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Quotients of Divisorial Toric Varieties

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Introduction

A frequently occurring question in algebraic geometry is whether an algebraic group action $G \times X \to X$ admits a categorical quotient—that is, a regular map $X \to Y$ that is universal with respect to G-invariant regular maps $X \to Z$. For example, moduli functors are often co-represented by categorical quotients. In general, it is a difficult problem to decide whether a categorical quotient exists. Some counterexamples for actions of the multiplicative group $\mathbb{C}^*$ are presented in [4].

As these examples show, difficulties already arise with subtorus actions on toric varieties. Such actions have been investigated by several authors, mainly focusing on the much more restrictive concept of a good quotient (see e.g. [13; 16; 21]). The description of toric varieties in terms of rational fans relates the problem of constructing quotients to problems of combinatorial convexity. Hence the class of toric varieties serves as a testing ground for more general ideas.

Let X be a toric variety and let H be a subtorus of the big torus of X. Our approach to categorical quotients for the induced action of H on X is to consider the problem in suitable subcategories. A first step is to construct a quotient in the category of toric varieties itself: in [2], we showed that there always exists a *toric quotient*

$$p\colon X \to X/_{\mathrm{tq}} H.$$

This is a toric morphism that is universal with respect to H-invariant toric morphisms. The essential part of the proof is an explicit algorithm in terms of combinatorial data. The toric quotient is a canonical starting point for quotients in further categories. For example, in [3] we gave an explicit method to decide by means of the toric quotient when a subtorus action on a quasiprojective toric variety admits a categorical quotient in the category of quasiprojective varieties.

In this paper we give a considerable generalization of the results of [3]; namely, we solve the analogous problem in the category of divisorial varieties. Recall that an irreducible variety X is called *divisorial* if every point $x \in X$ has an affine neighborhood of the form $X \setminus \mathrm{Supp}(D)$ with an effective Cartier divisor D on X (see e.g. [10] and [8, II.2.2]).

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The class of divisorial varieties contains the quasiprojective varieties as well as all $\mathbb{Q}$-factorial varieties. It has nice functorial properties (see [10]), and, moreover, often provides a natural framework to extend statements known to hold for quasiprojective varieties on the one hand and for smooth varieties on the other.

A connection to toric geometry is provided by the embedding results of [14]: A variety is divisorial if and only if it admits a closed embedding into a smooth toric prevariety Z having an affine diagonal map $Z \to Z \times Z$. The equivariant version of this statement implies in particular that a toric variety is divisorial if and only if it has enough invariant effective Cartier divisors in the sense of Kajiwara [15]; see Section 1.

Now, given a divisorial toric variety X and a subtorus H of the big torus of X, when does the action of H on X admit a categorical quotient in the category of divisorial varieties? As mentioned, we start with the toric quotient

$$p\colon X \to X/_{\mathrm{tq}} H.$$

A first problem is that in general the toric quotient space $X/_{\mathrm{tq}} H$ is not a divisorial variety. To deal with this effect, we construct a *toric divisorial reduction*. This is a toric morphism

$$q\colon X/_{\mathrm{tq}} H \to (X/_{\mathrm{tq}} H)^{\mathrm{tdr}},$$

which is universal with respect to toric morphisms to divisorial toric varieties. The question then is: How do these toric constructions behave in the essentially larger category of arbitrary divisorial varieties? Our main result gives the following answer (see Corollary 6.3).

THEOREM. *The action of H on X admits a categorical quotient in the category of divisorial varieties if and only if the composition $q \circ p$ is surjective. Moreover, in the latter case, $q \circ p$ is the desired categorical quotient.*

The paper is organized as follows. In Section 1 we discuss divisoriality in the context of G-varieties and provide some general statements used in the subsequent constructions. Sections 2 and 3 are devoted to the construction of the toric divisorial reduction. This is done in the language of combinatorial convexity. The main tool is convex support maps that extend the notion of a convex support function on a fan.

Generalizing the corresponding well-known statement on projectivity and support functions, we show that divisoriality of a given toric variety is characterized by the existence of a strictly convex support map on its fan. Moreover, we relate convex support maps to toric morphisms to divisorial toric varieties. This allows the construction of the toric divisorial reduction. Finally, we present some examples in Section 3.

In Sections 4 and 5 we prepare the proof of the main results. The essential task is to reduce arbitrary H-invariant regular maps to H-invariant toric morphisms. This is done by the decomposition lemma (presented in Section 5): Given an H-invariant regular map $f\colon X \to Y$ to a divisorial variety, we construct a decomposition $f = h \circ g$ with an H-invariant toric morphism g followed by a rational map h defined near $g(X)$.

The ingredients for the proof of this decomposition lemma are the aforementioned embedding of Y into a certain smooth toric prevariety Z provided by [14] and the following lifting result (presented in Section 4): There exist quasiaffine toric varieties $\tilde{X}$ and $\tilde{Z}$ "above" X and Z, respectively, such that the map f admits a lifting $\tilde{f}\colon \tilde{X} \to \tilde{Z}$. In essence, this reduces the decomposition problem to the case of quasiaffine toric varieties.

In Section 6 we give statements and proofs of the main results. Finally, in Section 7 we formulate an open problem on categorical quotients for subtorus actions on toric varieties.

1. Divisorial *G*-Varieties

Throughout the entire paper, we work over a fixed, algebraically closed field $\mathbb{K}$. Hence, a prevariety is a reduced irreducible scheme of finite type over $\mathbb{K}$, and a variety is a separated prevariety. We say that a prevariety X is of *affine intersection* if its diagonal morphism $X \to X \times X$ is affine. The word "point" refers to a closed point.

As usual, when we speak of a G-(pre)variety, where G is an algebraic group, we mean an algebraic (pre)variety X together with a G-action given by a regular map $G \times X \to X$. See [5; 12] for the basic notions on toric varieties and prevarieties.

In this section, we provide some general facts on group actions on divisorial varieties. Following Borelli [10], we call a prevariety X *divisorial* if every point $x \in X$ has an affine open neighborhood of the form $U = X \setminus \operatorname{Supp}(D)$ with an effective Cartier divisor D on X.

REMARK 1.1.

(i) Quasiprojective varieties are divisorial.
(ii) Locally closed subspaces of divisorial prevarieties are divisorial.
(iii) Every divisorial prevariety X is of affine intersection.
(iv) Every $\mathbb{Q}$-factorial prevariety of affine intersection is divisorial.

A *geometric quotient* for the action of a reductive group G on a variety X is an affine regular map $p\colon X \to Y$ such that the fibers of p are precisely the G-orbits and the canonical homomorphism $\mathcal{O}_Y \to p_*(\mathcal{O}_X)^G$ is bijective. The analogous notion in the setting of prevarieties (i.e., for possibly nonseparated X and Y) is called a *geometric prequotient*.

In the sequel, we shall make use of the following characterization of divisoriality in terms of geometric quotients and closed embeddings (see [14, Thm. 3.1]).

THEOREM 1.2. *A variety X is divisorial if and only if one of the following statements holds.*

(i) *X is a geometric quotient of a quasiaffine variety by a free algebraic torus action.*
(ii) *X admits a closed embedding into a smooth toric prevariety of affine intersection.*

Here a torus action is called *free* if every orbit map is a locally closed embedding. This theorem has the following equivariant version (see [14, Thm. 3.4]).

THEOREM 1.3. *Let X be a normal divisorial T-variety, where T is an algebraic torus acting effectively.*

(i) *There is a quasiaffine variety $\hat{X}$ with a regular action of a torus $T \times H$ such that H acts freely with a T-equivariant geometric quotient $\hat{X} \to X$.*
(ii) *There is a T-equivariant closed embedding $X \to Z$ into a smooth toric prevariety Z of affine intersection, where T acts as a subtorus of the big torus.*

A first consequence is that divisorial varieties with torus actions always have many invariant effective Cartier divisors. This means that a toric variety is divisorial if and only if it has *enough invariant effective Cartier divisors* in the sense defined by Kajiwara [15].

