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Boundary Effects and Large Time Behavior for the System of Compressible Adiabatic Flow through Porous Media

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1. Introduction

In 1-dimensional porous media, the motion of compressible adiabatic flow can be modeled by the compressible Euler equations with frictional damping terms, say, the following balance laws:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ \left(e(v,s) + \frac{1}{2}u^2\right)_t + (pu)_x = -\alpha u^2, \end{cases} \tag{1.1}$$

under Lagrangian coordinates. For smooth solutions, the system (1.1) is equivalent to

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ s_t = 0. \end{cases} \tag{1.2}$$

Here, v denotes the specific volume, u is the velocity, s stands for entropy, p denotes the gas pressure with $p_v(v,s) < 0$ for $v > 0$, and e is the specific internal energy. For sake of simplicity, we assume that $\alpha = 1$ and $p(v,s) = (\gamma-1)v^{-\gamma}e^s$ with $\gamma > 1$.

In the case of isentropic flow where $s = \text{const.}$, (1.2) takes the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -u. \end{cases} \tag{1.3}$$

In [5], Hsiao and Liu proved that the large time behavior of the solutions of the Cauchy problem to (1.3) were captured by those of Darcy's law:

$$\begin{cases} v_t = -p(v)_{xx}, \\ u = -p(v)_x. \end{cases} \tag{1.4}$$

The better convergence rates were obtained in [21], [22], and [23]. To understand the boundary effects, the system (1.3) on $\mathbf{R}^+ = (0,\infty)$ with different kinds of boundary conditions were studied in [16] and [24], where the corresponding asymptotic behavior of solutions and the relation to the nonlinear diffusion waves

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of (1.4) were obtained. For more references on this topic, we refer to [4] and [6] for smooth solutions and to [1; 4; 8; 13; 14; 15; 17; 18; 20; 25] for weak solutions.

For the adiabatic flow where $s \neq$ const., much less is known. The global existence of smooth solutions to the Cauchy problem has been proved in [11] and [26] for small initial data. The problem of large time behavior of these solutions is known only for some particular initial data; see [7], [12], and [19]. However, the initial boundary value problem for (1.2) on bounded domain with fixed boundary conditions has been understood very well in [9; 10] by the combination of characteristic analysis and the energy method. As far as we know, there is not any work on the initial boundary value problems on $\mathbf{R}^+$ for the full system (1.2).

In this paper, we consider the initial boundary value problems of (1.2) on $\mathbf{R}^+$ with the initial data

$$(v, u, s)(x, 0) = (v_0, u_0, s_0)(x) \to (v_+, u_+, s_+), \ \ v_+ > 0, \ \text{ as } x \to \infty, \tag{1.5}$$

and boundary conditions

$$v(0, t) = v_0(0) = v_- > 0 \tag{1.6}$$

or

$$u(0, t) = 0. \tag{1.7}$$

From now on, we will denote by (P1) the problem (1.2), (1.5), and (1.6) and by (P2) the problem (1.2), (1.5), and (1.7). We shall study the global existence and large time behavior of the solutions for these two problems and their related diffusive problems:

$$\begin{cases} \tilde{v}_t = -p(\tilde{v}, s)_{xx}, \\ \tilde{u} = -p(\tilde{v}, s)_x, \quad x \in \mathbf{R}^+, \\ s_t = 0, \\ \tilde{v}(0, t) = \tilde{v}_0(0) = v_-, \\ (\tilde{v}, s)(x, 0) = (\tilde{v}_0, s_0)(x) \to (v_+, s_+), \ \ v_+ > 0, \ \text{ as } x \to \infty; \end{cases} \tag{1.8}$$

$$\begin{cases} \tilde{v}_t = -p(\tilde{v}, s)_{xx}, \\ u = -p(\tilde{v}, s)_x, \ \ x \in \mathbf{R}^+, \\ s_t = 0, \\ p(\tilde{v}, s)_x(0, t) = 0, \\ (\tilde{v}, s)(x, 0) = (\tilde{v}_0, s_0)(x) \to (v_+, s_+), \ \ v_+ > 0, \ \text{ as } x \to \infty. \end{cases} \tag{1.9}$$

The nonlinear diffusive phenomena created by a damping mechanism with boundary effects is expected. In fact, we will prove that the solutions of (P1) and (P2) behave time-asymptotically as those of (1.8) and (1.9), respectively. Moreover, the large time asymptotic states are given by stationary solutions or similarity functions, depending on the boundary conditions.

