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Moduli Spaces of Vector Bundles on Higher-Dimensional Varieties

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1. Introduction

Let X be an n-dimensional, smooth, irreducible, algebraic variety over $\mathbb{C}$ and let L be an ample divisor on X. Let $M_{X,L}(r; c_1, \dots, c_{\min(r,n)})$ denote the moduli space of rank-r, L-stable (in the sense of Mumford and Takemoto) vector bundles E on X with Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$. Moduli spaces for stable vector bundles on smooth, irreducible, algebraic projective varieties were constructed in the 1970s. Many interesting results have been proved regarding these moduli spaces when the underlying variety is a surface, but very little is known if the variety has dimension greater than or equal to three. Until now there have been no general results about these moduli spaces concerning the number of connected components, dimension, smoothness, rationality, topological invariants, and so forth.

A major result in the theory of vector bundles on an algebraic surface S was the proof that, for large c_2, $M_{S,L}(r; c_1, c_2)$ is irreducible, generically smooth, and of the expected dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(O_S)$. For moduli spaces of vector bundles on a higher-dimensional variety, the situation differs drastically. The smoothness and irreducibility turn out to be false when $\dim X \geq 3$. For instance, in [BM, Thm. 0.1], Ballico and Miró-Roig prove that, under certain technical restrictions on c_1, the number of irreducible components of the moduli space $M_{X,L}(2; c_1, c_2)$ of L-stable, rank-2 vector bundles on a smooth projective 3-fold X, with fixed c_1 and c_2L going to infinity, grows to infinity. See [MO] for examples of singular moduli spaces of vector bundles on $\mathbb{P}^{2n+1}$ with $c_2 \gg 0$.

Let $X = \mathbb{P}(\mathcal{E}) \to C$ be a $\mathbb{P}^d$-bundle over a smooth projective curve C of genus $g \geq 0$. The goal of this paper is to compute the dimension, prove the irreducibility and smoothness, and describe the structure of the moduli space $M_{X,L}(2; c_1, c_2)$ for a suitable polarization L closely related to c_2. More precisely, we will cover the study of all moduli spaces $M_{X,L}(2; c_1, c_2)$ such that the general point $[E] \in M_{X,L}(2; c_1, c_2)$ is given as a nontrivial extension of line bundles (Theorems 3.4, 3.5, 3.8, and Remark 3.9). In particular, for rational normal scrolls (i.e., $\mathbb{P}^d$-bundles over $\mathbb{P}^1$) and for a certain choice of c_1, c_2 and L, we have that the moduli space $M_{X,L}(2; c_1, c_2)$ is rational (Corollary 3.6). Therefore, the geometry of the underlying variety and of the moduli spaces are intimately

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related. We hope that phenomena of this sort will be true for other high-dimensional varieties.

Next, we outline the structure of the paper. In Section 2 we recall some basic facts on $\mathbb{P}^d$-bundles over a smooth projective curve of genus $g \geq 0$ that will be needed later on. A crucial result in the proof of our main results is the existence of a section s of a suitable twist of a rank-2 vector bundle E on a $\mathbb{P}^d$-bundle, $X = \mathbb{P}(\mathcal{E}) \to C$, whose zero scheme, $(s)_0$, has codimension ≥ 2 (Proposition 2.6). Section 3 contains our main results on moduli spaces: the irreducibility, smoothness, and structure of the moduli space $M_{X,L}(2; c_1, c_2)$ of L-stable, rank-2 vector bundles on a $\mathbb{P}^d$-bundle $X = \mathbb{P}(\mathcal{E}) \to C$ with certain Chern classes and a suitable polarization L. Our approach is to write L-stable, rank-2 vector bundles E on X as an extension of two line bundles. A well-known result for vector bundles over curves is that any vector bundle of rank $r \geq 2$ can be written as an extension of lower-rank vector bundles. For higher-dimensional varieties we may not be able to attain such a nice result (e.g., it is not true for $X = \mathbb{P}^n$ with $n \geq 2$). However, it turns out to be true for certain L-stable, rank-2 vector bundles E on $\mathbb{P}^d$-bundles X. In Section 4, we illustrate by means of an example the changes of the moduli space $M_L(2; c_1, c_2)$ that occur when the polarization L varies (Theorem 4.4).

2. Generalities

Throughout this paper, we fix a smooth, irreducible, projective curve C of genus $g \geq 0$ and canonical divisor K_C. For any $e \in \mathbb{Z}$, we use $\mathfrak{e}$ and $\mathfrak{e}'$ to denote divisors on C of degree e. Let $\mathcal{E}$ be a rank-$(d+1)$ vector bundle on C and consider

$$X = \mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}\mathcal{E}) \xrightarrow{\pi} C,$$

the projectivized vector bundle associated to $\mathcal{E}$. The projective bundle X is a $(d+1)$-dimensional variety called a $\mathbb{P}^d$-bundle over C. Two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ on C define the same $\mathbb{P}^d$-bundle if and only if there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$. When $d = 1$, we simply say that X is a ruled surface.

Let $\mathcal{H} := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle; for any point $p \in C$ we write $\mathcal{F}_p := \pi^*\mathcal{O}_C(p)$ and $F_p := \pi^* p$. Let H (resp. F) be the numerical equivalence class associated to the tautological line bundle $\mathcal{H}$ (resp. $\mathcal{F}_p$) on X. We have

$$\operatorname{Pic}(X) \cong \mathbb{Z}H \oplus \pi^* \operatorname{Pic}(C), \qquad \operatorname{Num}(X) \cong \mathbb{Z}^2 \cong \mathbb{Z}H \oplus \mathbb{Z}F;$$
$$H^{d+1} = \deg(\mathcal{E}), \quad H^d F = 1, \quad F^2 = 0.$$

The canonical divisor of X is $K_X \sim -(d+1)H + \pi^*(\det(\mathcal{E}) + K_C)$.

Moreover, if $D \sim aH + \pi^*\mathfrak{b}$ with $a \in \mathbb{Z}$, then $D \equiv aH + bF$; if, in addition, $a \geq 0$, then $\pi_* D = S^a(\mathcal{E}) \otimes \mathcal{O}_C(\mathfrak{b})$, where $S^a(\mathcal{E})$ is the ath symmetric power of $\mathcal{E}$.

EXAMPLE 2.1. Let $\mathcal{E} = \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$ be a rank-$(d+1)$ vector bundle on $\mathbb{P}^1$ and assume that $0 = a_0 \leq a_1 \leq \cdots \leq a_d$ with $a_d > 0$. The line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is generated by its global sections and defines a birational map

$$Y(a_0, \dots, a_d) := \mathbb{P}(\mathcal{E}) \xrightarrow{f} \mathbb{P}^N$$

with $N := d + \sum_{i=0}^{d} a_i$. The image of this map is a variety of dimension $d+1$ and minimal degree $c = \sum_{i=0}^{d} a_i$; it is called a *rational normal scroll*. Sometimes Y is also called a rational normal scroll. In case $d = 1$, we get the so-called Hirzebruch surfaces.