PROPOSITION 1.4. *Let T be an algebraic torus, and let X be a normal algebraic T-variety X. Then X is divisorial if and only if there exist T-invariant effective Cartier divisors $D_1, \dots, D_r$ on X such that the sets $X \setminus \operatorname{Supp}(D_i)$ are affine and cover X.*

Proof. We may assume that T acts effectively. Let X be divisorial. By Theorem 1.3, there is a T-equivariant closed embedding of X into a smooth toric prevariety Z of affine intersection, where T acts as a subtorus of the big torus. Hence X inherits the desired property from Z. The reverse implication is trivial. □

As the example of the rational nodal curve with standard $\mathbb{K}^*$-action shows, the assumption of normality is essential in the statement of Proposition 1.4. Our next result states that divisoriality is inherited by geometric quotients for torus actions.

PROPOSITION 1.5. *Let T be an algebraic torus, and suppose that X is a normal T-variety with geometric quotient $p\colon X \to Y$. Then X is divisorial if and only if Y is divisorial.*

Proof. We may assume that the torus T acts effectively on X. If the quotient variety Y is divisorial, then we obtain the desired effective Cartier divisors on X by pulling back suitable divisors from Y. Conversely, suppose that X is divisorial. Then, by Theorem 1.3, we may assume in the proof that X is a quasiaffine T-variety.

Given $y \in Y$, we have to find an affine open neighborhood of y that is the complement of the support of an effective Cartier divisor on Y. Choosing any T-equivariant affine closure of X, we find a function $f \in \mathcal{O}(X)$, homogeneous with respect to some character $\chi_f \in \mathrm{X}(T)$, such that for $D := \operatorname{div}(f)$ the T-invariant set $U := X \setminus V(f) = X \setminus \operatorname{Supp}(D)$ is an affine neighborhood of the fiber $p^{-1}(y)$.

By T-closedness of $p\colon X \to Y$, the set $V := p(U)$ is an open neighborhood of $y \in Y$. Moreover, as a geometric quotient space of the affine T-variety U, the set V is again affine. Thus, to prove the assertion, we need only show that $p(\operatorname{Supp}(D))$ is the support of an effective Cartier divisor E on Y. We construct local equations for such an E.

First we claim that every point $z \in Y$ has an affine neighborhood $V_z \subset Y$ such that, on $U_z := p^{-1}(V_z)$, there is an invertible function $h_z \in \mathcal{O}(U_z)$ that is homogeneous with respect to some positive multiple $m_z \chi_f$. To check this, start with any affine neighborhood $V_z \subset Y$ of z and choose a point $x \in p^{-1}(z)$. Consider the sublattice $N \subset \mathrm{X}(T)$ of characters occurring as weights of homogeneous functions $g \in \mathcal{O}(U_z)$ with $g(x) = 1$.

The sublattice N is of full rank in $\mathrm{X}(T)$; otherwise, we find a nontrivial 1-parameter subgroup $\lambda\colon \mathbb{K}^* \to T$ such that $\chi \circ \lambda = 1$ holds for all $\chi \in N$. It follows that $\lambda(\mathbb{K}^*)$ is contained in the isotropy group T_x. On the other hand, the T-action on U_z is effective and closed. Hence T_x is finite, a contradiction. Thus N is of full rank. In particular, some positive multiple $m_z\chi_f$ lies in N and our claim follows.

Now cover Y by finitely many V_z as in our claim. Then we may assume that all the invertible functions $h_z \in \mathcal{O}(U_z)$ are homogeneous with respect to the same multiple $m\chi_f$. Every function $g_z := f^m/h_z$ is T-invariant, is regular on U_z, and vanishes precisely on $\mathrm{Supp}(D) \cap U_z$. Since it is T-invariant, g_z may be viewed as a regular function on $V_z = p(U_z)$, where its zero set is just

$$p(\mathrm{Supp}(D) \cap U_z) = p(\mathrm{Supp}(D)) \cap V_z.$$

Since every $g_z/g_{z'}$ is an invertible regular function on $V_z \cap V_{z'}$ it follows that the g_z are local equations for the desired Cartier divisor E on Y. □

As Kajiwara has shown (see [15, Thm. 1.9]), every toric variety X with enough invariant effective Cartier divisors arises as a geometric quotient of a quasiaffine toric variety $\hat{X}$ by an algebraic subgroup of the big torus of $\hat{X}$. In view of the preceding results, we can enhance Kajiwara's statement as follows.

COROLLARY 1.6. *A toric variety X is divisorial if and only if there is a quasiaffine toric variety $\hat{X}$ and a toric morphism $p\colon \hat{X} \to X$ such that* $\ker(p)$ *is a subtorus of the big torus of $\hat{X}$ and p is a geometric quotient for the action of* $\ker(p)$ *on $\hat{X}$.*

Proof. If X is divisorial, then Theorem 1.3 gives the desired quotient presentation. The converse follows from Proposition 1.5. □

Finally, we consider translates of divisorial open subsets with respect to an action of a connected group. If the complement of the subset is small enough, then the union of such translates is again divisorial.

LEMMA 1.7. *Let G be a connected linear algebraic group, and let X be a normal G-variety. If $U \subset X$ is a divisorial open subset with* $\mathrm{codim}(X \setminus U) \geq 2$, *then also $G \cdot U$ is divisorial.*

Proof. We may assume that $X = G \cdot U$ holds. Let $D_1^U, \dots, D_r^U$ be Cartier divisors on U such that the sets $U_i := U \setminus \mathrm{Supp}(D_i^U)$ form an affine cover of U. By closing components, each D_i^U extends to a Weil divisor D_i on X.

We claim that $X \setminus \mathrm{Supp}_{D_i} = U_i$. To see this, let $A_i := X \setminus U_i$. Since U_i is affine, A_i is of pure codimension 1. Clearly $\mathrm{Supp}(D_i^U) \subset A_i$ and hence $\mathrm{Supp}(D_i) \subset A_i$. Thus $\mathrm{Supp}(D_i)$ is a union of irreducible components of A_i. Moreover, we have

$$X \setminus U = X \setminus (U_i \cup \mathrm{Supp}(D_i^U)) = A_i \setminus \mathrm{Supp}(D_i^U).$$

Since $X \setminus U$ has codimension at least 2, it follows that the intersection of each irreducible component A_i' of A_i with $\mathrm{Supp}(D_i^U)$ is dense in A_i'. This implies $A_i = \mathrm{Supp}(D_i)$ and our claim is proved. In particular, we have

$$X = G \cdot U = G \cdot \bigcup_{i=1}^{r} X \setminus \mathrm{Supp}(D_i) = \bigcup_{i=1}^{r} \bigcup_{g \in G} X \setminus \mathrm{Supp}(g \cdot D_i).$$

Thus it suffices to show that, for each D_i, some multiple is Cartier on X. This is done as follows. The restriction D_i' of D_i to the regular locus $X' \subset X$ is Cartier. Since X' is G-invariant, we may apply G-linearization; that is, replacing D_i with a suitable multiple we achieve that $\mathcal{O}_{D_i'}$ is a G-sheaf (see e.g. [17, Prop. 2.4]).

We claim that this structure of a G-sheaf extends canonically to $\mathcal{O}_{D_i}$. For an open set $V \subset X$ let $V' := V \cap X'$. Given a section $s \in \mathcal{O}_{D_i}(V)$, we define its translates $g \cdot s$ as follows. Translate the restriction $s' \in \mathcal{O}_{D_i}(V')$ to a section $g \cdot s' \in \mathcal{O}_{D_i}(g \cdot V')$ and then extend $g \cdot s'$ to the desired section $g \cdot s \in \mathcal{O}_{D_i}(g \cdot V)$.

Using the G-sheaf structure on $\mathcal{O}_{D_i}$, we see that locally $\mathcal{O}_{D_i}$ is generated by a single function. That means D_i is a Cartier divisor. □

2. Support Maps

Projectivity of a given toric variety is characterized by the existence of a strictly convex support function on its fan (see e.g. [12]). Generalizing the notion of a support function, we here introduce the concept of a support map on a fan and define convexity properties for such maps. The main result of this section states that, for a given fan, existence of a strictly convex support map is equivalent to divisoriality of the associated toric variety.

For a lattice N, we denote the associated rational vector space by $N_{\mathbb{Q}}$. A *cone* in N is a polyhedral (not necessarily strictly) convex cone $\sigma \subset N_{\mathbb{Q}}$. A *quasifan* in N is a finite set Λ of cones in N such that for $\sigma \in \Lambda$, every face of σ belongs to Λ, and for $\sigma, \sigma' \in \Lambda$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'. A *fan* is a quasifan containing only strictly convex cones.

