On n-dimensional Uniform Distribution Modulo 1

by

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1. J. F. Koksma [1] has shown that if $g_k(x)$ is a function of the real continuous variable x and the positive integral variable k, which satisfies rather general conditions in the interval $a \leq x \leq b$, then the sequence of numbers $g_1(x), g_2(x), \ldots$ is uniformly distributed modulo 1, for almost all $x \in (a, b)$; i.e. that for almost all $x \in (a, b)$ the following statement is true: for arbitrary α and β with $0 \leq \alpha < \beta \leq 1$ the relation

$$\lim_{N\to\infty} \frac{N(\alpha,\beta)}{N} = \beta - \alpha$$

holds, where $N(\alpha,\beta)$ is the number of integers $k \leq N$ for which

$$\alpha \leq g_x(x) < \beta \pmod 1.$$

Defining the discrepancy D(N) as

$$D(N) = \sup_{(\alpha,\beta)} \left| \frac{N(\alpha,\beta)}{N} - (\beta - \alpha) \right|,$$

this is clearly equivalent to the assertion that $\lim_{N\to\infty} D(N) = 0$ for a. a. $x \in (a, b)$. More recently, P. Erdös and Koksma [2, 3] have gone further in this direction, and have not only relaxed the conditions on g but have given very sharp estimates for D(N) and the associated exponential sum

(1) $$\sum_{k=1}^{N} e^{2\pi i \rho g_k(x)} .$$

Using a lemma of Kac, Salem and Zygmund [4] on quasi-orthogonal functions, the present author [5] obtained a metric theorem in which somewhat more is assumed about g than in the general theorems of [1] and [3], and which yields slightly weaker estimates for D(N) and the quantity (1), but which asserts the convergence almost everywhere in (a, b) of certain sums

(2) $$\sum_{k=1}^{\infty} c_k e^{2\pi i \rho g_k(x)} ,$$

where c_k approaches zero at a suitable rate as $k \to \infty$. We propose now to generalize the method of [5] to the n-dimensional case.

This can be done in various ways: by introducing additional continuous or sequential variables, or by considering s-tuples $(g_k(x), \ldots, h_k(x))$ (which leads to uniform distribution over the s-dimensional unit cube). Eventually we shall consider s-tuples of functions of r continuous variables and n sequential variables, but first we note a trivial extension of [5] to a case of several continuous variables.

Suppose that $\{\phi_k(x)\}$ is a system of complex-valued functions in $L^2(a, b)$, and put

$$a_{jk} = \int_a^b \phi_j(x) \overline{\phi}_k(x)\, dx .$$

Kac, Salem and Zygmund have termed the system $\{\phi_k\}$ quasi-orthogonal in (a, b) if the complex quadratic form $\sum_{j,k=1}^{\infty} a_{jk} u_j \overline{u}_k$ is bounded in Hilbert space; that is if

$$\left|\sum_{j,k=1}^{\infty} a_{jk}\, u_j \bar{u}_k\right| \leq M \sum_{j=1}^{\infty} |u_j|^2$$

They showed that if

$$(3) \qquad |a_{jk}| < \frac{C}{\max(1, |j-k|^{\varepsilon})}$$

for some C, $\varepsilon > 0$, then the system $\{\phi_k\}$ (or $\{\phi_k / k^{(1-\varepsilon)/2}\}$ is quasi-orthogonal if $\varepsilon > 1$ (or $\varepsilon < 1$), and used an easy generalization of the Rademacher – Menchoff theorem (cf. [6], vol. 2, p. 763) to deduce the convergence of certain series. It was shown in [5] that if $g_k(x)$ is suitably restricted and

$$\phi_k(x) = e^{2\pi i \beta g_k(x)},$$

then (3) holds; this shows the convergence of the series (2), and this in turn gives bounds for the sum (1).

Now suppose that ϕ_k, instead of being a function of x alone, is a function of x, ..., z. Correspondingly, a_{jk} is now defined as a multiple integral:

$$a_{jk} = \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \phi_j \bar{\phi}_k \, dx \ldots dz .$$

With this new a_{jk}, quasi-orthogonality is defined just as before, and the remainder of the argument of [4] goes through just as before. If now $g_k(x, \ldots, z)$ is taken to be of the form $f_k(x) + h_k(y, \ldots, z)$, where f_k satisfies the conditions imposed in [5], and $\phi_k(x, \ldots, z) = \exp(2\pi i \beta g_k(x, \ldots, z))$, then

$$|a_{jk}| = \left|\int_{a_1}^{b_1} e^{2\pi i\beta(f_j(x)-f_k(x))}\,dx \,.\right.$$

$$\left.\int_{a_2}^{b_2}\cdots\int_{a_r}^{b_r} e^{2\pi i\beta(h_j(y,\ldots,z)-h_k(y,\ldots,z))}\,dy\ldots dz\right|$$

$$\leq (b_2-a_2)\ldots(b_r-a_r)\left|\int_{a_1}^{b_1} e^{2\pi i\beta(f_j(x)-f_k(x))}\,dx\right|$$

$$\leq \frac{C'}{\max(1, |j-k|^{\epsilon})} \,.$$

Thus, writing the conditions on f_k explicitly, we have the following theorem:

Theorem 1. *Suppose that for* $a_1 \leq x \leq b_1, \ldots, a_r \leq z \leq b_r$ *the functions* $f_k(x)$ *and* $h_k(y,\ldots,z)$ *have the following properties:*

(i) df_k/dx *and* d^2f_k/dx^2 *exist,*

(ii) $d(f_j(x) - f_k(x))/dx$ *is monotonic, and different from zero for* $j \neq k$, *and that* $|d(f_j(x) - f_k(x))/dx| \geq C\,|j-k|^{\epsilon}$ *for some* $C, \epsilon > 0$ *and all* j, k,

(iii) $\exp(ih_k(y,\ldots,z))$ *is integrable.*

Then the sequence $f_1(x) + h_1(y,\ldots,z)$, $f_2(x) + h_2(y,\ldots,z), \ldots$ *is uniformly distributed* (mod 1) *for all such r-tuples* $(x,\ldots,z)$ *except for a set of r-dimensional Lebesgue measure zero.*

For example, for almost all pairs (x, y) the sequence $\{xe^k + ye^{2k}\}$ is uniformly distributed (mod 1), and for almost all pairs (x, y) for which $\max(x, y) > 1$ the sequence $\{x^k + y^k\}$ is u.d. (mod 1).

The principle involved here, which depends on the factorization of a multiple integral, will recur in the general case.

2. For simplicity we abbreviate the n-tuple $(j_1, \ldots, j_n)$ of positive integers to (j), and write $(j) < (t)$ or $(j) \leq (t)$ if

$$\begin{array}{lcl} 1 \leq j_1 < t_1, & & 1 \leq j_1 \leq t_1, \\ 1 \leq j_2 < t_2, & \text{or} & 1 \leq j_2 \leq t_2, \\ \vdots & & \vdots \\ 1 \leq j_n < t_n, & & 1 \leq j_n \leq t_n, \end{array}$$

respectively. Moreover, we write $\sum_{(j)}$ for

$$\sum_{j_1, \ldots, j_n=1}^{\infty},$$

designate a point $(x_1, \ldots, x_r)$ of r-dimensional space by x, and write $(t) \to \infty$ for $\min(t_1, \ldots, t_n) \to \infty$. Finally, $e^{2\pi i x}$ will be abbreviated to $e(x)$ and

$$(\sum_{(j)} |u_{(j)}|^2)^{1/2}$$

to $S(u_{(j)})$.

Let R be a set in r-space, having positive r-dimensional Lebesgue measure, and let $\{\phi_{(j)}(x)\}$ be an n-fold sequence of complex-valued functions in L^2 in R. Define

$$a_{(j)(k)} = \int_R \phi_{(j)} \bar{\phi}_{(k)} \, dx,$$

where $\bar{\phi}$ is the complex conjugate of ϕ. We say that the system $\{\phi_{(j)}\}$ is quasi-orthogonal in R if there is an M such that the inequality

(4) $$\Big|\sum_{(j),(k)} a_{(j)(k)}\, u_{(j)}\, \bar{u}_{(k)}\Big| \leq M\, S^2(u_{(j)})$$

holds for all complex sequences $\{u_{(j)}\}$. As is pointed out in [4], quasi-orthogonal functions enjoy many properties similar to those of orthogonal functions. For example, if $f \in L^2(R)$ and

$$\gamma_{(j)} = \int_R f\, \bar{\phi}_{(j)}\, dx ,$$

then

$$\sum_{(j)} |\gamma_{(j)}|^2 \leq M \int_R |f|^2\, dx ,$$

which is Bessel's inequality except for the factor M. For if $\{c_{(j)}\}$ is an arbitrary complex sequence, then

$$0 \leq \int_R \Big|f - \sum_{(j)\leq(t)} c_{(j)}\, \phi_{(j)}\Big|^2 dx$$

$$= \int_R |f|^2\, dx - \sum_{(j)\leq(t)} (\gamma_{(j)}\, \bar{c}_{(j)} + \bar{\gamma}_{(j)}\, c_{(j)})$$

$$+ \sum_{(j),(k)\leq(t)} a_{(j)(k)}\, c_{(j)}\, \bar{c}_{(k)}$$

$$\leq \int_R |f|^2\, dx - \sum_{(j)\leq(t)} (\gamma_{(j)}\, \bar{c}_{(j)} + \bar{\gamma}_{(j)}\, c_{(j)})$$

$$+ M \sum_{(j)\leq(t)} |c_{(j)}|^2$$

$$= \int_R |f|^2\, dx + \sum_{(j)\leq(t)} |M^{1/2}\, c_{(j)} - M^{-1/2}\, \gamma_{(j)}|^2$$

$$- M^{-1} \sum_{(j)\leq(t)} |\gamma_{(j)}|^2 .$$

Putting $c_{(j)} = \gamma_{(j)} M^{-1}$, the desired inequality follows.