REMARK 2.2. Given $X = \mathbb{P}(\mathcal{E})$ a $\mathbb{P}^d$-bundle over C, we write $\gamma = \gamma(\mathcal{E}) := \max\{-\mu^-(\mathcal{E}) + 1, 1\}$. By [Miy, Thm. 3.1], the divisor $L \equiv H + \gamma F$ is ample. Hence, the following inequality holds for any effective divisor $D \equiv nH + mF$:

$$0 \le (nH + mF)(H + \gamma F)^d = nH^{d+1} + nd\gamma + m.$$

LEMMA 2.3. *For any* $\mathfrak{b} \in \operatorname{Pic}(C)$, *we have*

$$\begin{aligned} &H^i(X, O_X(aH + \pi^*\mathfrak{b})) \\ &\quad = \begin{cases} 0 & \text{if } -d-1 < a < 0, \\ H^i(C, S^a(\mathcal{E}) \otimes O_C(\mathfrak{b})) & \text{if } a \ge 0, \\ H^{d+1-i}(C, S^{-d-1-a}(\mathcal{E}) \otimes O_C(\tilde{\mathfrak{b}})) & \text{if } a \le -d-1, \end{cases} \end{aligned}$$

where $\tilde{\mathfrak{b}} := -\mathfrak{b} + \det(\mathcal{E}) + K_C$.

LEMMA 2.4. *Let* $X = \mathbb{P}(\mathcal{E})$ *be a* $\mathbb{P}^d$*-bundle over* C *and let* $\mathcal{L}$ *be the irreducible family of codimension-*2 *closed subschemes* Z *of* X *that are complete intersections of type* (H, F_p). *Then* $\dim \mathcal{L} = h^0\mathcal{E} + h^0 O_C(p) - h^0\mathcal{E}(-p) - 2$. *Moreover, if* $\mathcal{E}$ *is normalized* (*i.e., if* $h^0\mathcal{E} \neq 0$ *but* $h^0\mathcal{E}(L) = 0$ *for all* $L \in \operatorname{Pic}(C)$ *with* $\deg(L) <$ 0), *then* $\dim \mathcal{L} = h^0\mathcal{E} + h^0 O_C(p) - 2$.

Proof. From the exact sequence

$$0 \to O_X(-H - F_p) \to O_X(-H) \oplus O_X(-F_p) \to I_Z \to 0, \tag{1}$$

we deduce

$$\begin{aligned} \dim \mathcal{L} = {} & \dim \operatorname{Hom}(O_X(-H - F_p), O_X(-H) \oplus O_X(-F_p)) \\ & - \dim \operatorname{Aut}(O_X(-H - F_p)) \\ & - \dim \operatorname{Aut}(O_X(-H) \oplus O_X(-F_p)) + \dim I_f, \end{aligned}$$

where $f \in \operatorname{Hom}(O_X(-H - F_p), O_X(-H) \oplus O_X(-F_p))$ is a general element and I_f denotes its isotropy group under the action of

$$\operatorname{Aut}(O_X(-H - F_p)) \times \operatorname{Aut}(O_X(-H) \oplus O_X(-F_p)).$$

From Lemma 2.3 we obtain

$$\begin{aligned} \dim \operatorname{Aut}(O_X(-H) \oplus O_X(-F_p)) &= 2h^0 O_X + h^0 O_X(H - F_p) \\ &= 2 + h^0\mathcal{E}(-p), \\ \dim \operatorname{Aut}(O_X(-H - F_p)) &= h^0 O_X = 1, \\ \dim \operatorname{Hom}(O_X(-H - F_p), O_X(-H) \oplus O_X(-F_p)) &= h^0 O_X(H) + h^0 O_X(F_p). \end{aligned}$$

Finally, since $h^0 O_X(H) = h^0 \mathcal{E}$, $h^0 O_X(F_p) = h^0 O_C(p)$, and $\dim I_f = 1$, it follows that $\dim \mathcal{L} = h^0\mathcal{E} + h^0 O_C(p) - h^0\mathcal{E}(-p) - 2$. If $\mathcal{E}$ is normalized then we have $H^0\mathcal{E}(-p) = 0$ and hence $\dim \mathcal{L} = h^0\mathcal{E} + h^0 O_C(p) - 2$. □

We end this section with two results that will be very useful in the sequel.

LEMMA 2.5. *Let E be a rank-2 vector bundle on $\mathbb{P}^d$. If $c_2(E) \le 0$, then $H^0(\mathbb{P}^d, E) \neq 0$.*

Proof. Since $c_2(E) \le 0$, we have $c_1^2(E) - 4c_2(E) \ge 0$. Schwarzenberger's inequality ($c_1^2 - 4c_2 < 0$ for stable rank-2 vector bundles on $\mathbb{P}^2$) together with Barth's theorem (which states that the restriction of a stable rank-2 vector bundle on $\mathbb{P}^d$ to a general hyperplane is again stable, with the exception of the null-correlation bundle on $\mathbb{P}^3$) implies that E is not stable.

Since $E^* \cong E(-c_1)$, $c_1(E^*) = -c_1(E)$, and $c_2(E^*) = c_2(E)$, we may assume that $c_1(E) \le 0$ and $c_2(E) \le 0$. Let n be the least integer such that $H^0(\mathbb{P}^d, E(n)) \neq 0$. We have to show that $n \le 0$. Take $0 \neq s \in H^0E(n)$. Then the scheme of zeros of s represents the second Chern class of $E(n)$. Hence, $0 \le c_2(E(n)) = c_2(E) + nc_1(E) + n^2$. Since $c_2(E) \le 0$ it follows that $n(c_1(E) + n) \ge 0$. If $n > 0$, then $c_1(E) + 2n > c_1(E) + n \ge 0$. Let us see that, in such case, E is stable. Toward this end, we take a rank-1 subbundle $O_{\mathbb{P}^d}(r)$ of E. Since $h^0E(-r) \neq 0$ we have $n \le -r$. Therefore, $2r \le -2n < c_1(E)$ and E is stable, which is a contradiction. Therefore, $n \le 0$ and the lemma follows. □

Let E be a rank-2 vector bundle on a $\mathbb{P}^d$-bundle X. Since $H^2(X, \mathbb{Z})$ is generated by H and F and since $H^4(X, \mathbb{Z})$ is generated by H^2 and HF, one may write $c_1(E) \equiv aH + bF$ and $c_2(E) \equiv xH^2 + yHF$ for $a, b, x, y \in \mathbb{Z}$. We may assume without loss of generality that $c_1(E)$ is numerically equivalent to one of the following classes: H, $H + F$, F, or 0.

The following proposition is the key point for proving our main results on moduli spaces of vector bundles on $\mathbb{P}^d$-bundles. It assures us the existence of sections vanishing in codimension ≥ 2, sections that allow us to prove the irreducibility and smoothness of the moduli spaces we deal with.

PROPOSITION 2.6. *Let X be a $\mathbb{P}^d$-bundle over C, $c_2 \in \mathbb{Z}$, $L \equiv dH + bF$ an ample divisor on X, $e \in \{0, 1\}$, and E a rank-2, L-stable vector bundle on X. Assume that either*:

(i) $c_1E \equiv H + eF$, $c_2E \equiv (c_2 + e)HF$, $b = 2c_2 - H^{d+1} + e - 1$, *and* $c_2 > (d\gamma + H^{d+1})/2 + 1$; *or*

(ii) $c_1E \equiv eF$, $c_2E \equiv -H^2 + (2c_2 + e)HF$, $b = c_2 - H^{d+1} + e - 1$, *and* $c_2 > d\gamma + H^{d+1} + 2$.