The *support* of a quasifan Λ is the union of all its cones and is denoted by $|\Lambda|$. A *map of quasifans* Λ in a lattice N and Λ' in a lattice N' is a lattice homomorphism $N \to N'$ such that the associated linear map $N_{\mathbb{Q}} \to N'_{\mathbb{Q}}$ maps the cones of Λ into cones of Λ'.

For the definition of support maps, fix a lattice N and a quasifan Δ in N. We say that a map $N_{\mathbb{Q}} \to \mathbb{Q}^k$ is linear on a subset $A \subset N_{\mathbb{Q}}$ if its restriction to A is the restriction of a linear map.

DEFINITION 2.1. A *support map* on Δ is a map $h\colon |\Delta| \to \mathbb{Q}^k$ that is linear on every cone $\sigma \in \Delta$.

For a support map $h\colon |\Delta| \to \mathbb{Q}^k$, let γ be the cone in $\hat{N} := N \times \mathbb{Z}^k$ generated by the graph Γ_h of h, and let $\mathfrak{F}(\gamma)$ denote the quasifan consisting of all faces of γ. The *filled graph* of h is the minimal subquasifan Λ_h of $\mathfrak{F}(\gamma)$ with $\Gamma_h \subset |\Lambda_h|$. Thus, Λ_h is generated by the cones $\delta \prec \gamma$ whose relative interior δ° meets Γ_h.

DEFINITION 2.2. The support map $h\colon |\Delta| \to \mathbb{Q}^k$ is called *convex* if the projection $P\colon \hat{N}_{\mathbb{Q}} \to N_{\mathbb{Q}}$ is injective on the support $|\Lambda_h|$.

This notion of convexity includes the classical concept of a convex support function on a complete fan as defined, for example, in [12, p. 67].

REMARK 2.3. Let $h\colon |\Delta| \to \mathbb{Q}$ be a support map on a fan Δ. If there are linear forms u_σ ($\sigma \in \Delta$) on N such that for any pair $\sigma, \tau \in \Delta$ we have

$$h|_\sigma = u_\sigma|_\sigma \quad \text{and} \quad h|_\tau \leq u_\sigma|_\tau,$$

then h is a convex support map on Δ. Conversely, if Δ is complete and h is convex then h or $-h$ satisfies these conditions.

On noncomplete fans, the concept of convexity for a support function via the inequalities of Remark 2.3 is more restrictive than our concept.

EXAMPLE 2.4. (See Figure 1.) Consider the fan Δ in $\mathbb{Z}^2$ generated by the two maximal cones

$$\sigma_1 := \operatorname{cone}((1, 0), (1, -1)), \qquad \sigma_2 := \operatorname{cone}((0, 1), (1, 1))$$

and the support map $h\colon |\Delta| \to \mathbb{Q}$ determined by

$$h(v_1, v_2) := \begin{cases} 2v_1 + 2v_2 & \text{if } (v_1, v_2) \in \sigma_1, \\ -v_1 + v_2 & \text{if } (v_1, v_2) \in \sigma_2. \end{cases}$$

Then h is convex: The convex hull γ of the graph Γ_h is a strictly convex cone with four rays, namely,

$$\gamma = \operatorname{cone}((1, 0, 2), (1, -1, 0), (0, 1, 1), (1, 1, 0)).$$

Moreover, the maximal cones of Λ_h are precisely the two faces of γ above σ_1 and σ_2, respectively.

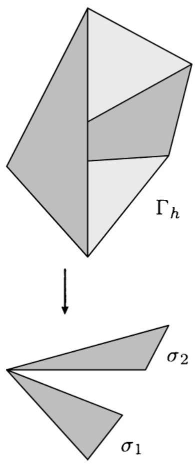

Figure 1

However, neither the function h nor the function $-h$ satisfies the inequalities of Remark 2.3, because

$$h((0,1)) = 1 < 2 \quad \text{and} \quad h((1,-1)) = 0 > -2.$$

In order to define the notion of strict convexity, we must note some observations on convex support maps. The first one is as follows.

LEMMA 2.5. *If the support map $h\colon |\Delta| \to \mathbb{Q}^k$ is convex, then the projected cones $P(\delta)$, $\delta \in \Lambda_h$, form a quasifan Σ_h in the lattice N.*

Proof. The projection P is injective on any given $\delta \in \Lambda_h$ and hence induces a bijection between the faces of δ and the faces of $P(\delta)$. Moreover, given $\delta_1, \delta_2 \in \Lambda_h$, injectivity of P on $|\Lambda_h|$ implies

$$P(\delta_1) \cap P(\delta_2) = P(\delta_1 \cap \delta_2).$$

Since $\delta_1 \cap \delta_2$ is a face of both δ_i, it follows that $P(\delta_1 \cap \delta_2)$ is a common face of $P(\delta_1)$ and $P(\delta_2)$. □

If $h\colon |\Delta| \to \mathbb{Q}^k$ is a convex support map, then we call Σ_h the *quasifan associated to h*. We need the following properties of this quasifan.

LEMMA 2.6. *Let Σ_h be the quasifan associated to a convex support map $h\colon |\Delta| \to \mathbb{Q}^k$. Then*:

(i) *every cone of Δ is contained in a cone of Σ_h*;
(ii) *every cone $\sigma \in \Sigma_h$ is generated by the cones $\tau \in \Delta$ with $\tau \subset \sigma$.*

DEFINITION 2.7. We say that a convex support map $h\colon |\Delta| \to \mathbb{Q}^k$ is *strictly convex* if its associated quasifan Σ_h equals Δ.

Using Remark 2.3, one may verify that, on a complete fan Δ, our notion of strict convexity for a support map $h\colon |\Delta| \to \mathbb{Q}$ concides with the usual one (as defined in [12, p. 67]). Again, for noncomplete fans the notions differ, as our Example 2.4 shows.

EXAMPLE 2.8. The convex support map $h\colon |\Delta| \to \mathbb{Q}$ of Example 2.4 is *strictly* convex.

We now come to the announced main result of this section—namely, the characterization of divisoriality of a toric variety via existence of a strictly convex support map.

PROPOSITION 2.9. *For a fan Δ in a lattice N, the following statements are equivalent*:

(i) *Δ admits a strictly convex support map*;
(ii) *the toric variety X associated to Δ is divisorial.*

In the proof of this statement, we make use of the following well-known characterization of existence of geometric quotients for subtorus actions in terms of fans (see e.g. [13, Thm. 5.1]).

PROPOSITION 2.10. *Let $\hat{\Delta}$ be a fan in a lattice $\hat{N}$ with associated toric variety $\hat{X}$, let $P\colon \hat{N} \to N$ be a surjective lattice homomorphism, and let H be the subtorus of the big torus of $\hat{X}$ corresponding to* $\ker(P)$. *Then the following statements are equivalent*:

(i) *P is injective on the support $|\hat{\Delta}|$;*
(ii) *the action of H on $\hat{X}$ has a geometric quotient.*

If one of these statements holds, then the quotient variety $\hat{X}/H$ is the toric variety determined by the fan $\{P(\sigma);\ \sigma \in \hat{\Delta}\}$ in N.

Proof of Proposition 2.9. Assume first that the fan Δ admits a strictly convex support map $h\colon |\Delta| \to \mathbb{Q}^k$. Then, since $\Delta = \Sigma_h$, all cones of Σ_h are strictly convex. As before, let $\hat{N} := N \times \mathbb{Z}^k$. By convexity of h, the projection $P\colon \hat{N}_{\mathbb{Q}} \to N_{\mathbb{Q}}$ is an injection on $|\Lambda_h|$. In particular, all cones of Λ_h are strictly convex. That means that Λ_h is a fan.

The toric variety $\hat{X}$ associated to Λ_h is quasiaffine, and the projection $P\colon \hat{N} \to N$ gives rise to a toric morphism $p\colon \hat{X} \to X$. According to Proposition 2.10, this toric morphism p is a geometric quotient for the subtorus action on $\hat{X}$ corresponding to $\ker(P) \subset \hat{N}$. Thus, Corollary 1.6 yields that X is divisorial.