3. The theorem of Rademacher and Menchoff asserts that, if $\{\Psi_k\}$ is an orthonormal system in (a, b), and $\{c_k\}$ is a sequence such that

$$\sum_{k=1}^{\infty} |c_k|^2 \log^2 k < \infty ,$$

then the series $\sum_1^{\infty} c_k \Psi_k(x)$ converges almost everywhere in (a, b). R. P. Agnew [7] has generalized this to a real double orthogonal series; he shows that such a series converges almost everywhere in the region of orthogonality if

$$\sum_{j,k=1}^{\infty} \left| c_{jk} \right|^2 \log^2(j+1) \log^2(k+1) < \infty .$$

An examination of the proof shows that, with no essential change in the ideas but with a considerable increase in complexity, his method suffices to prove the following theorem:

Theorem 2. *If* $\{\phi_{(j)}\}$ *is an arbitrary n-fold system of complex-valued functions, quasi-orthogonal in a region* R, *and if* $\{c_{(j)}\}$ *is a sequence such that*

$$\sum_{(j)} |c_{(j)}|^2 (\log(j_1 + 1) \cdots \log(j_n + 1))^2 < \infty ,$$

then the series

$$\sum_{(j)} c_{(j)} \phi_{(j)}(x)$$

converges almost everywhere in R.

The only theorem in the relevant portion of Agnew's work which must be modified because of the lack of orthogonality is the Riesz-Fischer theorem, which has the following analogue in the present case: if

$$\sum_{(j)} |c_{(j)}|^2 < \infty$$

and

$$S_{(t)}(x) = \sum_{(j)\leq(t)} c_{(j)} \phi_{(j)}(x) ,$$

then there is an $f \in L^2(R)$ such that

(i) $$\lim_{(t)\to\infty} \int_R |f - S_{(t)}|^2 dx = 0 ,$$

(ii) $$\int_R f \bar{\phi}_{(k)} dx = \sum_{(j)} a_{(j)(k)} c_{(j)} ,$$

(iii) $$\int_R |f|^2 dx = \sum_{(j),(k)} a_{(j)(k)} c_{(j)} \bar{c}_{(k)} .$$

The proof is along the same lines as in [7] and we omit it, since the result is not used in the proof of the Rademacher-Menchoff theorem.

4. Theorem 3. a) If $0 < \mathcal{E} < 1$, there is an M such that

$$\sum_{(j),(k)} \frac{u_{(j)} v_{(k)}}{\prod_{\nu=1}^{n} (j_\nu k_\nu)^{(1-\mathcal{E})/2} |j_\nu - k_\nu|^{\mathcal{E}})}$$

$$\leq M S(u_{(j)}) S(v_{(k)}) .$$

(b) If $\delta > 1$, there is an M for which

$$\left| \sum_{(j),(k)}{}' \frac{u_{(j)} v_{(k)}}{\prod_{\nu=1}^{n} (|j_\nu - k_\nu| \log^\delta (j_\nu + 1) \log^\delta (k_\nu + 1)} \right|$$

$$\leq M S(u_{(j)}) S(v_{(k)}),$$

and an M for which

$$\sum_{(j),(k)}{}' \frac{u_{(j)} v_{(k)}}{\prod_{\nu=1}^{n} |j_\nu - k_\nu|^\delta} \leq M S(u_{(j)}) S(v_{(k)}) .$$

<u>Here $\sum'$ means a summation over pairs</u> (j), (k) <u>for which</u> $j_\nu \neq k_\nu$ for $\nu = 1, \ldots, n$.

To simplify typography we carry out the proof of (a) only for n = 2; the method is general. We put

$$K(j, k; \ell, m)$$

$$= \begin{cases} 0 & \text{if } j = \ell \text{ or } k = m \\ \dfrac{1}{(jk\ell m)^{(1-\varepsilon)/2} |j-\ell|^\varepsilon |k-m|^\varepsilon} & \text{if } j \neq \ell \text{ or } k \neq m . \end{cases}$$

