On a Theorem of Frobenius

by

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1. Introduction. Let A be a matrix of order n with element a_{ij} in the i^{th} row and j^{th} column. The characteristic polynomial $P(\lambda)$ is the determinant $|A - \lambda I|$, where I is the identity matrix of order n. The roots of $P(\lambda)$ are called the eigenvalues of A. A theorem of Frobenius [1] states that if the a_{ij} are positive, then A has an eigenvalue which is positive, simple and exceeds the modulus of the other eigenvalues. Results about the eigenvalues of a matrix A for the case the elements a_{ij} are positive or zero can be deduced from this theorem by a limiting process. In section 2 a new, direct proof of a result for this case will be given. The extension of this result to the Fredholm integral equation will be discussed in section 3.

2. Statement and Proof of Theorem 1.

Definition 1. Let r be a positive integer. If r elements a_{ij} can be arranged to have the form

(1) $$a_{t_1 t_2}, a_{t_2 t_3}, \ldots, a_{t_r t_1},$$

they will be called a cycle of elements.

Theorem 1. Let A be a matrix of order n whose elements a_{ij} are positive or zero. The necessary and sufficient condition that A have a positive eigenvalue is that it have a cycle of elements, none of which vanish.

Proof. Necessity. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, each one being listed according to its multiplicity. Let

$$(2) \qquad f(\lambda) = \frac{1}{\lambda_1 - 1} + \cdots + \frac{1}{\lambda - \lambda_n} = \frac{n}{\lambda} + \sum_1^{\infty} \frac{m_k}{\lambda^{k+1}},$$

where

$$(3) \qquad m_k = \sum_1^n \lambda_i^k = \text{Trace } A^k$$

$$= \sum_1^n \cdots \sum_1^n a_{t_1 t_2} a_{t_2 t_3} \cdots a_{t_k t_1} .$$

From (3) it is seen that m_k is a sum of terms, each of which is a product of elements forming a cycle. If all cycles contain a vanishing term, $m_k = 0$, $k = 1, 2, \cdots$ and from (2), $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Sufficiency. Since $a_{ij} \geq 0$, it follows from (3) that $m_k \geq 0$. It then is a consequence of a theorem of Vivanti-Dienes [4] that if the series in (2) has circle of convergence $|\lambda| = R$, $f(\lambda)$ has a singularity at $\lambda = R$. To show R is positive and not zero, the condition that there is a cycle of r non-vanishing elements is used. Say the product of these elements is the positive number p. Let q be a positive integer, then from (3)

$$(4) \qquad m_{rq} \geq p^q$$

and therefore

$$(5) \qquad R = \overline{\lim} \, |m_k|^{1/k} \geq p^{1/r} > 0 .$$

Thus $f(\lambda)$ has a singularity with positive affix and from (2) it is seen that this can only happen if A has a positive eigenvalue.

3. Extension to Integral Equations. Let $K(x, y)$ be a continuous function for $0 \leq x \leq 1$, $0 \leq y \leq 1$. The values of λ for which the integral equation

$$(6) \qquad 0 = f(x) - \lambda \int_0^1 K(x, y)\, f(y)\, dy$$

has a non-trivial solution are called eigenvalues. It was shown by Fredholm [2] that there is an entire analytic function $D(\lambda)$ whose zeros coincide with the eigenvalues of (6). Furthermore, he found the following interesting expansion of the logarithmic derivative of $D(\lambda)$:

$$(7) \qquad \frac{D'(\lambda)}{D(\lambda)} = -(M_1 + M_2\lambda + \cdots),$$

where

$$(8) \qquad M_k = \int_0^1 \cdots \int_0^1 K(t_1, t_2)K(t_2, t_3) \cdots K(t_k, t_1)\, dt_1 \cdots dt_k .$$

Robert Jentzsch [3] extended the theorem of Frobenius by showing that if $K(x, y)$ is positive, then (6) has an eigenvalue which is positive, simple and smaller than the modulus of any other eigenvalue. Results about the eigenvalues when $K(x, y)$ is positive or zero seem not to have been treated in the literature. A theorem will be stated for this case, and the proof will be outlined.

Definition 2. Let r be positive integer. If the coordinates (x, y) of r points in the plane can be arranged to have the form

$$(9) \qquad (t_1, t_2)\,(t_2, t_3) \cdots (t_r, t_1),$$

they will be called a cycle of points.

T h e o r e m 2. The integral equation (6), with $K(x, y)$ continuous and positive or zero will have a positive eigenvalue if and only if $K(x, y)$ is positive on some set of points forming a cycle.

P r o o f . Necessity. If $K(x, y)$ is zero for at least one point of each cycle, from (8) it is seen that $M_k = 0$, $k = 1, 2, \ldots$, and $D(\lambda)$, from (7), turns out to be a constant. Since $D(0) \neq 0$, the constant is not z e r o, and $D(\lambda)$ therefore has no zeros. Hence (6) has no eigenvalues.

Sufficiency. If $K(x, y)$ is positive f o r points forming a cycle, then it can be shown by the method of section 2 that the series in (7) h a s a p o s i t i v e radius of convergence, say R, and that the function it defines has a singularity at $\lambda = R$. The singularity of $D'(\lambda)/D(\lambda)$ at $\lambda = R$ must result from $D(\lambda)$ having a zero at $\lambda = R$, since $D(\lambda)$ has no singularities. Therefore R is an eigenvalue of (6).

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Bibliography

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