Suppose now that the toric variety X determined by the fan Δ is divisorial. By Corollary 1.6, there is a quasiaffine toric variety $\hat{X}$ and a toric morphism $p\colon \hat{X} \to X$ such that $H := \ker(p)$ is a subtorus of the big torus of $\hat{X}$ and p is a geometric quotient for the action of H on $\hat{X}$.

Let $p\colon \hat{X} \to X$ arise from a map $P\colon \hat{N} \to N$ of fans $\hat{\Delta}$ and Δ. Since $H = \ker(p)$ is connected, the map P is surjective and we obtain a section $N \to \hat{N}$ for P. We may therefore assume that $\hat{N} = N \times \mathbb{Z}^k$ holds and that P is the projection onto the first factor. By Proposition 2.10, the projection P is injective on $|\hat{\Delta}|$. Thus, for each $\hat{\sigma} \in \hat{\Delta}$, the restriction

$$P|_{\hat{\sigma}}\colon \hat{\sigma} \mapsto \sigma := P(\hat{\sigma})$$

admits a uniquely determined linear inverse of the form $g_\sigma = (\mathrm{id}_{N_{\mathbb{Q}}}, h_\sigma)$. The maps $h_\sigma\colon \sigma \to \mathbb{Q}^k$ patch together to a support map h on Δ. By construction, $\Lambda_h = \hat{\Delta}$ and $\Sigma_h = \Delta$, so h is the desired strictly convex support map on Δ. □

In the remainder of this section we show that convex support maps in a canonical way define toric morphisms to divisorial toric varieties. Let Δ be a fan in a lattice N, and let $h\colon |\Delta| \to \mathbb{Q}^k$ be a convex support map.

There is a universal method to construct a fan from the associated quasifan Σ_h: Let $\sigma_{\min} \in \Sigma_h$ denote its minimal cone. This is a linear subspace of $N_{\mathbb{Q}}$. Let $N_0 := \sigma_{\min} \cap N$, set $N_h := N/N_0$, and denote the projection by $F_h\colon N \to N_h$. The *quotient fan* of Σ_h is the fan

$$\Delta_h := \{F_h(\sigma);\ \sigma \in \Sigma_h\}.$$

The projection $F_h\colon N \to N_h$ is a map of the quasifans Σ_h and Δ_h. Moreover, F_h is universal in the sense that every map of quasifans from Σ_h to a fan Δ' factors uniquely through F_h.

Now, let X and X_h denote the toric varieties associated to the fans Δ and Δ_h, respectively. Our precise statement is as follows.

PROPOSITION 2.11. *The toric variety X_h is divisorial, and the projection F_h induces a toric morphism $f_h\colon X \to X_h$.*

Proof. By Lemma 2.6(i) and the universal property of the quotient fan Δ_h, the projection $F_h\colon N \to N_h$ is a map of the fans Δ and Δ_h and hence induces a toric morphism $f_h\colon X \to X_h$. Thus we need only show that X_h is divisorial. In view of Proposition 2.9, we look for a strictly convex support map on Δ_h.

The first step is to construct a strictly convex support map g on the quasifan Σ_h associated to h. Consider a cone $\sigma \in \Sigma_h$. Then, denoting (as before) the projection by $P\colon \hat{N} \to N$, we have $\sigma = P(\delta)$ for some cone $\delta \in \Lambda_h$.

By convexity of h, the restriction $P\colon \delta \to \sigma$ has an inverse of the form (id, g_σ). The maps g_σ patch together to a support map g on Σ_h, and g extends h. Moreover, Γ_g equals Λ_h and hence the quasifan associated to g coincides with Σ_h.

Note that $\Sigma_g = \Sigma_h$ does not change if we add a global linear function to g. So we may assume that the support function g vanishes on the minimal cone of Σ_g. But then we can push down g to a strictly convex support function on the quotient fan Δ_h. □

3. Toric Divisorial Reduction

Fix a toric variety X. In [3], we presented a universal way to reduce X to a quasiprojective toric variety. In this section we give an analogous construction that reduces to divisorial toric varieties.

DEFINITION 3.1. A *toric divisorial reduction* of X is a toric morphism $r\colon X \to X^{\mathrm{tdr}}$ to a divisorial toric variety X^{tdr} such that every toric morphism $f\colon X \to Z$ to a divisorial toric variety Z has a unique factorization $f = \tilde{f} \circ r$ with a toric morphism $\tilde{f}\colon X^{\mathrm{tdr}} \to Z$.

THEOREM 3.2. *Every toric variety admits a toric divisorial reduction.*

The proof will be given shortly. We first need the following statement on the pullback of a convex support map.

LEMMA 3.3. *Let $F\colon N \to N'$ be a map of fans Δ and Δ' in lattices N and N', respectively. If $h'\colon |\Delta'| \to \mathbb{Q}^k$ is a convex support map on Δ', then $h := h' \circ F$ is a convex support map on Δ and F is a map of the associated quasifans Σ_h and $\Sigma_{h'}$.*

Proof. Clearly h is a support map on Δ. To prove convexity of h, we consider the filled graphs Λ_h, $\Lambda_{h'}$ and the map

$$\hat{F} := F \times \mathrm{id}_{\mathbb{Z}^k} \colon N \times \mathbb{Z}^k \to N' \times \mathbb{Z}^k.$$

We claim that $\hat{F}$ is a map of the quasifans Λ_h and $\Lambda_{h'}$. To verify this, note first that $\hat{F}$ maps the graph Γ_h to $\Gamma_{h'}$. Let $\delta \in \Lambda_h$. We need to show that the minimal face δ' of $\mathrm{conv}(\Gamma_{h'})$ containing $\hat{F}(\delta)$ belongs to $\Lambda_{h'}$. Let

$$G := \mathrm{id}_{|\Delta|} \times h, \qquad G' := \mathrm{id}_{|\Delta'|} \times h'.$$

By definition of Λ_h, the relative interior δ° of δ contains a point of the graph of h, that is, a point of the form $G(v)$ for some $v \in |\Delta|$. By the choice of δ' this means $\hat{F}(G(v)) \in (\delta')^\circ$. On the other hand, by the definitions of G, G', and $\hat{F}$, we have

$$\hat{F}(G(v)) = G'(F(v)) \in \Gamma_{h'}.$$

Hence $\Gamma_{h'} \cap (\delta')^\circ \neq \emptyset$. This implies $\delta' \in \Lambda_{h'}$, and our claim is proved.

For convexity of h, we must show that the projection $P \colon N \times \mathbb{Z}^k \to N$ is injective on $|\Lambda_h|$. Suppose $w_i = (v_i, t_i) \in |\Lambda_h|$ are two points such that $P(w_1)$ equals $P(w_2)$, which means that $v_1 = v_2$. Then we have

$$P'(\hat{F}(w_1)) = P'(\hat{F}(w_2)),$$

where $P' \colon N' \times \mathbb{Z}^k \to N'$ is the projection. Since $\hat{F}$ is a map of the quasifans Λ_h and $\Lambda_{h'}$ and since P' is injective on $|\Lambda_{h'}|$, it follows that $\hat{F}(w_1) = \hat{F}(w_2)$. In particular, we have $t_1 = t_2$ and thus $w_1 = w_2$.

Finally, the fact that F is a map of the quasifans $\Sigma_{h'}$ and Σ_h follows immediately from the fact that $\hat{F}$ is a map of the quasifans $\Lambda_{h'}$ and Λ_h. □

Proof of Theorem 3.2. Let X be a toric variety arising from a fan Δ in a lattice N. First we show that any given toric morphism $f \colon X \to Z$ from X to a divisorial variety Z factors uniquely through one of the toric morphisms f_h arising from a convex support map on Δ as in Proposition 2.11.

To see this, consider the map of fans $F \colon \Delta \to \Delta'$ associated to the given toric morphism f and choose a strictly convex support map h' on Δ'. Lemma 3.3 tells us that, by pulling back h' via F, we obtain a convex support map h on Δ. Moreover, F defines a map of quasifans from Σ_h to $\Sigma_{h'} = \Delta'$.

The map of fans F now factors as a map of fans through the projection $F_h \colon N \to N_h$; that is, F induces a map from the quotient fan Δ_h of Σ_h to Δ'. Obviously, the corresponding toric morphism is the desired factorization of $f \colon X \to Z$ through $f_h \colon X \to X_h$.