Then

$$\left| \sum_{j,k,\ell,m=1}^{\infty} K(j,k;\ell,m)\, u_{jk}\, v_{\ell m} \right|$$

$$= \left| \sum_{j,k,\ell,m=1}^{\infty} K^{1/2}\left(\frac{jk}{\ell m}\right)^{1/4} u_{jk}\, K^{1/2}\left(\frac{\ell m}{jk}\right)^{1/4} v_{\ell m} \right|$$

$$\leq \left(\sum_{j,k=1}^{\infty} |u_{jk}|^2 \sum_{\ell,m=1}^{\infty} K(j,k;\ell,m)\left(\frac{jk}{\ell m}\right)^{1/2} \right)^{1/2} .$$

$$\left(\sum_{\ell,m=1}^{\infty} |v_{\ell m}|^2 \sum_{j,k=1}^{\infty} K(j,k;\ell,m)\left(\frac{\ell m}{jk}\right)^{1/2} \right)^{1/2} ,$$

and since K is homogeneous of degree -1 in each of the pairs j, ℓ and k, m, the last expression is identical with the product of

$$\left(\sum_{j,k=1}^{\infty} |u_{jk}|^2 \sum_{\ell,m=1}^{\infty} K\left(1,1;\frac{\ell}{j},\frac{m}{k}\right)\left(\frac{\ell m}{jk}\right)^{-\frac{1}{2}} \frac{1}{jk} \right)^{1/2}$$

and

$$\left(\sum_{\ell,m=1}^{\infty} |v_{\ell m}|^2 \sum_{j,k=1}^{\infty} K\left(\frac{j}{\ell},\frac{k}{m};1,1\right) \frac{jk}{\ell m}^{-\frac{1}{2}} \frac{1}{\ell m} \right)^{\frac{1}{2}} .$$

Since the multiple integral

$$I = \int_0^{\infty} \int_0^{\infty} K(1,1;z,w)\,(zw)^{-1/2}\, dz\, dw$$

converges, we have

$$\lim_{j,k\to\infty} \sum_{\ell,m=1}^{\infty} K\left(1,1;\frac{\ell}{j},\frac{m}{k}\right)\left(\frac{\ell m}{jk}\right)^{-1/2} \frac{1}{jk} = I ,$$

and since the existence of the double integral implies that of the corresponding repeated integrals, we have that for all j, k,

$$\sum_{\ell, m=1}^{\infty} K\left(1, 1; \frac{\ell}{j}, \frac{m}{k}\right)\left(\frac{\ell m}{jk}\right)^{-1/2} \frac{1}{jk} < C_1$$

for some constant C_1. By symmetry, we conclude that

$$\left|\sum_{j,k,\ell,m=1}^{\infty} K(j,k;\ell,m)\, u_{jk}\, v_{\ell m}\right|$$

$$\leq C_1^{1/2}\, S(u_{jk}) \cdot C_1^{1/2}\, S(v_{\ell m})$$

and part (a) is proved with $M = C_1$.

To prove (b), we note first that, if $\{c_{(j)(k)}\}$ is an arbitrary sequence,

$$\left|\sum_{(j),(k)} c_{(j)(k)}\, u_{(j)}\, v_{(k)}\right|$$

$$\leq \sum_{(j),(k)} \left|c_{(j)(k)}\, u_{(j)}^2\right|^{1/2} \left|c_{(j)(k)}\, v_{(k)}^2\right|^{1/2}$$

$$\leq \sum_{(j),(k)} \left|c_{(j)(k)}\, u_{(j)}^2\right| \cdot \sum_{(j),(k)} \left|c_{(j)(k)}\, v_{(k)}^2\right|$$

$$= \left(\sum_{(j)} |u_{(j)}|^2 \sum_{(k)} |c_{(j)(k)}|\right)\left(\sum_{(k)} |v_{(k)}|^2 \sum_{(j)} |c_{(j)(k)}|\right),$$

so that to prove the desired inequalities it suffices to show that the quantities

$$\sum_{(j)}{}' \frac{1}{\prod_{\nu=1}^{n} (|j_\nu - k_\nu| \log^\delta (j_\nu+1) \log^\delta (k_\nu+1))}$$

$$= \prod_{\nu=1}^{n} \left\{ \frac{1}{\log^\delta (k_\nu+1)} \sum_{j_\nu=1}^{\infty}{}' \frac{1}{|j_\nu - k_\nu| \log^\delta (j_\nu+1)} \right\}$$

and

$$\sum_{(j)}{}' \frac{1}{\prod_{\nu=1}^{n} |j_\nu - k_\nu|^\delta} = \prod_{\nu=1}^{n} \sum_{j_\nu=1}^{\infty}{}' \frac{1}{|j_\nu - k_\nu|^\delta}$$

are bounded uniformly in (k). But

$$\frac{1}{\log^\delta (k+1)} \sum_{j=1}^{\infty}{}' \frac{1}{|j-k| \log^\delta (j+1)}$$

$$= \frac{1}{\log^\delta (k+1)} \left(\sum_{j=1}^{k-1} \frac{1}{|j-k| \log^\delta (j+1)} + \sum_{j=k+1}^{\infty} \frac{1}{|j-k| \log^\delta (j+1)} \right)$$

$$\leq \frac{1}{\log^\delta (k+1)} \left(\sum_{j=1}^{k-1} \frac{2}{|j-k|} + \sum_{j=k+1}^{\infty} \frac{1}{|j-k| \log^\delta (j-k)} \right).$$

$$\leq \frac{C_2 \log (k-1)}{\log^\delta (k+1)} + \frac{C_3 (\delta)}{\log^\delta (k+1)} < C_4 (\delta),$$

where the C's are independent of k, and

$$\sum_{j=1}^{\infty} \frac{1}{|j-k|^\delta} < \sum_{j=-\infty}^{\infty} \frac{1}{|j-k|^\delta} = 2 \sum_{j=1}^{\infty} j^{-\delta} = 2\zeta(\delta).$$

The proof is complete.