Our results here can be viewed as the generalization of [24] for the adiabatic case. However, the results of [24] are based strongly on the knowledge of the isentropic porous media equation $(1.4)_1$, which is well known (see [2; 3]). Essential difficulties occur in our problems since the theory for adiabatic porous media equations is unknown. The nonconstant entropy $s(x)$ makes the diffusive problems (1.8) and (1.9) highly nontrivial. In Section 2, we treat the problem (1.8) by the new approach introduced in [19] for the Cauchy problem with some modifications; see Lemmas 2.6–2.9. Our key elements are two. First, we introduce a new dependent variable w (see (2.2)) and consider the problems for w rather than the specific volume as in the isentropic case or the pressure as in [9; 10]; with the help of our new dependent variable w, we can control the total excessive mass of the problem near the desired diffusive profile by the L^1 technique. Then we can combine the L^1 technique with the weighted energy method to obtain the desired results for the solutions of (1.8). In Section 3, the global existence and large time behavior as well as the convergence rate for the solutions to (P1) are established by our results in Section 2 and the approach of [24] with some modifications; see Lemma 3.3. In Section 4, we apply the same approaches used in Sections 2 and 3 to the problem (1.9) and (P2).

In general, the initial boundary value problems are harder than Cauchy problems of hyperbolic systems. Our results are somewhat amazing in that the initial boundary value problem of (1.2) on $\mathbf{R}^+$ can be solved completely while the Cauchy problem remains open.

Notation. We denote by C the generic constants independent of t. For function spaces, $L^q = L^q(\mathbf{R}^+)$ $(1 \le q \le \infty)$ is a usual Lebesgue integrable function space with the norm $\|\cdot\|_{L^q}$. The L^2-norm on $\mathbf{R}^+$ is simply denoted by $\|\cdot\|$. By H^k we denote the standard Sobolev space on $\mathbf{R}^+$; its norm is $\|\cdot\|_{H^k}$ and $H^0 = L^2$. We will also use the norm

$$\|(g_1, \dots, g_n)\|_{H^k} = \sum_{i=1}^{n} \|g_i\|_{H^k}.$$

2. Diffusive Problem (1.8)

This section is devoted to study the diffusive problem (1.8). Clearly one has $s(x,t) = s_0(x) \equiv s(x)$ for $t > 0$, so it is sufficient to solve the following problem:

$$\begin{cases} \tilde{v}_t = -p(\tilde{v}, s)_{xx}, \quad x \in \mathbf{R}^+, \\ \tilde{v}(0,t) = v_-, \\ \tilde{v}(x,0) = \tilde{v}_0(x), \quad \tilde{v}_0(+\infty) = v_+ > 0. \end{cases} \tag{2.1}$$

Problem (2.1) is equivalent to the following porous media–type system of equations:

$$\begin{cases} w_t + a(x)(w^{-\gamma})_{xx} = 0, \quad x \in \mathbf{R}^+, \\ w(0,t) = w_-, \\ w(x,0) = w_0(x) = a(x)\tilde{v}_0(x), \\ w(+\infty) = w_+ > 0, \end{cases} \tag{2.2}$$

where $a(x) = (\gamma-1)^{-1/\gamma}e^{-(1/\gamma)s(x)}$, $w \equiv a(x)\tilde{v} = p(\tilde{v},s)^{-1/\gamma}$, and $w_- = a(0)v_-$.

Now let $a_1 = (\gamma-1)^{-1/\gamma}e^{-(1/\gamma)s_+}$ and $\tilde{w}(\eta)$ (where $\eta = x/\sqrt{1+t}$) be the unique similarity solution of

$$\begin{cases} w_t + a_1(w^{-\gamma})_{xx} = 0, \\ w(0,t) = w_-, \quad w(+\infty,t) = w_+. \end{cases} \tag{2.3}$$

Some properties of $\tilde{w}(\eta)$ are listed in the following lemma (see [2; 3]).