Now let us take a closer look at the toric morphisms $f_h \colon X \to X_h$ arising from convex support maps. Recall that the morphism f_h is already determined by the quasifan Σ_h associated to h. By Lemma 2.6(ii), each such quasifan has the property that all cones are generated by cones of Δ. Consequently there exist only finitely many of such quasifans, say $\Sigma_1, \dots, \Sigma_r$.

Let $f_i \colon X \to Y_i$ denote the toric morphisms to divisorial toric varieties determined by Σ_i, and consider their product $f := f_1 \times \cdots \times f_r$. Let Y denote the closure of the image $f(X)$ in $Y_1 \times \cdots \times Y_r$. The normalization $\tilde{Y}$ of Y is again a divisorial toric variety, and f lifts to a toric morphism to $\tilde{Y}$. In $\tilde{Y}$ we choose the smallest open toric subvariety Y' containing the image of f and, restricting f, we obtain a toric morphism $r\colon X \to Y'$.

By construction, for every i we have a unique factorization of f_i through r: namely, $f_i = \mathrm{pr}_i \circ r$, where $\mathrm{pr}_i\colon Y' \to Y_i$ denotes the restriction of the projection on the ith factor. This proves that r is the desired toric divisorial reduction. □

We conclude this section with some examples. Note that any 2-dimensional toric variety is simplicial and hence divisorial. Hence the minimal dimension for interesting examples is 3.

EXAMPLE 3.4. If a toric variety does not admit nontrivial effective Cartier divisors (see e.g. [12, p. 25]), then its toric divisorial reduction is a point.

EXAMPLE 3.5. (See Figure 2.) Consider the following eight vectors in $\mathbb{Q}^3$:

$$v_1 := (2,2,1),\quad v_2 := (-2,2,1),\quad v_3 := (-2,-2,1),\quad v_4 := (2,-2,1),$$
$$v_5 := (1,1,1),\quad v_6 := (-1,1,1),\quad v_7 := (-1,-1,1),\quad v_8 := (2/3,1/3,1).$$

Let Δ denote the fan in $\mathbb{Z}^3$ with maximal cones

$$\sigma_1 := \mathrm{cone}(v_1, v_2, v_5, v_6),\quad \sigma_2 := \mathrm{cone}(v_2, v_3, v_6, v_7),$$
$$\sigma_3 := \mathrm{cone}(v_3, v_4, v_7, v_8),\quad \sigma_4 := \mathrm{cone}(v_1, v_4, v_5, v_8),$$
$$\sigma_5 := \mathrm{cone}(v_5, v_6, v_7, v_8).$$

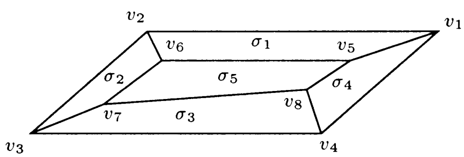

Figure 2 Intersection of Δ with the plane $x_3 = 1$

The identity on $\mathbb{Z}^3$ defines a map of fans from Δ to the fan of faces $\mathfrak{F}(\sigma)$ of the cone $\sigma := \mathrm{cone}(v_1, v_2, v_3, v_4)$. We claim that the corresponding toric morphism $r\colon X_\Delta \to X_\sigma$ is the toric divisorial reduction of X_Δ.

To see this, consider a convex support map $h\colon |\Delta| \to \mathbb{Q}^k$ and its associated quasifan Σ_h. Lemma 2.6 implies that we have only two possibilities: either $\Sigma_h = \mathfrak{F}(\sigma)$ or $\Sigma_h = \Delta$. Thus, to verify our claim, we need only exclude the latter possibility; that is, we must show that h cannot be strictly convex.

Otherwise, let $\delta_5 \in \Lambda_h$ be the maximal cone above σ_5 and choose a linear form $\lambda\colon N_{\mathbb{Q}} \times \mathbb{Q}^k \to \mathbb{Q}$ that is nonnegative on $\gamma := \mathrm{conv}(\Gamma_h)$ and fulfills $\delta_5 = \gamma \cap \lambda^\perp$. Pulling back λ via $\mathrm{id}_N \times h$, we obtain a nonnegative support function g on Δ vanishing precisely on σ_5. Note that

$$g(v_1) = g(v_2) = g(v_3).$$

Moreover, we have the relations

$$v_4 = 17v_3 - 28v_7 + 12v_8, \qquad v_4 = 5v_1 - 16v_5 + 12v_8.$$

Applying g, we obtain $17g(v_3) = 5g(v_1)$. This contradicts $g(v_1) = g(v_3)$, so h cannot be strictly convex and thus our claim is proved.

EXAMPLE 3.6. (See Figure 3.) We describe a toric variety with a nonsurjective toric divisorial reduction. Similarly to the preceding example, consider the vectors

$$\begin{aligned} &v_1 := (2,2,1,0), && v_2 := (-2,2,1,0), && v_3 := (-2,-2,1,0),\\ &v_4 := (2,-2,1,0), && v_5 := (1,1,1,0), && v_6 := (-1,1,1,0),\\ &v_7 := (-1,-1,1,0), && v_8 := (2/3,1/3,1,0) && \end{aligned}$$

in $\mathbb{Q}^4$. Furthermore, let e_4 be the fourth canonical base vector. Let Δ denote the fan in $\mathbb{Z}^4$ with maximal cones

$$\begin{aligned} &\sigma_1 := \text{cone}(v_1, v_2, v_5, v_6), && \sigma_2 := \text{cone}(v_2, v_3, v_6, v_7),\\ &\sigma_3 := \text{cone}(v_3, v_4, v_7, v_8), && \sigma_4 := \text{cone}(v_1, v_4, v_5, v_8),\\ &\sigma_5 := \text{cone}(v_5, v_6, v_7, v_8), && \sigma_6 := \text{cone}(v_5, v_6, e_4). \end{aligned}$$

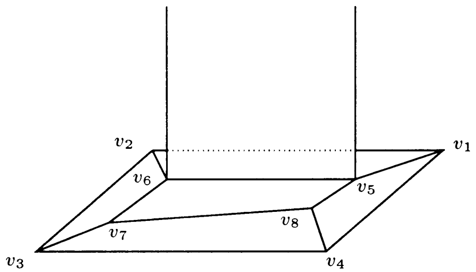

Figure 3 Intersection of Δ with the hyperplane $x_3 = 1$

The identity on $\mathbb{Z}^4$ defines a map of fans from Δ to the fan of faces $\mathfrak{F}(\sigma)$ of the cone $\sigma := \text{cone}(v_1, v_2, v_3, v_4, e_4)$. We claim that the corresponding toric morphism $r: X_\Delta \to X_\sigma$ is the toric divisorial reduction of X_Δ. Note that this map is not surjective.

Let us verify the claim. If h is a convex support map then it follows that $|\Sigma_h| \subset \sigma$. The restriction of h to the support of the subfan Δ' of Δ generated by the cones $\sigma_1, \ldots, \sigma_5$ defines a convex support map h' of Δ'. So by the previous example, $\Sigma_{h'} = \mathfrak{F}(\sigma')$, where σ' denotes the cone generated by $v_1, \ldots, v_4$.

Now Lemma 3.3 implies that the smallest cone τ in Σ_h containing σ_5 also contains all of σ'. That means (by Lemma 2.6) that either $\tau = \sigma'$ or $\tau = \sigma$. In any case, since σ' is a face of σ we obtain $\sigma' \in \Sigma_h$.

Next consider the smallest cone $\tau' \in \Sigma_h$ containing σ_6. We have $v_5, v_6 \in \sigma_6$, so the cone τ' meets σ' in its relative interior. Since Σ_h is a quasifan, we can conclude that σ' is in fact a face of τ'. Because $e_4 \in \tau$ this implies $\tau' = \sigma$, and we obtain $\Sigma_h = \mathfrak{F}(\sigma)$.

4. A Lifting Lemma

Here we relate regular maps between divisorial toric prevarieties to regular maps between quasiaffine toric varieties. For maps of projective spaces, the following example is a classical observation.