Using Theorem 3, we can prove the following analogue of Lemma 1 of [4]:

Theorem 4. Let R be a set in r-space, having positive r-dimensional Lebesgue measure, and let $\{\phi_{(j)}\}$ be any sequence of functions all belonging to $L^2(R)$. Suppose that there are positive constants C_5, ε such that

$$|a_{(j)(k)}| = \left|\int_R \phi_{(j)} \bar{\phi}_{(k)}\, dx\right|$$

$$\leq \frac{C_5}{\prod_{\nu=1}^{n} \max(1, |j_\nu - k_\nu|)}$$

for all sets (j), (k). Then

(i) If $\varepsilon < 1$, the series

$$\sum_{(j)} \frac{\phi_{(j)}(x)}{(j_1 \cdots j_n)^{1-\delta}}$$

converges almost everywhere in R provided $\delta < \varepsilon/2$.

(ii) If $\varepsilon = 1$, the series

$$\sum_{(j)} \frac{\phi_{(j)}(x)}{\{\log(j_1+1)\cdots\log(j_n+1)\}^{5/2+\vartheta}(j_1\cdots j_n)^{1/2}}$$

converges almost everywhere in R if $\vartheta > 0$.

(iii) If $\varepsilon > 1$, the series

$$\sum_{(j)} c_{(j)} \phi_{(j)}(x)$$

converges almost everywhere in R if

$$\sum_{(j)} |c_{(j)}|^2 \left\{\log (j_1+1)\cdots \log (j_n+1)\right\}^2 < \infty .$$

We first dispose of the trivial difficulty that arises when some of the indices j_ν are equal to the corresponding k_ν. Consider for example case iii) with n = 2; there we wish to show that

$$\left| \sum_{j,k,\ell,m=1}^{\infty}{}' \frac{u_{jk}\, v_{\ell m}}{\max(1, |j-\ell|^\delta)\cdot \max(1, |k-m|^\delta)} \right|$$

$$\leq M\, S(u_{jk})\, S(v_{\ell m}) .$$

We have

$$\left| \sum_{j,k,\ell,m=1}^{\infty}{}' \frac{u_{jk}\, v_{\ell m}}{\max(1, |j-\ell|^\delta)\cdot \max(1, |k-m|^\delta)} \right|$$

$$\leq \left| \sum_{j,k,\ell,m}{}' \frac{u_{jk}\, v_{\ell m}}{|j-\ell|^\delta\, |k-m|^\delta} \right| + \left| \sum_{j,k,m}{}' \frac{u_{jk}\, v_{jm}}{|k-m|^\delta} \right|$$

$$+ \left| \sum_{j,k,\ell}{}' \frac{u_{jk}\, v_{\ell k}}{|j-\ell|^\delta} \right| + \left| \sum_{j,k} u_{jk}\, v_{jk} \right| ,$$

the sums on the right being over all sets of indices for which the denominator of the corresponding summand is different from zero. The first of these sums is covered by Theorem 2 with n = 2, and the last by Hölder's inequality. As regards the second term, (there is a similar argument for the third term), we have

$$\left|\sum_{j,k,m} \frac{u_{jk}\, v_{jm}}{|k-m|^{\delta}}\right| \leq \sum_{j=1}^{\infty}{}' \left|\sum_{k,m} \frac{u_{jk}\, v_{jm}}{|k-m|^{\delta}}\right| ,$$

and by Theorem 3 with n = 1, this is

$$\leq \sum_{j=1}^{\infty} M\left(\sum_{k=1}^{\infty} |u_{jk}|^2\right)^{1/2}\left(\sum_{m=1}^{\infty} |v_{jm}|^2\right)^{1/2}$$

$$\leq S(u_{jk})\, S(v_{\ell m}) .$$

In the general case, just as here, terms with certain indices equal can be grouped together, and are then covered by Theorem 3 with smaller n. Clearly the number of such sums is 2^n; we designate $C_5 \cdot 2^n \cdot M$ by C_6.