Then $E(-H + \pi^*\mathfrak{c}_2)$ has a nonzero section whose scheme of zeros has codimension ≥ 2.

Proof. We prove case (i) and then leave the other case to the reader. By [Miy; Thm. 3.1], $L \equiv dH + bF$ with $b = 2c_2 - H^{d+1} + e - 1$, and $2c_2 > d\gamma + H^{d+1} + 2$

is an ample divisor on X. For any L-stable rank-2 vector bundle E on X with $c_1E \equiv H + eF$ and $c_2E \equiv (c_2 + e)HF$, we consider $\bar{E} := E(-H + \pi^*\mathfrak{c}_2)$. We have $c_1(\bar{E}) \equiv -H + (2c_2 + e)F$ and $c_2(\bar{E}) = 0$. Since $c_2(\bar{E}) = 0$ and $F \cong \mathbb{P}^d$, from Lemma 2.5 we deduce that $h^0(F, \bar{E}|_F) \neq 0$. Hence, there exists an integer $a \geq 0$ such that $O_{\mathbb{P}^d}(a) \hookrightarrow \bar{E}|_{\mathbb{P}^d}$. This injection induces an injection $O_X(aH + \pi^*\mathfrak{b}') \hookrightarrow \bar{E}$ for some divisor $\mathfrak{b}'$ on C. Take $0 \neq s \in H^0\bar{E}(-aH - \pi^*\mathfrak{b}')$ and let Y be its scheme of zeros. Let A be the maximal effective divisor contained in Y. Then s can be regarded as a section of $\bar{E}(-aH - \pi^*\mathfrak{b}' - A)$, and its scheme of zeros has codimension ≥ 2. If $l'H + \pi^*\mathfrak{m}' \equiv aH + \pi^*\mathfrak{b}' + A$ with $l' \geq 0$, then $\bar{E}(-l'H - \pi^*\mathfrak{m}')$ with $l' \geq 0$ has a nonzero section whose scheme of zeros has codimension ≥ 2. Therefore, $E(-lH - \pi^*\mathfrak{m})$ with $l > 0$ has a nonzero section whose scheme of zeros has codimension ≥ 2. To end the proof of (i) we need only show that $l = 1$ and $m = -c_2$.

Since E is L-stable and $O_X(lH + \pi^*\mathfrak{m}) \hookrightarrow E$, we have

$$(lH + mF)L^d = d^d(lH^{d+1} + lb + m) < \frac{c_1(E)L^d}{2} = \frac{d^d(H^{d+1} + b + e)}{2},$$

which is equivalent to $2m < -2(2l - 1)c_2 - (2l - 1)(e - 1) + e$. On the other hand, since $E(-lH - \pi^*\mathfrak{m})$ has a nonzero section whose scheme of zeros has codimension ≥ 2, we obtain

$$\begin{aligned} 0 &\leq c_2(E(-lH - \pi^*\mathfrak{m}))H^{d-1} \\ &= ((c_2 + e + 2lm - m - el)HF + l(l-1)H^2)H^{d-1} \\ &= c_2 + e(1 - l) + (2l - 1)m + l(l - 1)H^{d+1}. \end{aligned}$$

Therefore, $m(2l - 1) \geq -l(l - 1)H^{d+1} - c_2 + e(l - 1)$. By hypothesis, $2c_2 > d\gamma + H^{d+1} + 2$. We thus have

$$\begin{aligned} &\frac{-2l(l-1)c_2}{2l-1} - \frac{c_2}{2l-1} + \frac{l(l-1)d\gamma}{2l-1} + \frac{2l(l-1)}{2l-1} + \frac{e(l-1)}{2l-1} \\ &\quad \leq m \\ &\quad < -(2l-1)c_2 - \frac{(2l-1)(e-1)}{2} + \frac{e}{2}, \end{aligned} \tag{2}$$

which implies that $l^2(2c_2 + d\gamma + 2e) - l(2c_2 + d\gamma + 2e) - \frac{1}{2} < 0$ with $l \geq 1$. Hence $l = 1$, and using (2) again we obtain $m = -c_2$, which proves (i). □

3. Moduli Spaces of Vector Bundles on $\mathbb{P}^d$-Bundles

We will denote by $M_{X,L}(2; c_1, c_2)$ the moduli space of L-stable, rank-2 vector bundles on X with Chern classes c_1 and c_2. If there is no possible confusion then we will write $M_L(2; c_1, c_2)$ instead of $M_{X,L}(2; c_1, c_2)$. The goal of this section is to compute the dimension, prove the irreducibility and smoothness, and describe the structure of moduli spaces $M_L(2; c_1, c_2)$ of L-stable, rank-2 vector bundles with certain Chern classes and for a suitable polarization L closely related to c_2. We want to stress that the polarization L that we choose depends strongly on c_2; our results turn out to be false if we fix c_1 and L and if c_2L^{d-1} goes to infinity. Indeed, for $d = 2$ and fixed L, the minimal number of irreducible components of the moduli space $M_L(2; c_1, c_2)$ of L-stable, rank-2 vector bundles with fixed c_1 and c_2L going to infinity grows to infinity [BM, Thm. 0.1].

One way to study rank-2 vector bundles over an algebraic variety X is to use extensions of line bundles. Using this idea, we construct the following families.

CONSTRUCTION 3.1. For $c_1 \equiv H + eF$ with

$$e \in \{0,1\} \quad \text{and} \quad \mathbb{Z} \ni c_2 > (H^{d+1} + d\gamma)/2 + 1,$$

we construct a rank-2 vector bundle E on X as a nontrivial extension

$$\varepsilon\colon 0 \to O_X(H - \pi^*\mathfrak{c}_2') \to E \to O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \to 0, \tag{3}$$

where $\mathfrak{c}_2, \mathfrak{c}_2' \in \operatorname{Pic}(C)$ are divisors on C of degree c_2 and $\mathfrak{e} \in \operatorname{Pic}(C)$ is a divisor of degree e. We shall call $\mathcal{F}$ the *irreducible family* of rank-2 vector bundles constructed in this way.

PROPOSITION 3.2. *Let X be a $\mathbb{P}^d$-bundle over C, let $\mathbb{Z} \ni c_2 > (H^{d+1}+d\gamma)/2+1$, and let $L \equiv dH + bF$ be an ample divisor on X with $b = 2c_2 - H^{d+1} - (1-e)$, $e \in \{0,1\}$. For any $E \in \mathcal{F}$, we have the following.*

(a) $H^0E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$.

(b) *E is a rank-2, L-stable vector bundle with $c_1(E) \equiv H + eF$ and $c_2(E) \equiv (c_2 + e)HF$.*

(c) *$\mathcal{F}$ is a $\mathbb{P}^N$-bundle over* $\operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$, *where*

$$N = \dim \operatorname{Ext}^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2')) - 1.$$

In particular, $\dim \mathcal{F} = h^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1$.

Proof. Observe first of all that, since $b = 2c_2 - H^{d+1} - (1-e) > d\gamma$, it follows (by [Miy, Thm. 3.1]) that L is an ample divisor on X.