EXAMPLE 4.1. Let $f\colon \mathbb{P}_n \to \mathbb{P}_m$ be a regular map of projective spaces. Then f is of the form

$$[z_0, \dots, z_n] \mapsto [f_0(z_0, \dots, z_n), \dots, f_m(z_0, \dots, z_n)]$$

with homogeneous polynomials f_i that are pairwise of the same degree. In other words, there is a lifting

$$\begin{array}{ccc} \mathbb{K}^{n+1} \setminus \{0\} & \xrightarrow{\hat{f}} & \mathbb{K}^{m+1} \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathbb{P}_n & \xrightarrow{f} & \mathbb{P}_m. \end{array}$$

The main result of this section is the following generalization of the previous lifting statement.

LEMMA 4.2. *Let $f\colon X_1 \to X_2$ be a regular map of divisorial toric prevarieties such that $f(X_1)$ intersects the big torus of X_2. Then there exists a commutative diagram*

$$\begin{array}{ccc} \hat{X}_1 & \xrightarrow{\hat{f}} & \hat{X}_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ X_1 & \xrightarrow{f} & X_2, \end{array}$$

where $\hat{X}_1, \hat{X}_2$ are quasiaffine toric varieties, $q_i\colon \hat{X}_i \to X_i$ are geometric prequotients for free subtorus actions on $\hat{X}_i$, and $\hat{f}\colon \hat{X}_1 \to \hat{X}_2$ is a regular map.

Proof. We use the ideas and methods presented in [14, Sec. 2]. Choose effective T_i-invariant Cartier divisors $D_1^i, \dots, D_{r_i}^i$ on X_i such that the complements $X_i \setminus \mathrm{Supp}(D_j^i)$ form an affine cover of X_i. Let $W_i \subset \mathrm{CDiv}(X_i)$ denote the subgroup generated by $D_1^i, \dots, D_{r_i}^i$. The pullback via f gives rise to a group homomorphism

$$\psi\colon W_2 \to \mathrm{CDiv}(X_1), \qquad D \mapsto f^*(D).$$

Enlarge W_1 by adding the image $\psi(W_2)$. Note that the line bundles determined by the divisors of W_i are T_i-linearizable (see [17, p. 67, Remark]). We shall regard ψ in the sequel as a homomorphism from W_2 to W_1. Consider the $\mathcal{O}_{X_i}$-algebras

$$\mathcal{A}_i := \bigoplus_{D \in W_i} \mathcal{O}_D(X_i)$$

and their associated relative spectra $\hat{X}_i := \mathrm{Spec}(\mathcal{A}_i)$. By [14, Remark 2.1], the inclusion $\mathcal{O}_{X_i} \subset \mathcal{A}_i$ gives rise to a geometric prequotient $q_i \colon \hat{X}_i \to X_i$ for the free action of the algebraic torus $H_i := \mathrm{Spec}(\mathbb{K}[W_i])$ on $\hat{X}_i$ induced by the W_i-grading of $\mathcal{A}_i$.

Since W_1 and W_2 define ample groups of line bundles in the sense of [14, Def. 2.2], each $\hat{X}_i$ is in fact a quasiaffine variety. Moreover, by [14, Prop. 2.3], the variety $\hat{X}_i$ carries a regular action of the algebraic torus T_i commuting with the action of H_i such that $q_i \colon \hat{X}_i \to X_i$ becomes T_i-equivariant. It follows that $\hat{X}_i$ is a toric variety with big torus $\hat{T}_i = T_i \times H_i$.

We still have to construct the lifting $\hat{f} \colon \hat{X}_1 \to \hat{X}_2$. Toward this end, note that for every affine open subset $U \subset X_2$ we may obtain a homomorphism of W_i-graded algebras by setting

$$\mathcal{A}_2(U) \to \mathcal{A}_1(f^{-1}(U)), \qquad \mathcal{O}_D(U) \ni h \mapsto f^*(h) \in \mathcal{O}_{\psi(D)}(U) \quad (D \in W_2).$$

Note that, on the homogeneous component $\mathcal{A}_2(U)_0$, this is just the comorphism of the map f. By definition of $\hat{X}_i$ and the maps $q_i \colon \hat{X}_i \to X_i$, each of the preceding homomorphisms gives rise to a lifting

$$\hat{f}_U \colon q_1^{-1}(f^{-1}(U)) \to q_2^{-1}(U)$$

of the restriction $f \colon f^{-1}(U) \to U$. By construction, the maps $\hat{f}_U$ patch together to the desired lifting $\hat{f} \colon \hat{X}_1 \to X_2$ of $f \colon X_1 \to X_2$. □

The following observation will be needed later to obtain equivariance properties for the lifting $\hat{f} \colon \hat{X}_1 \to \hat{X}_2$ constructed in Lemma 4.2.

LEMMA 4.3. *For $i = 1, 2$, let T_i be algebraic tori and let Y_i be irreducible T_i-varieties such that T_2 acts freely on Y_2. If $f \colon Y_1 \to Y_2$ is regular and maps the orbits of T_1 into orbits of T_2, then there is a homomorphism $\varphi \colon T_1 \to T_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ holds for all $(t, x) \in T_1 \times Y_1$.*

Proof. By Sumihiro's theorem [20, Cor. 2], we may assume that Y_2 is affine. Thus, there is an algebraic quotient $Y_2 \to Y$ for the action of T_2 on Y_2. Since T_2 acts freely, the quotient map $Y_2 \to Y$ is equivariantly locally trivial. Thus, shrinking Y, we may even assume that $Y_2 = T_2 \times Y$ holds. In particular, one has $f = (f_1, f_2)$ with regular maps $f_1 \colon Y_1 \to T_2$ and $f_2 \colon Y_1 \to Y$. So, we obtain a regular map

$$\Phi \colon T_1 \times Y_1 \to T_2, \qquad (t, x) \mapsto f_1(t \cdot x) f_1(x)^{-1}.$$

For fixed $x \in Y_1$, the map $t \mapsto \Phi(t, x)$ maps the neutral element of T_1 to the neutral element of T_2 and thus is necessarily a homomorphism of the tori T_1 and

T_2. By rigidity of tori [9, III.8.10], the map Φ does not depend on x. Hence, there is a homomorphism $\varphi\colon T_1 \to T_2$ with $\Phi(t,x) = \varphi(t)$ for all $(t,x) \in T_1 \times Y_1$. Clearly, φ is as desired. □

A different aspect of the lifting problem is discussed extensively in [7]: Given two quotient presentations $\hat{X}_i \to X_i$ of toric varieties in the sense of [6] and a regular map $f\colon X_1 \to X_2$, when can this map be lifted to a regular map $F\colon \hat{X}_1 \to \hat{X}_2$?

5. Decomposition of Regular Maps

Let X be a toric variety with big torus T and consider the action of a closed subgroup $H \subset T$ on X. Here we provide the key for relating H-invariant regular maps $X \to Y$ to H-invariant toric morphisms.

LEMMA 5.1. *Let $f\colon X \to Y$ be an H-invariant regular map to a divisorial variety Y. Then there exists a dominant H-invariant toric morphism $g\colon X \to X'$ to a divisorial toric variety X', an open subset $U \subset X'$ with $g(X) \subset U$, and a regular map $h\colon U \to Y$ such that $f = h \circ g$.*

Proof. First we reduce the problem to the case where H is connected. Suppose that $g\colon X \to X'$ and $h\colon U \to Y$ satisfy the assertion for the identity component H^0 of H. Then g induces an action of the finite abelian group $\Gamma := H/H^0$ on X'. Let $p\colon X' \to X''$ be the geometric quotient for this action. Note that p is a toric morphism. Using Corollary 1.6, we see that the variety X'' is again divisorial.

By appropriate shrinking, we achieve that U is Γ-invariant. Since p is geometric, $p(U)$ is open in X'' and the restriction $p\colon U \to p(U)$ is again a geometric quotient for the action of Γ. Since h is Γ-invariant, we have $h = h' \circ p$ for some regular map $h'\colon p(U) \to Y$. It follows that $f = h' \circ (p \circ g)$ is the desired decomposition. Consequently, it suffices to give the proof for connected H.

The next simplification provides the link to the toric setting. As mentioned before, we can realize Y as a closed subvariety of a smooth toric prevariety Z of affine intersection (see Theorem 1.2). Let $Z' \subset Z$ denote the minimal orbit closure of the big torus of Z such that $f(X) \subset Z'$ holds. Then Z' is again a smooth toric prevariety of affine intersection, but in Z' the image $f(X)$ intersects the big torus.