We now prove Theorem 4. In case (i), we have from Theorem 3 that

$$\left|\sum_{(j),(k)} \frac{a_{(j)(k)}\, u_{(j)}\, v_{(k)}}{\prod_{\nu=1}^{n} (j_\nu k_\nu)^{(1-\varepsilon)/2}}\right| \leq C_6\, S(u_{(j)})\, S(v_{(k)})$$

so that the system $\{\phi_{(j)}(x)/(j_1 \cdots j_n)^{(1-\varepsilon)/2}\}$ is quasi-orthogonal in R, and so by Theorem 2 the series

$$\sum_{(j)} c_{(j)} \frac{\phi_{(j)}(x)}{(j_1 \cdots j_n)^{(1-\varepsilon)/2}}$$

converges almost everywhere if

$$\sum_{(j)} \left(|c_{(j)}|^2 \prod_{\nu=1}^{n} \log^2 (j_\nu + 1)\right) < \infty .$$

Take $c_{(j)} = (j_1 \cdots j_n)^{-1/2-\eta}$; the desired result is obtained with $\delta = \varepsilon/2 - \eta$.

In case (ii), the same reasoning shows that the system

$$\left\{ \phi_{(j)} / \prod_{\nu=1}^{n} \log^{\delta} (j_\nu + 1) \right\}$$

is quasi-orthogonal in R if $\delta > 1$, and here we take

$$c_{(j)} = \left(\prod_{\nu=1}^{n} j_\nu^{1/2} \log^{3/2+\eta} (j_\nu + 1) \right)^{-1},$$

which leads to the desired inequality with $\vartheta = \eta + \delta - 1$.

Part (iii) is a direct consequence of Theorems 2 and 3.

5. We come now to the principal theorems. We first consider a special case in which $r = n$.

Theorem 5. Suppose that C_7, ε are positive numbers, and that, for $\nu = 1, \ldots, n$, the function $f_\nu(z, j)$ has the following properties in the interval $a_\nu \le x \le b_\nu$:

(i) $df_\nu(z, j)/dx$, $d^2 f_\nu(z, j)/dz$ exist,

(ii) $\frac{d}{dz}(f_\nu(z, j) - f_\nu(z, k))$ is monotonic, and is different from zero for $j \neq k$,

(iii) for $z = a_\nu$ and $z = b_\nu$ the inequality

$$(5) \quad \left| \frac{d}{dz}(f_\nu(z, j) - f_\nu(z, k)) \right| \ge C_7 \, |j - k|^{\varepsilon} ; \; (\nu = 1, \ldots, n)$$

holds.

Suppose finally that $\beta_1, \ldots, \beta_n$ are integers, and that certain of them, say $\beta_1, \ldots, \beta_h (h \ge 1)$ are different from zero. Then if R' is the paralleletope $a_1 \le x_1 \le b_1, \ldots, a_h \le x_h \le b_h$, then

$$\left| \int_{R'} e(\beta_1 f_1(x_1, j_1) + \cdots + \beta_h f_h(x_h, j_h)) \right.$$

$$\left. \bar{e}(\beta_1 f_1(x_1, k_1) + \cdots + \beta_h f_h(x_h, k_h))\, dx' \right|$$

$$\leq \frac{C_8}{\prod_{\nu=1}^{h} \max(1, |j_\nu - k_\nu|^{\varepsilon})},$$

where C_8 depends only on C_7.

(Here, as in what follows, we use a dash to indicate that what would otherwise have been an n-tuple is now, on account of the vanishing of $\beta_{k+1}, \ldots, \beta_n$, only an h-tuple. Also, z is a one-dimensional real variable.)

Since the integral in (6) factors into a product of single integrals, and since this theorem with n = 1 was proved in [5], there is nothing to show here.

Combining Theorems 5 and 4 (the latter with n = h), we have the first part of

Theorem 6. _If $f_1, \ldots f_n$ satisfy the conditions of Theorem 4, then according as $\varepsilon < 1$, $\varepsilon = 1$ or $\varepsilon > 1$, the series_

$$\sum_{(j)'} \frac{e(\beta_1 f_1(x_1, j_1) + \cdots + \beta_h f_h(x_h, j_h))}{(j_1 \cdots j_h)^{1-\delta}}, \quad (\delta < \tfrac{\varepsilon}{2})$$

$$\sum_{(j)'} \frac{e(\beta_1 f_1(x_1, j_1) + \cdots + \beta_h f_h(x_h, j_h))}{\prod_{\nu=1}^{n} (j_\nu^{1/2} \log^{5/2+\vartheta}(j_\nu + 1))}, \quad (\vartheta > 0)$$

or

$$\sum_{(j)'} c_{(j)'}\, e(\beta_1 f_1(x_1, j_1) + \cdots + \beta_h f_h(x_h, j_h))$$

$$\left(\text{where} \sum_{(j)'} |c_{(j)'}|^2 \prod_{\nu=1}^{h} \log^2 (j_\nu + 1) < \infty\right)$$