(a) We start proving that $H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = 0$. By [Miy, Thm. 3.1], $\bar{L} \equiv H + \gamma F$ is an ample divisor. If $H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) \neq 0$, applying Remark 2.2 we get $0 \le (H - (2c_2 + e)F)(H + \gamma F)^d = H^{d+1} + d\gamma - 2c_2 - e$, which contradicts the assumption $2c_2 > d\gamma + H^{d+1} + 2$. Therefore, $H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = 0$.

We consider the exact cohomology sequence associated to (3). Since

$$H^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = \operatorname{Ext}^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2')),$$

the map $\delta\colon H^0O_X \to H^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e})$ given by $\delta(1) = \varepsilon$ is an injection. This fact, together with $H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = 0$, gives us $H^0E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$, which proves (a).

(b) It is easy to see that $c_1(E) \equiv H + eF$ and $c_2(E) \equiv (c_2 + e)HF$ for any $E \in \mathcal{F}$. Let us see that E is L-stable; that is, for any rank-1 subbundle $O_X(D)$ of $E \in \mathcal{F}$, we obtain $DL^d < (c_1(E)L^d)/2$. For any subbundle $O_X(D)$ of E we have

(i) $O_X(D) \hookrightarrow O_X(H - \pi^*\mathfrak{c}_2')$ or

(ii) $O_X(D) \hookrightarrow O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})$.

In the first case, $D \equiv H - c_2 F - C$, with C numerically equivalent to an effective divisor. Hence,

$$\begin{aligned} DL^d = (H - c_2 F - C)L^d \leq (H - c_2 F)L^d &= d^d(H^{d+1} + b - c_2) \\ &< \frac{d^d(H^{d+1}+b+e)}{2} = \frac{c_1(E)L^d}{2}. \end{aligned}$$

Assume $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})$. From (a) we have $H^0E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$. Therefore, $D \equiv (c_2+e)F - C'$, with $C' \equiv nH + mF$ numerically equivalent to a nonzero effective divisor. Consequently,

$$\begin{aligned} DL^d = ((c_2+e)F - C')L^d &= ((c_2+e)F - nH - mF)L^d \\ &= d^d(c_2 + e - 2nc_2 + n(1-e) - m) \\ &< \frac{c_1(E)L^d}{2} = \frac{d^d(2c_2+2e-1)}{2} \end{aligned}$$

if and only if $-4nc_2 + 2n(1-e) - 2m < -1$. Since C' is numerically equivalent to a nonzero effective divisor, we have $-m \leq n(H^{d+1} + d\gamma)$ and $n > 0$ or $n = 0$ and $m > 0$. By hypothesis, $c_2 > (H^{d+1} + d\gamma)/2 + 1$; therefore, $-4nc_2 + 2n(1-e) - 2m < -1$ and E is L-stable.

(c) Let p_1 and p_2 be the projections of $X \times \operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$ to X and $\operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$, respectively. We define $G_1 := p_1^*\mathcal{O}_X(H - \pi^*\mathfrak{c}_2')$ and $G_2 := p_1^*\mathcal{O}_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})$. Set $\mathcal{H} := \operatorname{Ext}^1_{p_2}(G_2, G_1)$, where $\operatorname{Ext}^1_{p_2}(G_2, \cdot)$ is the right derived functor of $\operatorname{Hom}_{p_2}(G_2, \cdot) := p_{2_*}\mathcal{H}om(G_2, \cdot)$. Note that $\mathcal{H}$ is a locally free sheaf over $\operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$ of rank $h^1\mathcal{O}_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e})$ and is compatible with arbitrary base change. Consider the projective bundle $\gamma : \mathbb{P}(\mathcal{H}) \to \operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$ and the morphism $p := \gamma \times \mathrm{id}_X : \mathbb{P}(\mathcal{H}) \times X \to \operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C) \times X$. Over $\mathbb{P}(\mathcal{H}) \times X$ there is a tautological extension

$$0 \to p^*(G_1) \to \mathcal{V} \to p^*(G_2) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{H})}(-1) \to 0$$

such that, for each $t \in \mathbb{P}(\mathcal{H})$, the restriction to $\{t\} \times X$ is isomorphic to the extension corresponding to t. That is,

$$0 \to \mathcal{O}_X(H - \pi^*\mathfrak{c}_2') \to E \to \mathcal{O}_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \to 0$$

and hence there is a natural bijective morphism $\mathbb{P}(\mathcal{H}) \to \mathcal{F}$. Thus, $\mathcal{F}$ is a $\mathbb{P}^N$-bundle over $\operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$, where $N = \dim \operatorname{Ext}^1(\mathcal{O}_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), \mathcal{O}_X(H - \pi^*\mathfrak{c}_2')) - 1$ and

$$\begin{aligned} \dim \mathcal{F} &= \dim \operatorname{Ext}^1(\mathcal{O}_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), \mathcal{O}_X(H - \pi^*\mathfrak{c}_2')) + 2\dim \operatorname{Pic}^0(C) - 1 \\ &= h^1\mathcal{O}_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1. \qquad \square \end{aligned}$$

REMARK 3.3. The existence of large families of indecomposable rank-2 vector bundles over $\mathbb{P}^d$-bundles of arbitrary dimension faces up to Hartshorne's conjecture [H] on the nonexistence of indecomposable rank-2 vector bundles on projective spaces $\mathbb{P}^n$, $n \geq 6$.

THEOREM 3.4. *Given the assumptions of Proposition 3.2, the moduli space* $M_L(2; H + eF, (c_2+e)HF)$ *is a* $\mathbb{P}^N$*-bundle over* $\operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C)$ *with* $N := h^1\mathcal{O}_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) - 1$.

Proof. Using Proposition 3.2 and the universal property of $M_L(2; H + eF, (c_2 + e)HF)$, we obtain a morphism

$$\phi\colon \mathcal{F} \to M_L(2; H + eF, (c_2 + e)HF),$$

which is a bijective map. In fact, we first prove that ϕ is injective. Assume that there are two nontrivial extensions:

$$0 \to O_X(H - \pi^*\mathfrak{c}_2') \xrightarrow{\alpha_1} E \xrightarrow{\beta_1} O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \to 0;$$
$$0 \to O_X(H - \pi^*\bar{\mathfrak{c}}_2') \xrightarrow{\alpha_2} E \xrightarrow{\beta_2} O_X(\pi^*\bar{\mathfrak{c}}_2 + \pi^*\bar{\mathfrak{e}})) \to 0.$$

Since

$$\begin{aligned}\operatorname{Hom}(O_X(H - \pi^*\mathfrak{c}_2'), O_X(\pi^*\bar{\mathfrak{c}}_2 + \pi^*\bar{\mathfrak{e}}))\\ = \operatorname{Hom}(O_X(H - \pi^*\bar{\mathfrak{c}}_2'), O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})) = 0\end{aligned}$$

(Lemma 2.3), we have $\beta_2 \circ \alpha_1 = \beta_1 \circ \alpha_2 = 0$. Consequently, there must exist a $\lambda \in \operatorname{Aut}(O_X(H - \pi^*\mathfrak{c}_2')) \cong \mathbb{C}$ such that $\alpha_2 = \alpha_1 \circ \lambda$. Therefore, ϕ is an injection.