For the moment, regard f as a map from X to Z' and suppose that $g\colon X \to X'$ and $h\colon U \to Z'$ satisfy the assertion for $f\colon X \to Z'$. Taking closures in U and Z' (respectively), we obtain

$$h(U) \subset h\big(\overline{g(X)}\big) \subset \overline{h(g(X))} = \overline{f(X)} \subset Y.$$

This means that h is, in fact, a map from U to Y. Thus X', g, h, and U also provide the desired data for the original $f\colon X \to Y$. Consequently, we can assume in the sequel that Y is a smooth toric prevariety of affine intersection and that $f(X)$ intersects the big torus of Y. But then, according to Lemma 4.2, there is a commutative diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y, \end{array}$$

where $\hat{X}$, $\hat{Y}$ are quasiaffine toric varieties and where the vertical maps are geometric prequotients for free actions of subtori H_X and H_Y of the big tori of $\hat{X}$ and $\hat{Y}$, respectively. We may even assume that $\hat{X} = X$ holds.

Let $H' := p^{-1}(H)$ and suppose that the H'-invariant regular map $f' := f \circ p$ admits a decomposition of the form $f' = h' \circ g'$ with a dominant H'-invariant toric morphism $g' \colon \hat{X} \to X'$ and a regular map $h' \colon U \to Y$ defined on an open neighborhood U of the image of g'.

Then, by the universal property of p, there is a toric morphism $g \colon X \to X'$ with $g' = g \circ p$. Clearly this morphism is dominant. Moreover, since p is surjective, g is H-invariant and $g(X) \subset U$ holds. Consequently, $f = h' \circ g$ is a decomposition as desired. Hence it suffices to prove the assertion for the case when $\hat{X} = X$ and $H_X = 1$ hold and p is the identity map.

Now we consider the regular map $\hat{f} \colon X \to \hat{Y}$ as a map from an H-variety to an H_Y-variety. Because $q \circ \hat{f} = f$ is H-invariant, every H-orbit is mapped by $\hat{f}$ into a fiber of q. On the other hand, the fibers of q are precisely the H_Y-orbits. We can therefore apply Lemma 4.3 and conclude that $\hat{f}$ is H-equivariant with respect to a homomorphism $H \to H_Y$.

Choosing a locally closed toric embedding $\hat{Y} \subset \mathbb{K}^s$, we obtain a homomorphism $H_Y \to \mathbb{K}^s$, and the induced map $\hat{f} \colon X \to \mathbb{K}^s$ is H-equivariant with respect to the homomorphism $H \to H_Y \to \mathbb{K}^s$. Hence the components of $\hat{f}$ are H-homogeneous regular functions. By writing the components of $\hat{f}$ as linear combinations of character functions of the big torus $T \subset X$ and using the summands to define a toric morphism $g' \colon X \to \mathbb{K}^r$, we obtain a decomposition of $\hat{f}$ in the form $\hat{f} = s \circ g'$, with a linear map $s \colon \mathbb{K}^r \to \mathbb{K}^s$. Note that g' induces an action of H on $\mathbb{K}^r$, making $s \colon \mathbb{K}^r \to \mathbb{K}^s$ into an H-equivariant map.

Let W be the normalization of the closure of $g'(X)$ in $\mathbb{K}^r$. Then W is an affine toric variety with big torus $g'(T)$. We can lift g' to a dominant toric morphism $\hat{g} \colon X \to W$ and pull back s to a regular map $\hat{s} \colon W \to \mathbb{K}^s$. Both $\hat{g}$ and $\hat{s}$ are again equivariant for the induced H-action on W. The set $V := \hat{s}^{-1}(\hat{Y})$ is H-invariant and open in W. Moreover, we have $\hat{g}(X) \subset V$. So far, we are in the following situation:

$$\begin{array}{ccc} V & \xrightarrow{\hat{s}} & \hat{Y} \\ {\scriptstyle \hat{g}}\uparrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

Since $\hat{s} \colon W \to \mathbb{K}^s$ is an affine map, its restriction $\hat{s} \colon V \to \hat{Y}$ is also affine. Thus $q \circ \hat{s} \colon V \to Y$ is an affine H-invariant regular map. Existence of an affine

H-invariant map $V \to Y$ already implies existence of a good quotient $p\colon V \to V/\!\!/H$ for the action of H (see e.g. [19, Prop. 3.12]), so we obtain the following commutative diagram of regular maps:

$$\begin{array}{ccc} V & \xrightarrow{\;p\;} & V/\!\!/H \\ {\scriptstyle \hat{g}}\uparrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\;f\;} & Y. \end{array}$$

Note that $g := p \circ \hat{g}\colon X \to V/\!\!/H$ is H-invariant and that $V/\!\!/H$ is divisorial, because Y is divisorial and h is an affine morphism, so the decomposition $f = h \circ g$ is almost what we need. To complete the proof it suffices to show that we can embed $V/\!\!/H$ as an open subset into a divisorial toric variety X' such that g, when viewed as a morphism from X to X', is toric.

For this last step we argue as follows. Note that we constructed V as an open H-invariant subset of the toric variety W. In [21], Święcicka showed that "maximal" open subsets with a good quotient by a given subtorus in a toric variety are in fact toric subvarieties.

More precisely, according to [21, Cor. 2.4], V is contained in an open toric subvariety $V' \subset W$ with a good toric quotient $p'\colon V' \to V'/\!\!/H$ such that the induced map $V/\!\!/H \to V'/\!\!/H$ is an open inclusion. Of course, we can choose V' in such a manner that $V'/\!\!/H = T' \cdot (V/\!\!/H)$ holds, where T' denotes the big torus of $V'/\!\!/H$. We set $X' := V'/\!\!/H$ and $U := V/\!\!/H$ to arrive at the following commutative diagram:

$$\begin{array}{ccccccc} & & V' & \xrightarrow{\;p'\;} & V'/\!\!/H & = & X' \\ & & \cup & & \cup & & \cup \\ X & \xrightarrow{\;\hat{g}\;} & V & \xrightarrow{\;p\;} & V/\!\!/H & = & U. \end{array}$$

The morphism $X \to V'$ sending x to $\hat{g}(x)$ is a dominant toric morphism because $\hat{g}\colon X \to W$ is; hence the same is true for $g = p' \circ \hat{g}\colon X \to X'$. Moreover, because $\hat{g}(X) \subset V$ holds, we conclude that the big torus T' of X' is contained in U. It follows that the complement $X' \setminus U$ is of codimension at least 2 in X'. Thus, Lemma 1.7 yields that the toric variety X' is also divisorial. This completes the proof. □

6. Divisorial Reduction and Categorical Quotients

In this section we come to the main results of this article. Recall from [18] that a *categorical quotient* for a G-variety X is a G-invariant regular map $X \to Y$ such that any G-invariant regular map $X \to Z$ factors uniquely through $X \to Y$. Clearly, this notion can be restricted to any subcategory of the category of algebraic varieties as soon as the G-variety X belongs to this subcategory.

We give an answer to the problem of existence of categorical quotients for subtorus actions in the divisorial category. In fact, our method of proof solves the existence problem of a more general universal object. Consider a toric variety X with big torus T and the action of a subtorus $H \subset T$.

DEFINITION 6.1. An *H-invariant divisorial reduction* of X is a regular map $r\colon X \to Y$ to a divisorial variety Y such that every H-invariant regular map $f\colon X \to Z$ to a divisorial variety Z admits a unique factorization $f = \tilde{f} \circ r$ with a regular map $\tilde{f}\colon Y \to Z$. If $H = 1$, then we simply speak of a *divisorial reduction*.

A candidate for such a reduction is constructed in two steps. First, recall from [2] that there is a toric quotient for the action of H on X, which means there is a toric morphism

$$p\colon X \to X/_{\mathrm{tq}}H$$

that is a categorical quotient for the action of H on X in the category of toric varieties. In a second step, construct the toric divisorial reduction of the toric quotient space as described in Section 3:

$$q\colon X/_{\mathrm{tq}}H \to (X/_{\mathrm{tq}}H)^{\mathrm{tdr}}.$$

THEOREM 6.2. *For a toric variety X, the following statements are equivalent*:

(i) *X admits an H-invariant divisorial reduction*;
(ii) *the composition $q \circ p\colon X \to Z$ is surjective.*

Moreover, if one of these statements holds, then $q \circ p$ is the H-invariant divisorial reduction.