LEMMA 2.1. *For $i+j \geq 1$ and $i \geq 0,\ j \geq 0$, we have*

$$\begin{cases} |\tilde{w}(\eta) - w_+| + |\tilde{w}'(\eta)| + |\tilde{w}''(\eta)| \leq C|w_+ - w_-|\exp\{-C_2\eta^2\}, \\ \tilde{w}_x = (1+t)^{-1/2}\tilde{w}'(\eta), \quad \tilde{w}_t = -\frac{1}{2}(1+t)^{-1}\eta\tilde{w}'(\eta), \\ \|D_t^iD_x^j\tilde{w}(\cdot,t)\|^2 \leq C|w_+ - w_-|^2(1+t)^{-(2i+j)+1/2}, \\ \|D_t^iD_x^j\tilde{w}(\cdot,t)\|_{L^\infty} \leq C|w_+ - w_-|^2(1+t)^{-(i+j/2)}. \end{cases}$$

We now solve problem (2.2) by comparing $w(x,t)$ with $\tilde{w}(\eta)$.

Let $\phi = w - \tilde{w}$; then, from (2.2) and (2.3), we have the following system for ϕ:

$$\begin{cases} \phi_t + a(x)(\psi(\tilde{w})\phi)_{xx} + (a-a_1)(\tilde{w}^{-\gamma})_{xx} + a(x)(g(\phi,\tilde{w})\phi^2)_{xx} = 0, \\ \phi(0,t) = 0, \\ \phi(x,0) = \phi_0(x) = w_0(x) - \tilde{w}(x,0), \quad x \in \mathbf{R}^+, \end{cases} \tag{2.4}$$

where we have set

$$\psi(\tilde{w}) = -\gamma\tilde{w}^{-(\gamma+1)} \quad \text{and} \quad g(\phi,\tilde{w})\phi^2 = (\phi+\tilde{w})^{-\gamma} - \tilde{w}^{-\gamma} - \psi(\tilde{w})\phi.$$

Let $F = -\psi(\tilde{w})\phi$; then the corresponding problem on F is given by

$$\begin{cases} F_t + a(x)\psi(\tilde{w})F_{xx} - \psi(\tilde{w})(a-a_1)(\tilde{w}^{-\gamma})_{xx} \\ \qquad - \psi_1(\tilde{w})F\tilde{w}_t - a\psi(\tilde{w})(fF^2)_{xx} = 0, \\ F(0,t) = 0, \\ F(x,0) = F_0(x) = -\psi(\tilde{w}(x,0))\phi_0(x), \quad x \in \mathbf{R}^+, \end{cases} \tag{2.5}$$

where

$$-\psi_1(\tilde{w})F = \psi'(\tilde{w})\phi \quad \text{and} \quad fF^2 = g\phi^2.$$

Now we define the Banach space $X(0,t)$ for all $T > 0$ by

$$X(0,t) = \{F \in C^0(0,t;H^2),\ 0 \leq t \leq T\},$$

equipped with the norm $N^2(t) = \sup_{0\leq\tau\leq t}\|F(\tau)\|_{H^2}^2$.

The main result of this section is the following theorem.

THEOREM 2.2. *Assume that $F_0(x)$ and $s(x) = s_0(x)$ are smooth functions such that $F_0 \in H^2(\mathbf{R}^+) \cap L^1(\mathbf{R}^+)$ and $x(s(x) - s_+) \in L^1(\mathbf{R}^+)$. Then there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that, if $|w_+ - w_-| \leq \delta$ and $\|F_0\|_{H^2} \leq \varepsilon_0$, then* (2.5) *has a unique global smooth solution F satisfying*

$$\sum_{j=0}^{2} w_{j+1}(t)\|\partial_x^jF(\cdot,t)\|^2 + \int_0^t \sum_{j=1}^{3} w_j(\tau)\|\partial_x^jF(\cdot,\tau)\|^2\,d\tau \leq C, \tag{2.6}$$

where the weight functions are defined as

$$w_1(t) = (1+t)^{1/2}(1+\log(1+t))^{-k} \quad \textit{and} \quad w_j(t) = (1+t)^{j-1}w_1(t)$$

for $j, k > 1$.

The local existence and uniqueness of the solution to (2.5) in $X(0, T)$ is standard. From now on, we assume a priori the existence in $X(0, T)$ for some $T > 0$.

The following L^1-estimate follows from the standard contraction property of the porous media–type equation and will play a fundamental role in the rest of this section.