Let us see that ϕ is surjective. Take $E \in M_L(2; H + eF, (c_2 + e)HF)$. By Proposition 2.6, $E(-H + \pi^*\mathfrak{c}_2')$ has a nonzero section s whose scheme of zeros has codimension ≥ 2. Since $c_2 E(-H + \pi^*\mathfrak{c}_2') = 0$, the section s defines an exact sequence

$$0 \to O_X(H - \pi^*\mathfrak{c}_2') \to E \to O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \to 0$$

of type (3). Therefore, ϕ is surjective and it follows that ϕ is bijective.

CLAIM. *For any $E \in M_L(2; H + eF, (c_2 + e)HF)$ we have*

$$\begin{aligned}\dim T_{[E]} M_L(2; H + eF, (c_2 + e)HF)\\ = h^1 O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1.\end{aligned}$$

Proof. By deformation theory, $T_{[E]} M_L(2; H + eF, (c_2 + e)HF) \cong \operatorname{Ext}^1(E, E)$. Let us compute $\dim \operatorname{Ext}^1(E, E)$. We have already seen that any $E \in M_L(2; H+eF, (c_2 + e)HF)$ sits in an extension of type (3). Applying $\operatorname{Hom}(\cdot, E)$ to the exact sequence (3) yields

$$\begin{aligned}0 &\to \operatorname{Hom}(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \to \operatorname{Hom}(E, E)\\ &\to \operatorname{Hom}(O_X(H - \pi^*\mathfrak{c}_2'), E) \to \operatorname{Ext}^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E)\\ &\to \operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1(O_X(H - \pi^*\mathfrak{c}_2'), E)\\ &\to \operatorname{Ext}^2(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \to \cdots.\end{aligned} \tag{4}$$

Since $h^1 O_X = g$, $H^2 O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = H^2 O_X = 0$, and also $H^0 E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$ (Proposition 3.2(a)), we have

$$\begin{aligned}h^1 E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = h^1 O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + g - 1,\\ \operatorname{Ext}^2(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) = H^2 E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0.\end{aligned} \tag{5}$$

Consider the exact cohomology sequence associated to the exact sequence (3). Since $H^0 O_X(-H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) = H^1 O_X(-H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) = 0$ and since $h^1 O_X = g$, we have $h^0 E(-H+\pi^*\mathfrak{c}_2') = 1$ and $h^1 E(-H+\pi^*\mathfrak{c}_2') = g$. Therefore, from the exact sequence (4) we obtain

$$\begin{aligned}\dim \operatorname{Ext}^1(E,E) &= h^1 E(-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) + h^1 E(-H+\pi^*\mathfrak{c}_2') \\ &\quad - h^0 E(-H+\pi^*\mathfrak{c}_2') + \dim \operatorname{Hom}(E,E) \\ &= h^1 O_X(H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) + 2g - 1,\end{aligned}$$

where the second equality follows from (5) and the fact that E is L-stable (and thus simple), proving our Claim. □

Since ϕ is a bijective map and $\dim \mathcal{F} = h^1 O_X(H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) + 2g-1$ (Proposition 3.2), it follows from the Claim that the moduli space $M_L(2; H+eF, (c_2+e)HF)$ is smooth. Finally, we may now deduce from Proposition 3.2(c) that $M_L(2; H+eF, (c_2+e)HF)$ is a $\mathbb{P}^N$-bundle over $\operatorname{Pic}^0(C)\times\operatorname{Pic}^0(C)$, where $N := h^1 O_X(H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) - 1$. In particular, $M_L(2; H+eF, (c_2+e)HF)$ is a nonempty, smooth, irreducible, and projective variety whose dimension is $h^1 O_X(H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) + 2g-1$. Thus we have proved Theorem 3.4. □

THEOREM 3.5. *Let X be a $\mathbb{P}^d$-bundle over C with $d>1$ and $\mathbb{Z}\ni c_2 > H^{d+1} + d\gamma + 2$. We fix the ample divisor $L \equiv dH + bF$ on X, $b = c_2 - H^{d+1} - (1-e)$, and $e \in \{0,1\}$. Then $M_L(2; eF, -H^2 + (2c_2+e)HF)$ is a $\mathbb{P}^M$-bundle over $\operatorname{Pic}^0(C)\times\operatorname{Pic}^0(C)$ with $M := h^1 O_X(2H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{e}) - 1$.*

Proof. We consider the irreducible family $\mathcal{G}$ of rank-2 vector bundles E on X given by a nontrivial extension,

$$\varepsilon\colon 0 \to O_X(H-\pi^*\mathfrak{c}_2') \to E \to O_X(-H+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) \to 0. \tag{6}$$

Arguing as in Theorem 3.4, we prove that $M_L(2; eF, -H^2+(2c_2+e)HF)$ is a $\mathbb{P}^M$-bundle over $\operatorname{Pic}^0(C)\times\operatorname{Pic}^0(C)$ with $M = h^1 O_X(2H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) - 1$. □

As a corollary, we obtain the rationality of the following moduli spaces of stable vector bundles on rational normal scrolls.

COROLLARY 3.6. *Let $Y := Y(a_0,\dots,a_d)$ be a $(d+1)$-dimensional, rational, normal scroll. Let $L = dH + bF$ be an ample divisor, $c_2\in\mathbb{Z}$, and $e\in\{0,1\}$.*

(a) *If $d>0$, $c_2 > (H^{d+1}+d)/2 + 1$, and $b = 2c_2 - H^{d+1} - (1-e)$, then $M_L(2; H+eF, (c_2+e)HF) \cong \mathbb{P}^N$ with*

$$N = 2(d+1)c_2 - H^{d+1} + e(d+1) - (d+2).$$

(b) *If $d>1$, $c_2 > H^{d+1} + d + 2$, and $b = c_2 - H^{d+1} - (1-e)$, then $M_L(2; eF, -H^2+(2c_2+e)HF) \cong \mathbb{P}^M$ with*

$$M = 2(ed+1)c_2 - e(d+2)H^{d+1} + 2(e-1).$$

REMARK 3.7. Using Bogomolov's inequality $(c_1(E)^2 - 4c_2(E))H^{d-1} < 0$, one can see that the hypothesis $2c_2 > H^{d+1} + d\gamma + 2$ (resp., $c_2 > H^{d+1} + d\gamma + 2$) when $c_1(E) \equiv H + eF$ (resp., $c_1(E) \equiv eF$) with $e \in \{0, 1\}$ is not too restrictive.

In the following theorem we generalize Theorems 3.4 and 3.5 to other classes of c_2. Since the proof is essentially the same, we will omit it.

THEOREM 3.8. *Let X be a $\mathbb{P}^d$-bundle over C with $d > 1$ and integers b, a, e, with $e \in \{0, 1\}$. Fix an ample divisor $L \equiv \alpha H + \beta F$. Assume*

$$0 > -2a > -d - 1, \quad \alpha = ad, \quad \beta = -b - aH^{d+1} + e - 1,$$

and $-ab > a^2 H^{d+1} + a^2 d\gamma + a(a + 2)$ (*respectively,*

$$0 > 1 - 2a > -d - 1, \quad \alpha = (2a - 1)d, \quad \beta = -2b - (2a - 1)H^{d+1} + e - 1,$$

and $-2ab > (2a - 1)aH^{d+1} + (2a - 1)ad\gamma + a(2a - 1) + 1$.)