Applying this result to divisorial toric varieties X, we obtain the following solution to the quotient problem.

COROLLARY 6.3. *The action of a subtorus H on a divisorial toric variety X admits a categorical quotient in the category of divisorial varieties if and only if the composition of $X \to X/_{\mathrm{tq}}H$ and $X/_{\mathrm{tq}}H \to (X/_{\mathrm{tq}}H)^{\mathrm{tdr}}$ is a surjective map.*

A further special case of Theorem 6.2 is the case of a trivial torus $H = 1$. Here we obtain the following.

COROLLARY 6.4. *A toric variety admits a divisorial reduction if and only if its toric divisorial reduction is surjective.*

Proof of Theorem 6.2. Assume first that $q \circ p$ is surjective. We show that a given H-invariant regular map $f\colon X \to Z$ to a divisorial variety Z factors through $q \circ p$. Lemma 5.1 yields a decomposition $f = h \circ g$ with an H-invariant dominant toric morphism $g\colon X \to X'$ to a divisorial toric variety X'.

By the universal properties of p and q, the toric morphism g has a factorization $g = g' \circ (q \circ p)$. By surjectivity of $q \circ p$, the map h is defined on a neighborhood of the image of g'. Hence $f = (h \circ g') \circ (q \circ p)$ is the desired factorization. Thus $q \circ p$ is the H-invariant divisorial reduction of X.

Conversely, suppose that X has an H-invariant divisorial reduction $r\colon X \to Y$. Since the normalization of a divisorial variety is again divisorial, we can conclude that Y is normal. Moreover, the universal property of $r\colon X \to Y$ implies that r is surjective and that Y inherits a set-theoretical action of the big torus $T \subset X$, making r equivariant. Note that it is not clear a priori that this action is regular, so we cannot treat Y as a toric variety.

Let $Z := (X/_{\text{tq}}H)^{\text{tdr}}$. We shall compare the H-invariant divisorial reduction $r\colon X \to Y$ with the toric morphism $q \circ p\colon X \to Z$. On the one hand, because of the universal property of r, the map $q \circ p$ factors uniquely through r. So there is a unique regular map $\alpha\colon Y \to Z$ with $q \circ p = \alpha \circ r$.

On the other hand, Lemma 5.1 provides a decomposition $r = h \circ g$ with a dominant toric morphism $g\colon X \to X'$ to a divisorial toric variety X' and a rational map h from X' to Y that is defined on the image of g. By the universal properties of p and q, we have $g = g' \circ q \circ p$ with a toric morphism $g'\colon Z \to X'$. We thus arrive at the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ r\ } & Y \\ {\scriptstyle q\circ p}\downarrow & \overset{\alpha}{\swarrow} & \uparrow{\scriptstyle h} \\ Z & \xrightarrow[\ g'\ ]{} & X'. \end{array}$$

Note that $g'(q(p(X))) = g(X)$ is contained in the domain of definition of the rational map h. Since r is surjective, we have $q(p(X)) = \alpha(Y)$ and so obtain that h is defined on $g'(\alpha(Y))$. It follows that $(h \circ g') \circ \alpha$ is the identity on Y. This shows that α is injective. Moreover, on the big torus of Z, the map $\alpha \circ (h \circ g')$ is the identity.

Consequently, $\alpha\colon Y \to Z$ is a birational injection. Since Z is normal, Zariski's main theorem tells us that α is in fact an open embedding. Since the image $\alpha(Y)$ is invariant under the induced set-theoretical action of T on Y, the map α is an isomorphism. In particular, $r\colon X \to Y$ is surjective. □

We conclude this section with some examples. In many situations, the foregoing results give positive answers to the problem of existence of quotients. A typical case are toric varieties defined by fans with convex support.

COROLLARY 6.5. *Let X be a toric variety arising from a fan with convex support. Then X admits a divisorial reduction.*

Proof. Let the toric divisorial reduction $q\colon X \to X'$ arise from a map $Q\colon N \to N'$ of fans Δ and Δ'. Then $\sigma := Q(|\Delta|)$ is a convex cone in N' and $\sigma \subset |\Delta'|$. Intersecting the cones of Δ' with σ, we obtain a further fan in N', namely,

$$\Delta'' := \bigcup_{\tau' \in \Delta'} \mathfrak{F}(\tau' \cap \sigma).$$

Let X'' be the associated toric variety. The identity map $N \to N'$ defines an affine toric morphism $g\colon X'' \to X'$. In particular, X'' is divisorial. Moreover, $Q\colon N \to N'$ is also a map of the fans Δ and Δ''. The corresponding toric morphism $q' \colon X \to X''$ is surjective because $Q(|\Delta|)$ equals $|\Delta''|$. Consider the decomposition

$$\begin{array}{ccc} X & \xrightarrow{\quad q \quad} & X' \\ & {\scriptstyle q'}\searrow \qquad \nearrow{\scriptstyle g} & \\ & X''. & \end{array}$$

The universal property of the toric divisorial reduction implies that $g \colon X'' \to X'$ is an isomorphism. Hence $q \colon X \to X'$ is surjective, and the assertion follows from Corollary 6.4. □

Corollary 6.6. *Let X be a divisorial toric variety arising from a fan with convex support. Then every subtorus action on X admits a categorical quotient in the category of divisorial varieties.*

Proof. Let the toric quotient $p \colon X \to X'$ arise from a map $P \colon N \to N'$ of fans Δ and Δ'. By [2, Remark 2.5], each cone $\sigma' \in \Delta'$ is generated by images $P(\sigma)$ of certain $\sigma \in \Delta$. Thus Δ', too, has convex support and $p \colon X \to X'$ is surjective, so Corollaries 6.3 and 6.5 give the claim. □

However, Corollary 6.3 also provides counterexamples to existence of quotients. There can be different reasons for nonsurjectivity of $q \circ p$, as the following examples show.

Example 6.7. For the toric variety X described in Example 3.6, the toric divisorial reduction is not surjective; hence X does not admit a divisorial reduction. Moreover, by Cox's construction (see [11]), X is a good quotient of an open subset $\hat{X} \subset \mathbb{K}^9$ by a 5-dimensional subtorus $H \subset (\mathbb{K}^*)^9$. Thus, the action of H on $\hat{X}$ admits no categorical quotient in the category of divisorial varieties.

Example 6.8. Let Δ be the fan in $\mathbb{Z}^4$ having the following maximal cones:

$$\sigma_1 := \operatorname{cone}((1,0,0,0),(0,1,0,0)), \qquad \sigma_2 := \operatorname{cone}((0,0,1,0),(0,0,0,1)).$$

The associated toric variety X is an open toric subset of $\mathbb{K}^4$. Define a projection $P \colon \mathbb{Z}^4 \to \mathbb{Z}^3$ by

$$\begin{aligned} P((1,0,0,0)) &:= (1,0,0), \quad & P((0,1,0,0)) &:= (0,1,0),\\ P((0,0,1,0)) &:= (0,0,1), \quad & P((0,0,0,1)) &:= (1,1,0). \end{aligned}$$

By [2], the toric morphism $p \colon X \to \mathbb{K}^3$ defined by P is the toric quotient for the action of the subtorus $H := \ker(p)$ on X. Since p is not surjective, the action of H on X has no categorical quotient in the category of divisorial varieties.

7. An Open Problem

In this article we have solved the problem of existence of categorical quotients for subtorus actions on toric varieties in the divisorial category. For the analogous question in the category of all algebraic varieties, we have partial results.

For example, the toric quotient $p\colon X \to X/_{\mathrm{tq}} H$ is a categorical quotient in the category of algebraic varieties if the subtorus H is of codimension at most 2, or if the map p satisfies a certain curve lifting property and $X/_{\mathrm{tq}} H$ is of expected dimension [1; 4].

However, the general question still remains open. Therefore, we pose it here as a problem.

PROBLEM 7.1. Give necessary and sufficient conditions for subtorus actions on toric varieties to admit a categorical quotient in the category of algebraic varieties.

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