LEMMA 2.3. *Under the conditions of Theorem 2.2, if the solution exists in* $X(0, T)$ *then, for* $0 \le t \le T$, *the following estimate holds*:

$$\|\phi(\cdot, t)\|_{L^1} \le C_1\|F(\cdot, t)\|_{L^1} \le C_2(\|\phi_0\|_{L^1} + \delta). \tag{2.7}$$

Proof. We present here a formal argument that can easily be made rigorous by using any sequence approximating the sign function and then passing to the limit by means of the Lebesgue dominated convergence theorem. Observe that $h = \operatorname{sign}(\phi) = \operatorname{sign}(F)$ and $h(0, t) = 0$. Let us multiply the equation $(2.4)_1$ by $a^{-1}h$; then, by integrating over $[0, t] \times (0, +\infty)$, it follows that

$$\begin{aligned}
&\int_0^{+\infty} a^{-1}|\phi|(x, t)\,dx + \int_0^t \int_0^{+\infty} \operatorname{sign}'(F)F_x^2\,dx\,d\tau \\
&\quad \le C\int_0^{+\infty} a^{-1}|\phi_0|(x)\,dx + C\left|\int_0^t \int_0^{+\infty} (a - a_1)\tilde{w}_t \operatorname{sign}(F)\,dx\,d\tau\right| \\
&\qquad + \left|\int_0^t \int_0^{+\infty} (fF^2)_x F_x \operatorname{sign}'(F)\,dx\,d\tau\right| \\
&\quad \le C(\|\phi_0\|_{L^1} + \delta).
\end{aligned} \tag{2.8}$$

Here, we have used the following facts:

$$\begin{aligned}
&\left|\int_0^t \int_0^{+\infty} (a - a_1)\tilde{w}_t \operatorname{sign}(F)\,dx\,d\tau\right| \\
&\qquad \le C\int_0^t \int_0^{+\infty} |s - s_+||\tilde{w}_t|\,dx\,d\tau \\
&\qquad \le C\int_0^t \int_0^{+\infty} (1+t)^{-3/2}|x(s - s_+)||\tilde{w}'(\eta)|\,dx\,d\tau \\
&\qquad \le C\delta;
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\int_0^t \int_0^{+\infty} (fF^2)_x F_x \operatorname{sign}'(F)\,dx\,d\tau \\
&\qquad = \int_0^t \int_0^{+\infty} F_x(2fF_x + f_F F F_x + f_{\tilde{w}} F\tilde{w}_x)F\delta_{\{F=0\}}\,dx\,d\tau \\
&\qquad = 0.
\end{aligned} \tag{2.10}$$

Hence, (2.8) gives the proof of (2.7). □

With the help of Lemma 2.3, we can make the energy estimates on F.

LEMMA 2.4. *Under the conditions in Theorem 2.2, suppose that* $N(T) \le \varepsilon$ *for some suitable small constant* $\varepsilon > 0$. *Then, for* $0 \le t \le T$, *we have that*

$$\|F(\cdot,t)\|^2 + \int_0^t \|F_x(\cdot,\tau)\|^2\,d\tau \le C(\|F_0\|^2 + \delta). \tag{2.11}$$

Proof. Multiplying the equation $(2.4)_1$ by $a^{-1}F$ and then integrating the result over $[0,t] \times (0,+\infty)$ yields

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{2} a^{-1} F\phi(x,t)\,dx + \int_0^t \int_0^{+\infty} F_x^2\,dx\,d\tau \\
&\quad \le \int_0^{+\infty} \frac{1}{2} a^{-1} F_0\phi_0\,dx + \left| \int_0^t \int_0^{+\infty} a^{-1}(a-a_1)(\tilde{w}^{-\gamma})_{xx} F\,dx\,d\tau \right| \\
&\qquad + \left| \int_0^t \int_0^{+\infty} \frac{1}{2} a^{-1}\psi_2(\tilde{w}) F^2 \tilde{w}_t\,dx\,d\tau \right| + \left| \int_0^t \int_0^{+\infty} (fF^2)_x F_x\,dx\,d\tau \right| \\
&\quad \equiv \int_0^{+\infty} \frac{1}{2} a^{-1} F_0\phi_0\,dx + I_1 + I_2 + I_3,
\end{aligned} \tag{2.12}$$

with $\psi_2(\tilde{w})F^2 = \phi^2\psi'(\tilde{w})$.

