On the Definition of Clifford Algebras

by

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Clifford algebras are usually defined in one of two ways. Let K be a field of characteristic not two. One method is to give a basis of the algebra [1]. The basis consists of the elements e_A where A ranges through the subsets of the set $N = \{1, 2, \ldots, n\}$, including the null set $\emptyset$. We write e_i for $e_{\{i\}}$ and define

(1) $\quad e_i^2 = a_i e_\emptyset \quad (i = 1, \ldots, n)$

where the a_i are elements of K; also

(2) $\quad e_i e_j = -e_j e_i \quad (i \neq j)$.

Then if $A = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$, we require that $e_A = e_{i_1} \cdots e_{i_r}$ and $e_\emptyset = 1$. From (1) and (2) products of the e_A can be defined. That multiplication is associative needs to be verified by computation.

A second method of definition is more intrinsic [2]. Let V be an n-dimensional vector space over K. Let T(V) be the tensor algebra of V, i.e., the free associative algebra over K consisting of sums of products of vectors in V, where it is assumed that the product with a scalar is commutative. Let f be a symmetric bilinear scalar function on V. Let J be the ideal of T(V) generated by all $vw + wv - 2f(v, w)$, where v and w range through V. The difference algebra $T(V)/J$ is defined to be a Clifford algebra.

The two definitions are connected by choosing an orthogonal basis in the space V with the metric defined by f, i.e., a basis $u_1, \dots, u_n$ of V such that

$$f(u_i, u_j) = \delta_{ij} a_j .$$

Let $\overline{u_i}$ be the residue class of u_i modulo J. The mapping

$$\theta : e_i \to \overline{u}_i$$

is clearly a homomorphism onto. In order to show that it is an isomorphism it is necessary to prove that the $\overline{u}_{i_1} \dots \overline{u}_{i_r}$ $(i_1 < \cdots < i_r)$ are linearly independent. This can be done by considering the inverse mapping θ^{-1} but then one must already have the algebra as given by the first definition. We shall prove directly that the $\overline{u}_{i_1} \cdots \overline{u}_{i_r}$, which we shall denote by $\overline{u}_A$ $(A = \{i_1, \dots, i_r\})$, are linearly independent.

The proof is by contradiction. Suppose $\sum c_A \overline{u}_A = 0$ (c_A in K). Then $\sum c_A u_A$ is in J and so

$$(3) \qquad \sum c_A u_{i_1} \cdots u_{i_r} = \sum a_{ij}(u_i^2 - a_i)b_{ij}$$

$$+ \sum c_{ijk}(u_i u_j + u_j u_i)\, d_{ijk} ,$$

where the a_{ij}, b_{ij}, c_{ijk}, d_{ijk} are non-commutative polynomials in the u_i. Suppose for some $B = \{j_1, \dots, j_s\}$ we have $c_B \neq 0$; we may assume $c_B = 1$. Since (3) is an identity in the indeterminates u_i, we can equate those terms in which $u_{j_1}, \dots, u_{j_s}$ appear to odd powers and the other u_i to even powers. Hence

$$u_{j_1} \cdots u_{j_s} = F ,$$

where

$$F = \sum a'_{ij}(u_i^2 - a_i)b'_{ij}$$

$$+ \sum c'_{ijk}(u_i u_j + u_j u_i)d'_{ijk};$$

consequently

(4) $\quad u_{j_1} \cdots u_{j_s} u_{j_1} \cdots u_{j_s} = (u_{j_1} \cdots u_{j_s})F$

and every term in this expression contains each u_i to an even power and every $u_{j_1}, \ldots, u_{j_s}$ to a power at least 2.

Let $x_1, \ldots, x_n$ be n commutative independent indeterminates over K. To each expression $\sum cu_{i_1} \cdots u_{i_p}$ where each subscript appears an even number of times in each term we make correspond $\sum(-1)^v cx_{i_1} \cdots x_{i_p}$ where v is the number of inversions from the natural order in $i_1, \ldots, i_p$. This is a homomorphism onto $K[x_1, \ldots, x_n]$. Clearly addition is preserved. In order to show that multiplication is preserved it is sufficient to prove that if $i = (i_1, \ldots, i_p)$ and $k = (k_1, \ldots, k_q)$ are both in natural order, then $(i_1, \ldots, i_p, k_1, \ldots, k_q)$ has an even number of inversions. But this is true because the numbers in i appear in pairs of adjacent equal numbers.

Under this mapping $c''(u_i u_j + u_j u_i)d''$ has image 0. Hence from (4) we find that

$$(x_{j_1} \cdots x_{j_s})^2 = (x_{j_1} \cdots x_{j_s})^2 \sum g_i(x_i^2 - a_i),$$

where g_i is a polynomial in $x_1^2, \ldots, x_n^2$. Division by $(x_{j_1} \cdots x_{j_s})^2$ gives

$$1 = \sum g_i(x_i^2 - a_i).$$

But this is impossible because under the mapping

$$x_i \to a_i^{1/2} \quad (i = 1, \ldots , n)$$

into the field $K(a_1^{1/2}, \ldots, a_n^{1/2})$, the right side has image 0.

References

1. C. Chevalley, Theory of Lie groups, vol. 1, p. 61 (1946).

2. M. Eichler, Quadratische Formen und orthogonale Gruppen, p. 22 (1952).

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