Then $M_L(2; eF, -a^2H^2 + (ae - 2ab)HF)$ (*resp.,* $M_L(2; H + eF, a(1-a)H^2 + (b + ae - 2ab)HF)$) *is a* $\mathbb{P}^N$*-bundle* (*resp.,* $\mathbb{P}^M$*-bundle*) *over* $\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$, *where* $N := h^1 O_X(2aH + \pi^*(\mathfrak{b} + \mathfrak{b}' - \mathfrak{e})) - 1$ (*resp.,* $M := h^1 O_X((2a - 1)H + \pi^*(\mathfrak{b} + \mathfrak{e})) - 1$).

REMARK 3.9. We want to stress that with Theorem 3.8 we have covered the study of all moduli spaces $M_L(2; c_1, c_2)$ such that the general point $[E]$ of $M_L(2; c_1, c_2)$ is given as a nontrivial extension of line bundles. Indeed, the Chern classes of vector bundles E studied in Theorem 3.8 are the only ones that can be obtained as Chern classes of a vector bundle E constructed as a nontrivial extension of line bundles.

We will finish this section by computing the Kodaira dimension and the Picard group of moduli spaces studied previously.

COROLLARY 3.10. *Under the assumptions of Theorems 3.4 and 3.5, we have*:

$$\mathrm{Kod}(M_L(2; H + eF, (c_2 + e)HF)) = -\infty;$$
$$\mathrm{Kod}(M_L(2; eF, -H^2 + (2c_2 + e)HF)) = -\infty.$$

That $M_L(2; H + eF, (c_2 + e)HF)$ (resp., $M_L(2; eF, -H^2 + (2c_2 + e)HF)$) is a $\mathbb{P}^N$-bundle (resp., $\mathbb{P}^M$-bundle) over $\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ with natural projection Π (resp., Π') allows us to prove the following corollary.

COROLLARY 3.11. *Under the assumptions of Theorems 3.4 and 3.5, we have*:

$$\mathrm{Pic}(M_L(2; H + eF, (c_2 + e)HF)) \cong \mathbb{Z} \oplus \Pi^* \mathrm{Pic}(\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C));$$
$$\mathrm{Pic}(M_L(2; eF, -H^2 + (2c_2 + e)HF)) \cong \mathbb{Z} \oplus \Pi'^* \mathrm{Pic}(\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)).$$

In particular, if X is a rational normal scroll then

$$\mathrm{Pic}(M_L(2; H + eF, (c_2 + e)HF)) \cong \mathrm{Pic}(M_L(2; eF, -H^2 + (2c_2 + e)HF)) \cong \mathbb{Z}.$$

4. Change of Polarizations

Let X be a smooth, irreducible, projective variety of dimension n. In [Q], Qin considered the following problem: What is the difference between $M_{X,L_1}(2; c_1, c_2)$ and $M_{X,L_2}(2; c_1, c_2)$ where L_1 and L_2 are two different polarizations? In Section 3 we studied the moduli space $M_L(2; c_1, c_2)$ for fixed c_1, c_2 and a suitable polarization L on a $\mathbb{P}^d$-bundle X over a smooth projective curve. In this section, we will illustrate some of the changes of the moduli space $M_L(2; c_1, c_2)$ that occur when the polarization L varies.

We keep the notation introduced in Sections 2 and 3. For technical reasons, we also assume that $\mathcal{E}$ is normalized. Hence, for any integer $n \gg 0$ we have

(i) $\deg(O_C(-\mathfrak{n} - \mathfrak{n}' + p + p' + K_C)) < 0$,
(ii) $h^0\mathcal{E} + h^0 O_C(p'') \leq h^1\mathcal{E}(-\mathfrak{n} - \mathfrak{n}')$,
(iii) $n > (H^{d+1} + d\gamma)/2 + 3$,

where $\mathfrak{n}, \mathfrak{n}' \in \mathrm{Pic}(C)$ are divisors on C of degree n and where $p, p', p'' \in C$ are points of C.

We start by recalling some results (due to Qin [Q]) about walls and chambers.

DEFINITION 4.1. (i) Let $S \in A^{d-1}_{\mathrm{num}}(X)$ and $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ with $d+1 = \dim X$. We define $W^{(\xi,S)} := C_X \cap \{x \in \mathrm{Num}(X) \otimes \mathbb{R} \mid x\xi S = 0\}$.

(ii) Define $\mathcal{W}(c_1, c_2)$ to be the set whose elements consist of $W^{(\xi,S)}$, where S is a complete intersection surface in X and where ξ is the numerical equivalence class of a divisor G on X such that $G + c_1$ is divisible by 2 in $\mathrm{Pic}(X)$, $G^2 S < 0$, and $c_2 + (G^2 - c_1^2)/4 = [Z]$ for some locally complete intersection codimension-2 cycle Z in X.

(iii) A *wall of type* (c_1, c_2) is an element in $\mathcal{W}(c_1, c_2)$. A *chamber of type* (c_1, c_2) is a connected component of $C_X \setminus \mathcal{W}(c_1, c_2)$. A $\mathbb{Z}$*-chamber of type* (c_1, c_2) is the intersection of $\mathrm{Num}(X)$ with some chamber of type (c_1, c_2).

We say that a wall $W^{(\xi,S)}$ of type (c_1, c_2) separates two polarizations L and L' if and only if $\xi SL < 0 < \xi SL'$.

REMARK 4.2. In [Q, Cor. 2.2.2], Qin proves that $M_{X,L}(2; c_1, c_2)$ depends only on the chamber of L and that the study of moduli spaces of rank-2 vector bundles stable with respect to a polarization lying on walls may be reduced to the study of moduli spaces of rank-2 vector bundles stable with respect to a polarization lying in $\mathbb{Z}$-chambers.

EXAMPLE 4.3. Let X be a $\mathbb{P}^d$-bundle over C. We fix $c_1 \equiv H \in \mathrm{Num}(X)$ and, for any $\mathbb{Z} \ni n > (H^{d+1} + d\gamma)/2 + 3$, $c_2 \equiv nHF \in H^4(X, \mathbb{Z})$. We consider $S, S' \in A^{d-1}_{\mathrm{num}}$ and $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defined by

$$S := dH^{d-1} + \beta H^{d-2}F, \quad S' := dH^{d-1} + (\beta - 2)H^{d-2}F, \quad \xi := H - 2nF,$$

with $\beta = (2n - H^{d+1})(d-1) + 2$. It is easy to see that $W^{(\xi,S)}$ and $W^{(\xi,S')}$ define a wall of type (c_1, c_2). Moreover, $W^{(\xi,S)}$ and $W^{(\xi,S')}$ are nonempty. In fact, we have

$$\tilde{L} \equiv dH + (2n - H^{d+1} - 2)F \in W^{(\xi,S)},$$
$$\tilde{L}' \equiv dH + (2n - H^{d+1})F \in W^{(\xi,S')}.$$

Finally, consider the ample divisors on X;

$$L \equiv dH + (2n - H^{d+1} - 1)F,$$
$$L' \equiv dH + (2n - H^{d+1} - 3)F,$$
$$L'' \equiv dH + (2n - H^{d+1} + 1)F.$$

Since $L'\xi S < 0 < L\xi S$ and $L\xi S' < 0 < L''\xi S'$, it follows that the wall $W^{(\xi,S)}$ separates L and L' and that the wall $W^{(\xi,S')}$ separates L and L''. We will denote by $\mathcal{C}$ (resp., $\mathcal{C}'$ and $\mathcal{C}''$) the chamber containing L (resp., L' and L'').