We estimate I_1, I_2, and I_3 step-by-step as follows:

$$\begin{aligned}
I_1 &= \left| \int_0^t \int_0^{+\infty} a^{-1}(a-a_1)(\tilde{w})_{xx}^{-\gamma} F\,dx\,d\tau \right| \\
&\le C \left| \int_0^t \int_0^{+\infty} (s-\bar{s}) F\tilde{w}_t\,dx\,d\tau \right| \\
&\le C\delta\varepsilon \int_0^t (1+\tau)^{-3/2} \|x(s-\bar{s})\|_{L^1}\,d\tau \\
&\le C\delta\varepsilon;
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
I_2 &= \left| \int_0^t \int_0^{+\infty} \frac{1}{2} a^{-1}\psi_2(\tilde{w}) F^2 \tilde{w}_t\,dx\,d\tau \right| \\
&\le C \int_0^t \|F\|_{L^\infty} \|\tilde{w}_t\|_{L^\infty} \|F\|_{L^1}\,dx\,d\tau \\
&\le C\delta \int_0^t \|F\|^{1/2} \|F_x\|^{1/2} (1+\tau)^{-1}\,d\tau \\
&\le C\delta \left(\int_0^t \|F\|^2 \|F_x\|^2\,d\tau + \int_0^t (1+\tau)^{-4/3}\,d\tau \right) \\
&\le C\delta \left(1 + \varepsilon^2 \int_0^t \|F_x\|^2\,d\tau \right);
\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
I_3 &= \left| \int_0^t \int_0^{+\infty} (fF^2)_x F_x \, dx \, d\tau \right| \\
&\le C \int_0^t \int_0^{+\infty} (|F| + |F^2|) F_x^2 + |F^2 \tilde{w}_x F_x| \, dx \, d\tau \\
&\le \left(\frac{1}{2} + C\varepsilon \right) \int_0^t \|F_x\|^2 \, d\tau + C\delta^2 \int_0^t \|F\|_{L^\infty}^4 \, d\tau \\
&\le \left(\frac{1}{2} + C\varepsilon \right) \int_0^t \|F_x\|^2 \, d\tau + C\delta^2 \int_0^t \|F\|^2 \|F_x\|^2 \, d\tau \\
&\le \left(\frac{1}{2} + C\varepsilon \right) \int_0^t \|F_x\|^2 \, d\tau.
\end{aligned} \tag{2.15}
$$

Owing to the smallness of δ and ε, we conclude from (2.12)–(2.15) that

$$
\|F(\cdot, t)\|^2 + \int_0^t \|F_x(\cdot, \tau)\|^2 \, d\tau \le C(\|F_0\|^2 + \delta), \tag{2.16}
$$

which completes the proof of Lemma 2.4. □

For higher-order estimates, we use the problem (2.5) to obtain the following results.

LEMMA 2.5. *Under the same conditions as in Lemma 2.4, we have*

$$
\begin{aligned}
\|(F_x, F_t, F_{xx})(\cdot, t)\|^2 &+ \int_0^t \|(F_{xx}, F_{xxx}, F_{tx})(\cdot, \tau)\|^2 \, d\tau \\
&\le C(\|F_0\|_{H^2}^2 + \delta).
\end{aligned} \tag{2.17}
$$

Proof. Let us multiply the equation $(2.5)_1$ by F_{xx}. Then

$$
\begin{aligned}
\int_0^{+\infty} F_x^2(x, t) \, dx &+ \int_0^t \int_0^{+\infty} F_{xx}^2(x, \tau) \, dx \, d\tau \\
&\le C \Bigg(\|F_{0x}\|^2 + \left| \int_0^t \int_0^{+\infty} \tilde{w}_t F_{xx} \, dx \, d\tau \right| \\
&\quad + \left| \int_0^t \int_0^{+\infty} (fF^2)_{xx} F_{xx} \, dx \, d\tau \right| \Bigg),
\end{aligned} \tag{2.18}
$$

which implies, with the help of the Cauchy–Schwartz inequality and Lemma 2.1, that

$$
\begin{aligned}
\int_0^{+\infty} F_x^2(\cdot, t) \, dx &+ \int_0^t \int_0^{+\infty} F_{xx}^2(\tau, x) \, dx \, d\tau \\
&\le C(\|F_{0x}\|^2 + \delta^2) + C \int_0^t \int_0^{+\infty} (fF^2)_{xx}^2 \, dx \, d\tau.
\end{aligned} \tag{2.19}
$$