respectively, converge almost everywhere in R'. Consequently, the bounds

$$\sum_{(j) \leq (N)} e(\beta_1 f_1 + \cdots + \beta_n f_n)$$

$$= \begin{cases} 0((N_1 \ldots N_h)^{1-\delta} N_{h+1} \ldots N_n), \\ 0(N_1^{1/2} \ldots N_h^{1/2} N_{h+1} \ldots N_n (\log N_1 \ldots \log N_h)^{\frac{5}{2}+\vartheta}), \\ 0(N_1^{1/2} \ldots N_h^{1/2} N_{h+1} \ldots N_n (\log N_1 \ldots \log N_h)^{\frac{3}{2}+\vartheta}) \end{cases}$$

respectively, hold for almost all $x \in R$, <u>where</u> R <u>is the paralleletope</u> $a_1 \leq x_1 \leq b_1, \ldots, a_n \leq x_n \leq b_n$.

The second part follows easily from the first by repeated partial summation, since by that device one easily shows that the convergence of $\sum_1^\infty a_k/f(k)$, where f(k) increases monotonically and without bound, implies that $\sum_1^N a_k = 0(f(N+1))$. This proves the result for almost all $x' \in R'$, and the extension to $x \in R$ follows by noting that if x' is a point of R' for which the bound holds, then for all $x \in R$ for which the first h coordinates are those of x', the bound also holds.

The importance of Theorem 6 lies in the fact that, according to van der Corput's generalization (cf. [8], pp. 92-94) of Weyl's criterion for uniform distribution (mod 1), a necessary and sufficient condition that the system $(g_1(x,(j)),\ldots,g_s(x,(j))$ be u. d. (mod 1) (for fixed x) is that for every set of integers $\beta_1,\ldots,\beta_s$, not all zero,

$$\lim_{(N)\to\infty} \frac{1}{N_1\ldots N_n} \sum_{(j)\leq(N)} e(\beta_1 g_1(x,(j))+\cdots+\beta_n g_n(x,(j))) = 0 .$$

Theorem 6 asserts that this is the case when $g_\nu(x,(j)) = f_\nu(x_\nu, j_\nu)(\nu = 1,\ldots,n)$, for almost all $x \in R$, and describes the growth of the exponential sums. For example, designating by $<z>$ the fractional part of z, we have that the points $(<z^j>, <ky>)$ are u. d. over the unit square $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$, for almost all (z,y) for which $|z| > 1$.

Theorems 5 and 6 can easily be extended to the case $g_\rho(x,(j)) = f_\rho(x_\rho,(j))$ for $\rho = 1,\ldots,r$, for here the integral (6) still factors into a product of single integrals. All that is necessary to obtain u. d. (mod 1) is to replace (5) by the condition that to each ρ there corresponds a ν such that

$$\left|\frac{d}{dz}(f_\rho(z,(j)) - f_\rho(z,(k)))\right| \geq C_7 \, |j_\nu - k_\nu|^{\varepsilon} ,$$

and that, for each such pair ρ, ν,

$$\frac{d}{dz}(f_\rho(z,(j)) - f_\rho(z,(k)))$$

is monotonic, and different from zero for $j_\nu \neq k_\nu$. For then it is clear that any sum

$$\sum_{(j)\leq(N)} e(\beta_1 f_1 + \cdots + \beta_h f_h))$$

will be $o(N_1, \ldots, N_n)$. In any particular case, bounds analogous to those in Theorem 6 can be given for these sums, but a general statement is awkward to formulate.

Finally, we can consider the general case of a set of s functions $g_\sigma(x,(j))$ of r continuous variables x_ρ and n sequential variables j_ν.

Theorem 7. Suppose that for each choice of integers $\beta_1, \ldots \beta_s$ not all zero, the expression

$$L_{(j)} = \beta_1 g_1(x,(j)) + \cdots + \beta_s g_s(x,(j))$$

can be written as a sum of two functions having no x_ρ as common argument, so that

$$(7) \qquad L_{(j)} = f(x_{\rho_1}, \ldots, x_{\rho_h}, (j)) + F(x,(j))$$

where

$$\frac{\partial F}{\partial x_{\rho_1}} \equiv \cdots \equiv \frac{\partial F}{\partial x_{\rho_h}} \equiv 0 .$$

(The numbers $h, \rho_1, \ldots, \rho_h$ may, of course, depend on $\beta_1, \ldots, \beta_s$, and F may vanish identically, in which case h = r) For simplicity, write $f(x', (j))$ instead of $f(x_{\rho_1}, \ldots, x_{\rho_h}, (j))$.