Now we determine and compare the moduli spaces $M_L(2; H, nHF)$ corresponding to polarizations L lying in the chambers $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$ (respectively). Keeping the notation introduced in Example 4.3, we have the following theorem.

THEOREM 4.4. *Let X be a $\mathbb{P}^d$-bundle over C and let $0 \ll n \in \mathbb{Z}$.*

(a) *For all $\bar{L}'' \in \mathcal{C}''$, $M_{\bar{L}''}(2; H, nHF)$ is empty.*
(b) *For all $\bar{L} \in \mathcal{C}$, $M_{\bar{L}}(2; H, nHF)$ is a $\mathbb{P}^N$-bundle over* $\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ *with* $N := h^1\mathcal{O}_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) - 1$.
(c) *For all $\bar{L}' \in \mathcal{C}'$, $M_{\bar{L}'}(2; H, nHF)$ is a nonempty open subset of $M_{\bar{L}}(2; H, nHF)$ and*

$$\dim(M_{\bar{L}}(2; H, nHF) \setminus M_{\bar{L}'}(2; H, nHF)) = h^0\mathcal{E} + h^0\mathcal{O}_C(p) + 2(g-1),$$

with p a point of C.

Proof. (a) follows from Proposition 4.5 and Remark 4.6. (b) follows from Theorem 3.4 and Remark 4.2. (c) follows from Proposition 4.7 and Remark 4.2. □

We now discuss what happens for polarizations lying on the chamber $\mathcal{C}''$.

PROPOSITION 4.5. *Let X be a $\mathbb{P}^d$-bundle over C, let $0 < a \in \mathbb{Z}$, let $(H^{d+1} + d\gamma)/2 + 3 < n \in \mathbb{Z}$, and let $L_0 \equiv aH + bF$ be an ample divisor such that $b/a \geq (2n - H^{d+1})/d$. Then $M_{L_0}(2; H, nHF)$ is empty.*

REMARK 4.6. Keeping the notation introduced in Example 4.3, we have that $b/a \geq (2n - H^{d+1})/d$ is equivalent to $\xi S' L_0 \geq 0$. Hence, $M_L(2; H, nHF)$ is empty for any $L \in \mathcal{C}''$.

Proof of Proposition 4.5. Let E be a rank-2, L_0-stable vector bundle on X with $c_1E \equiv H$ and $c_2E \equiv nHF$. Since $c_2(E(-H + \pi^*\mathfrak{n})) = 0$, we may argue as in Proposition 2.6 to obtain that $E(-lH - \pi^*\mathfrak{m})$, $l > 0$, has a nonzero section whose scheme of zeros has codimension ≥ 2.

Since E is L_0-stable and since $\mathcal{O}_X(lH + \pi^*\mathfrak{m}) \hookrightarrow E$, it follows that $2am < -a(2l-1)H^{d+1} - (2l-1)db$. Since $b/a \geq (2n - H^{d+1})/d$ by hypothesis, we get

$m < -(2l-1)n$. On the other hand, since $E(-lH - \pi^*\mathfrak{m})$ has a nonzero section whose scheme of zeros has codimension ≥ 2, we have

$$0 \leq c_2(E(-lH-\pi^*\mathfrak{m}))H^{d-1} = ((n+2lm-m)HF + l(l-1)H^2)H^{d-1}$$
$$= n + (2l-1)m + l(l-1)H^{d+1}.$$

Hence $m \geq (-l(l-1)H^{d+1} - n)/(2l-1)$, and since (by hypothesis) $2n > H^{d+1} + d\gamma + 6$ we obtain $l^2(2n + d\gamma + 6) - l(2n + d\gamma + 6) < 0$. Since $l > 0$, we arrive at a contradiction and thus $M_{L_0}(2; H, nHF)$ is empty. □

In the next proposition, we compare moduli spaces corresponding to polarizations L lying in the chambers $\mathcal{C}$ and $\mathcal{C}'$.

PROPOSITION 4.7. *Let X be a $\mathbb{P}^d$-bundle over C and let $0 \ll n \in \mathbb{Z}$. We fix an ample divisor $\bar{L}' \equiv dH + bF \in \mathcal{C}'$ on X with $b = 2n - H^{d+1} - 3$. Then $M_{\bar{L}'}(2; H, nHF)$ is a nonempty open subset of $M_{\bar{L}}(2; H, nHF)$ with $\bar{L} \equiv dH + (2n - H^{d+1} - 1)F \in \mathcal{C}$. In particular, $M_{\bar{L}'}(2; H, nHF)$ is a smooth, irreducible, quasiprojective variety of dimension*

$$h^1\mathcal{O}_X(H - \pi^*\mathfrak{n} - \pi^*\mathfrak{n}') + 2g - 1 = h^1\mathcal{E}(-\pi^*\mathfrak{n} - \pi^*\mathfrak{n}') + 2g - 1.$$

Proof. We consider the open subset $\mathcal{U}$ of $M_{\bar{L}}(2; H, nHF)$ defined by

$$\mathcal{U} := \{E \in M_{\bar{L}}(2; H, nHF) \mid H^0E(-\pi^*(\mathfrak{n} - p')) = 0\},$$

with p' a point of C. To prove the proposition it is enough to see that $\mathcal{U}$ is nonempty, $\mathcal{U} \cong M_{\bar{L}'}(2; H, nHF)$, and

$$\dim(M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}) = h^0\mathcal{E} + h^0\mathcal{O}_C(p'') + 2g - 2 < \dim M_{\bar{L}}(2; H, nHF),$$

where p'' is a point of C.

CLAIM 1. *$\mathcal{U}$ is a nonempty open subset of $M_{\bar{L}}(2; H, nHF)$ and*

$$\dim(M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}) = h^0\mathcal{E} + h^0\mathcal{O}_C(p'') + 2g - 2,$$

where p'' is a point of C.

Proof. For any $E \in M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}$, we take $0 \neq s \in H^0E(-\pi^*(\mathfrak{n} - p'))$ and the associated exact sequence

$$0 \to \mathcal{O}_X(D + \pi^*(\mathfrak{n} - p')) \to E \to I_Z(H - \pi^*(\mathfrak{n}' - p) - D') \to 0. \quad (7)$$

Here $D' \equiv D \equiv xH + yF$ are numerically equivalent to an effective divisor, $\mathfrak{n}' - p$ is a divisor on C of degree $n-1$, and $[Z]$ is a codimension-2 closed subscheme of X. The $\bar{L}$-stability of E implies that $4xn - 2x + 2y - 1 < 0$. Since D is numerically equivalent to an effective divisor, it follows that $x = 0$ and $y \geq 0$ or $x > 0$ and $-y \leq x(H^{d+1} + d\gamma)$ (see Remark 2.2). By hypothesis $n \gg 0$; in particular, $n > (H^{d+1} + d\gamma)/2 + 3$ and so the only solution is $x = y = 0$. Thus $D \equiv D' \equiv 0$ and we have the exact sequence

$$0 \to \mathcal{O}_X(\pi^*(\mathfrak{n} - p')) \to E \to I_Z(H - \pi^*(\mathfrak{n}' - p)) \to 0, \quad (8)$$

where $[Z] = c_2(E(-\pi^*(\mathfrak{n} - p')))$ is a complete intersection of type $(H, F_{p''})$, with p'' a point of C and where $\mathfrak{n}' - p$ and $\mathfrak{n} - p'$ are two divisors on C of degree $n-1$.

