If z is not one of the points x_k, let $\varepsilon > 0$. Since $|T_{kj}(z)| < 1/k$, for all index pairs (k, j), there exist at most finitely many indices k for which the inequality $|T_{kj}(z)| > \varepsilon$ can be satisfied; for each of these indices k, $\lim_{j\to\infty} T_{kj}(z) = 0$, and therefore $|T_n(z)| < \varepsilon$, except for a finite number of indices n. This concludes the proof.

THEOREM 6. *If the set* E *is dense in a domain* B *and is an* SD, *then the set* $E \cap B$ *is not denumerable.*

COROLLARY. *Not every denumerable set is an* SD.

In proving the theorem, we may suppose that E is the SD of a sequence (4), and that $d_n \to 0$. Under these conditions, each point of E is a limit point of the sequence $\{s_n\}$ associated with (4). Let K_n denote the point set where $|T_n(z)| \geq 1$, that is, the closed circular disk $|z - z_n| \leq d_n$; then every open subset of B contains one of the sets K_n.

Let Q be one of the disks K_n in B; then Q contains two disjoint disks Q_0 and Q_1 which belong to the sequence $\{K_n\}$. Similarly, Q_0 contains two disjoint disks Q_{00} and Q_{01}, and Q_1 contains two disjoint disks Q_{10} and Q_{11} (all four belonging to the sequence $\{K_n\}$), and so forth. With each dyadic fraction $a = 0.a_1a_2\cdots$ we associate the point p_a which lies in each of the disks $Q_{a_1}, Q_{a_1a_2}, \ldots$. Since $p_a \neq p_b$ for $a \neq b$, and since each point p_a lies in the SD of (4), the theorem is proved.

THEOREM 7. *The property of being an* SD *is not invariant under topological transformations of the plane onto itself.*

We shall actually prove more than the theorem asserts. We shall show that there exists a Jordan arc A such that some denumerable set on A is not an SD. The theorem will then follow trivially from Theorem 5.

The crucial geometric property which we need for our arc A is the following: the arc must contain a point set $\{t_k\}$, dense on A, such that, for every arc Γ in the plane which passes through a point t_k, the ratio $\rho(z, A)/|z - t_k|$ (where $\rho(z, A)$ denotes the distance between the point z and the arc A) tends to zero as $z \to t_k$ along Γ. An example of such an arc is given by the graph of the function

$$y = F(x) = \sum f(x - x_k)/k^2 \qquad (0 \leq x \leq 1),$$

where

$$f(t) = |t|^{1/2} \sin 1/t \qquad (t \neq 0),$$

$$f(0) = 0,$$

and where $\{x_k\}$ is dense on $[0, 1]$.

Let A and $\{t_k\}$ be an arc and a point set on the arc which have the required geometric property. We shall establish the theorem by proving that $\{t_k\}$ is not the SD of any sequence (4).

Suppose that the sequence (4) converges to zero on the complement of A, and that it diverges on $\{t_k\}$. If every subarc of A contains points of the set $\{z_n\}$ associated with (4), then the SD of (4) is not denumerable (proof as in the preceding theorem), and therefore it contains points which do not belong to $\{t_k\}$.

If some open subarc A^* of A contains no points of $\{z_n\}$, then to each point t_k on A^* there corresponds a subsequence $\{T_{n(k,j)}\}$ $(j = 1, 2, \cdots)$ which has the two properties that $z_{n(k,j)} \to t_k$ and that $|T_{n(k,j)}(t_k)| > a_k$, where a_k is some positive constant. Because of the special geometric property of A, there exists a closed arc B on A^* on which the inequality $|T_n(z)| \geq 1$ is satisfied, for some $n = n(B)$. Since B contains in its interior at least two points of the sequence $\{t_k\}$, the argument can be repeated to establish two disjoint closed subarcs B_0 and B_1 of B on which two corresponding elements of $T_n(z)$, with $n(B_0)$ and $n(B_1)$ both greater than $n(B)$, have modulus greater than 1. The remainder of the proof is similar to the proof of Theorem 6.

One might suspect that if the Jordan arc A used in our proof is sufficiently wicked, then no denumerable set which is dense on A is an SD. The following theorem shows that this is not the case.

THEOREM 8. *If the set* F *is closed and nowhere dense in the plane, then there exists a denumerable subset of* F *which is dense on* F *and which is an* SD.

To prove this theorem, we choose a sequence $\{t_k\}$ of points which is dense in the plane and lies in the complement of F. From each of the points t_k we draw a line segment L_k to one of the nearest points of F, and we denote the endpoint of L_k by z_k. As in the proof of Theorem 5, it is easy to construct a sequence $\{T_n\}$ whose SD is the set $\{z_k\}$.

THEOREM 9. *If the closure of the set* E *is denumerable, then* E *is an* SD.

In the proof, we may assume that $\overline{E}$ omits the point $z = \infty$. Suppose then that

$$\overline{E} = \{t_k\} \qquad (k = 1, 2, \cdots),$$

and let

$$\{z_n\} = \{t_1, t_2, t_1, t_2, t_3, t_1, \cdots\}.$$

We can assume that t_1 belongs to E. Let

$$T_n(z) = c_n/(z - z_n) \quad \text{if } z_n \in E,$$

$$T_n(z) = c_n/(z - z_1) \quad \text{if } z_n \notin E.$$

Here the positive constants c_n are chosen so that $|T_n(z_n)| < 1/n$ for

$$k = 1, 2, \cdots, n - 1 \text{ and } z_k \neq z_n, \quad \text{if } z_n \in E,$$

$$k = 1, 2, \cdots, n \quad \text{and } z_k \neq z_1, \quad \text{if } z_n \notin E.$$

Since $c_n \to 0$, $T_n(z) \to 0$ for every z in the complement of $\overline{E}$. If z is a fixed point in $\overline{E}$, and if $\{T_n(z)\}^*$ denotes the subsequence of $\{T_n(z)\}$ for which $z_n = z$, then the elements of $\{T_n(z)\}^*$ have the value ∞ if $z_n \in E$; otherwise, they tend to 0. In either case, the sequence complementary to $\{T_n(z)\}^*$ is a null sequence. This completes the proof.

5. UNSOLVED PROBLEMS

1. Is the intersection of two SD's an SD?

2. Is every subset of a denumerable SD an SD?

3. What are necessary and sufficient conditions on a denumerable set in order that it be an SD?

4. Although the SD's can not be characterized in terms of topological properties alone, it may be possible to find all the topological restrictions which a point set must satisfy in order to be an SD. More precisely: it may be possible to find a topological property P such that every SD has the property P, and such that every set with the property P can be transformed into an SD by means of an appropriate topological mapping of the plane onto itself.

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