Let R be the r-dimensional region

$$a_1 \leq x_1 \leq b_1, \ldots, a_r \leq x_r \leq b_r,$$

and put $A = (a_1, \ldots, a_r)$, $B = (b_1, \ldots, b_r)$. Suppose that f and F have the following properties:

(i) For some ν_1 the inequality

$$\left|\frac{\partial^h}{\partial x_{\rho_1} \cdots \partial x_{\rho_h}} (f(x', (j)) - f(x', (k)))\right|$$

$$\geq C_9 \left|j_{\nu_1} - k_{\nu_1}\right| \qquad (C_9, \varepsilon > 0)$$

holds for $x = A$ and $x = B$, for all (j), (k).

(ii) For all $(j), (k)$ for which $j_{\nu_1} \neq k_{\nu_1}$; the quantities

$$f^* = \frac{\partial^h}{\partial x_{\rho_1} \cdots \partial x_{\rho_h}} (f(x', (j)) - f(x', (k)))$$

and

$$f^{**} = \frac{\partial^{2h}}{\partial x^2_{\rho_1} \cdots \partial x^2_{\rho_h}} (f(x', (j)) - f(x', (k)))$$

are different from zero for $x \in R$.

(iii) $e(F(x, (j)))$ is integrable over R, for all (j).

Then the system $(g_1, (x, (j)), \ldots, g_s(x, (j)))$ is uniformly distributed (mod 1) for almost all $x \in R$.

For by the decomposition (7), we may write

$$\int_R e(L_{(j)} - L_{(k)})\, dx$$

$$= \int_{R'} e(f(x', (j)) - f(x', (k)))\, dx' \cdot$$

$$\int_{R''} e(F(x'', (j)) - F(x'', (k)))\, dx'' ,$$

where $x' = (x_{\rho_1}, \ldots, x_{\rho_h})$ and R' is the h-dimensional region

$$a_{\rho_1} \leq x_{\rho_1} \leq b_{\rho_1}, \ldots, a_{\rho_h} \leq x_{\rho_h} \leq b_{\rho_h}$$

and x'', R'' have analogous (r-h)-dimensional meanings. The second integral on the right has absolute vale at most equal to the (r-h)-dimensional volume of R'', which is smaller than some constant C_{10} depending only on R. To the first integral on the right we apply multiple integration by parts (cf. [6] vol. 1, p. 493), supposing that $j_{\nu_1} \neq k_{\nu_1}$:

$$\left| \int_{R'} e(f(x', (j)) - f(x', (k))) \frac{f^*(x', (j), (k))}{f^*(x', (j), (k))} \, dx' \right|$$

$$= \left| \left[\frac{1}{2\pi i} \cdot \frac{e(f(x', (j)) - f(x', (k)))}{f^*(x', (j), (k))} \right]_{A'}^{B'} \right.$$

$$\left. + \frac{1}{2\pi i} \int_{R'} e(f(x', (j)) - f(x', (k))) \cdot \frac{f^{**}}{(f^*)^2} \, dx' \right|$$

$$\leq \frac{1}{2\pi} \frac{1}{|f^*(A', (j), (k))|} + \frac{1}{|f^*(B', (j), (k))|}$$

$$+ \int_{R'} \left| \frac{f^{**}}{(f^*)^2} \right| dx'$$

$$= \frac{1}{2\pi} \left(\frac{1}{|f^*(A', (j), (k))|} + \frac{1}{|f^*(B', (j), (k))|} \right.$$

$$\left. + \left| \int_{R'} \frac{f^{**}}{(f^*)^2} \, dx' \right| \right)$$

$$\leq \frac{2C_9}{\pi |j_{\nu_1} - k_{\nu_1}|} .$$

Hence, applying Theorem 4 with $C_5 = 2C_9C_{10}/\pi$ and $n = 1$, we deduce the convergence of a certain simple series (summed over j_{ν_1}), and partial summation shows that

$$\sum_{j_{\nu_1}=1}^{N_{\nu_1}} e(L_{(j)}) = o(N_{\nu_1}),$$

whence it follows that

$$\sum_{(j)\leq(N)} e(L_{(j)}) = o(N_1 \ldots N_n),$$

all this being for almost all $x \in R$. The same reasoning can be applied for each choice of $\beta_1, \ldots, \beta_s$, and the theorem now follows from the generalized Weyl criterion.

Here again better estimates for the associated exponential sum can easily be deduced from Theorem 4 in any particular case.

Theorems 5 - 7 list very special cases in which the hypotheses of Theorem 4 are satisfied, and as is pointed out in the introduction, more is assumed in these theorems in the case $n = r = s = 1$ than in the theorems of Erdős and Koksma. If Theorems 5 - 7 were all that can be deduced from Theorem 4, the advantage of the present method would depend solely on the ease with which it can be applied to generalize the simple case. In a later paper, however, we show how Theorem 4 can be applied to some cases not amenable to the Erdős-Koksma argument; there we show, for example, that the sequence $\{z^k\}$ is uniformly distributed (mod 1) for almost all complex z with $|z| > 1$, and that $\{k \cos k\alpha\}$ is uniformly distributed (mod 1) for almost all real α.

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