Let us call $\mathcal{M}$ the irreducible family of rank-2 vector bundles given by an exact sequence of type (8). We have

$$\begin{aligned}
&\dim(M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}) \\
&\quad = \dim \mathcal{M} = \dim \operatorname{Ext}^1(I_{HF}(H - \pi^*(\mathfrak{n}' - p)), O_X(\pi^*(\mathfrak{n} - p'))) \\
&\qquad - h^0 E(-\pi^*(\mathfrak{n} - p')) + 2 \dim \operatorname{Pic}^0(C) + \dim \mathcal{L},
\end{aligned}$$

where $\mathcal{L}$ is the family of codimension-2 closed subschemes Z of X and complete intersections of type $(H, F_{p''})$, with p'' a point of C. Applying Lemma 2.4, we obtain $\dim \mathcal{M} = h^0\mathcal{E} + h^0 O_C(p'') + 2g - 2$. By hypothesis, $h^0\mathcal{E} + h^0 O_C(p'') \le h^1\mathcal{E}(-\mathfrak{n} - \mathfrak{n}')$ and so

$$\begin{aligned}
h^0\mathcal{E} + h^0 O_C(p'') + 2g - 2 = \dim \mathcal{M} &< \dim M_{\bar{L}}(2; H, nHF) \\
&= h^1\mathcal{E}(-\mathfrak{n} - \mathfrak{n}') + 2g - 1.
\end{aligned}$$

Hence $\mathcal{U}$ is a nonempty, open dense subset of $M_{\bar{L}}(2; H, nHF)$ and so we have $\dim(M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}) = h^0\mathcal{E} + h^0 O_C(p'') + 2g - 2$, which proves Claim 1. □

CLAIM 2. *For $E \in M_{\bar{L}}(2; H, nHF)$, E is $\bar{L}'$-stable if and only if $E \in \mathcal{U}$.*

Proof. Arguing as in Proposition 3.2(b), we can see that any $E \in \mathcal{U}$ is $\bar{L}'$-stable. Assume that $E \in M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}$. Let us show that E is not $\bar{L}'$-stable. Since $E \in M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}$, we have $O_X(\pi^*(\mathfrak{n} - p')) \hookrightarrow E$ with $\mathfrak{n} - p' \in \operatorname{Pic}(C)$ of degree $n-1$. If E is $\bar{L}'$-stable then

$$(n-1)F\bar{L}'^d = d^d(n-1) < \frac{c_1(E)\bar{L}'^d}{2} = \frac{d^d(2n-3)}{2},$$

which is a contradiction. Therefore, E is not $\bar{L}'$-stable and we have proved Claim 2. □

CLAIM 3. *Any $E \in M_{\bar{L}'}(2; H, nHF)$ sits in a nontrivial exact sequence*

$$0 \to O_X(H - \pi^*\mathfrak{n}') \to E \to O_X(\pi^*\mathfrak{n}) \to 0.$$

In particular, $M_{\bar{L}'}(2; H, nHF) \subset M_{\bar{L}}(2; H, nHF)$.

Proof. Since $c_2(E(-H + \pi^*\mathfrak{n}')) = 0$, it follows that $E(-lH - \pi^*\mathfrak{m})$, $l > 0$, has a nonzero section whose scheme of zeros has codimension ≥ 2. To end the proof of Claim 3 we need only show that $l = 1$ and $m = -n$.

Since E is $\bar{L}'$-stable and since $O_X(lH + \pi^*\mathfrak{m}) \hookrightarrow E$, it follows that $m < -(2l-1)n + 3(2l-1)/2$. On the other hand, since $E(-lH - \pi^*\mathfrak{m})$ has a nonzero section whose scheme of zeros has codimension ≥ 2, we get

$$\begin{aligned}
0 \le c_2(E(-lH - \pi^*\mathfrak{m}))H^{d-1} &= ((n + 2lm - m)HF + l(l-1)H^2)H^{d-1} \\
&= n + (2l-1)m + l(l-1)H^{d+1}.
\end{aligned}$$

Therefore, $m \geq (-l(l-1)H^{d+1} - n)/(2l-1)$. Using the hypothesis $2n > H^{d+1} + d\gamma + 6$, we obtain $m = -n+1$ or $m = -n$.

In the first case, let $\mathfrak{n}_1 \in \mathrm{Pic}(C)$ be a divisor on C of degree $n-1$. Since $c_2(E(-H+\pi^*\mathfrak{n}_1)) \equiv Z \equiv HF$ and since $E(-H+\pi^*\mathfrak{n}_1)$ has a nonzero section whose scheme of zeros has codimension ≥ 2, we have the exact sequence

$$0 \to O_X(H-\pi^*\mathfrak{n}_1) \to E \to I_Z(\pi^*\mathfrak{n}_1') \to 0,$$

where $\mathfrak{n}_1, \mathfrak{n}_1' \in \mathrm{Pic}(C)$ are two divisors on C of degree $n-1$. Hence, $c_3(E) = c_3(I_Z(\pi^*\mathfrak{n}_1')) + c_2(I_Z(\pi^*\mathfrak{n}_1'))c_1(O_X(H-\pi^*\mathfrak{n}_1)) \equiv 2H^2F$, which contradicts the fact that $c_3(E) = 0$ for any rank-2 vector bundle E. Therefore, $m = -n$ and E sits in the exact sequence

$$0 \to O_X(H-\pi^*\mathfrak{n}') \to E \to O_X(\pi^*\mathfrak{n}) \to 0,$$

where $\mathfrak{n}$ and $\mathfrak{n}'$ are divisors on C of degree n.

Since $n > (H^{d+1} + d\gamma)/2 + 1$, by Proposition 3.2, E is $\bar{L}$-stable; this proves Claim 3. □

Proof of Proposition 4.7 (cont.). From Claims 2 and 3, we deduce that

$$M_{\bar{L}'}(2; H, nHF) \cong \mathcal{U} \subset M_{\bar{L}}(2; H, nHF)$$

is a nonempty open dense subset. Indeed,

$$\begin{aligned}\dim(M_{\bar{L}}(2; H, nHF) \setminus \mathcal{U}) &= h^0\mathcal{E} + h^0 O_C(\bar{\mathfrak{p}}) + 2g - 2\\ &< \dim(M_{\bar{L}}(H, nHF))\\ &= h^1\mathcal{E}(-\pi^*\mathfrak{n} - \pi^*\mathfrak{n}') + 2g - 1,\end{aligned}$$

and this proves what we want. □

Final Remark. Let X be a 3-dimensional rational normal scroll and let $L = H + 2F$ be an ample divisor on X. It follows from [BM, Thm. 0.1] that the number of irreducible components of the moduli space $M_{H+2F}(2; H, c_2HF) \cong M_{H+2F}(2; H+2F, (c_2+1)HF)$ grows to infinity when c_2 goes to infinity. This shows again that the moduli space $M_L(2; H, c_2HF)$ strongly depends on the fixed polarization L. The authors hope to describe (in a forthcoming paper) different irreducible components of the moduli space $M_L(2; H, c_2HF)$, where L is a polarization lying on a chamber far from the chamber $\mathcal{C}$ described in Example 4.3.

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