We bound the last term in (2.19) as follows:

$$
\begin{aligned}
\int_0^t \int_0^{+\infty} & (fF^2)_{xx}^2 \, dx \, d\tau \\
& \le C \int_0^t \int_0^{+\infty} (|F| + |F_x| + |w_x|)^2 F_x^2 + F^2 F_{xx}^2 + F^4 (\tilde{w}_{xx}^2 + \tilde{w}_x^4) \, dx \, d\tau \\
& \le C\varepsilon^2 \delta^2 + C\varepsilon \int_0^t \int_0^{+\infty} F_{xx}^2(\tau, x) \, dx \, d\tau. \qquad (2.20)
\end{aligned}
$$

Then, (2.19)–(2.20) and the estimates in Lemma 2.4 give the first part of (2.17).

We now differentiate the equation $(2.5)_1$ in t to obtain

$$
\begin{aligned}
F_{tt} + a\psi(\tilde{w}) F_{xxt} + a\psi'(\tilde{w}) \tilde{w}_t F_{xx} - (\psi(\tilde{w})(a - a_1)(\tilde{w}^{-\gamma})_{xx})_t \\
+ (\psi_1(\tilde{w}) F \tilde{w}_t)_t - (a\psi(\tilde{w})(fF^2)_{xx})_t = 0. \qquad (2.21)
\end{aligned}
$$

Multiplying (2.21) by F_t and then integrating over $[0, t] \times (0, +\infty)$, we have

$$
\|F_t(\cdot, t)\|^2 + \int_0^t \|F_{tx}(\cdot, \tau)\|^2 \, d\tau \le C(\|F_0\|_{H^2}^2 + \delta). \qquad (2.22)
$$

The balance of the estimates in (2.17) can be easily proved by differentiating (2.5) in x. □

By the standard continuity argument (see [19]), we now conclude from Lemmas 2.3–2.5 as follows.

THEOREM 2.6. *Under the conditions of Theorem 2.2, there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that, if $\|F_0\|_{H^2} < \varepsilon_0$ and $|w_+ - w_-| \le \delta$, then there is a unique global classical solution $F(x, t)$ of* (2.5) *such that*

$$
\|F(t)\|_{H^2}^2 + \int_0^t (\|F_t\|_{H^1}^2 + \|F_x\|_{H^2}^2)(\tau) \, d\tau \le C_0(\|F_0\|_{H^2}^2 + \delta) \qquad (2.23)
$$

for all $t > 0$ and $C_0 > 0$ independent of t. Furthermore,

$$
\lim_{t \to \infty} \|F(\cdot, t)\|_{H^1} \to 0.
$$

By using the weighted energy method, we can prove the following decay rates.

LEMMA 2.7. *Let F be the solution to* (2.5) *obtained in Theorem 2.6. Then*

$$
\begin{aligned}
w_1(t)\|F(t)\|^2 + w_2(t)\|F_x(t)\|^2 \\
+ \int_0^t \big(w_1(\tau)\|F_x(\tau)\|^2 + w_2(\tau)\|F_{xx}(\tau)\|^2 \big) \, d\tau \le C. \qquad (2.24)
\end{aligned}
$$

Proof. Let us multiply (2.4) by $a^{-1} w_1(t) F$ to obtain

$$
\begin{aligned}
& \left(\tfrac{1}{2} F\phi a^{-1} w_1(t) \right)_t + w_1(t) F_x^2 - \tfrac{1}{2} w_1'(t) a^{-1} \psi_1(\tilde{w}) F^2 \\
& = \tfrac{1}{2} a^{-1} w_1(t) F^2 \tilde{w}_t - a^{-1} w_1(t)(a - a_1) F(\tilde{w}^{-\gamma})_{xx} + w_1(t) F_x (fF^2)_x + \{\cdots\}_x.
\end{aligned}
$$

When integrated on $[0, t] \times (0, +\infty)$, this yields

$$
\begin{aligned}
& w_1(t)\|F(\cdot,t)\|^2 + \int_0^t w_1(t)\|F_x(\tau)\|^2\,d\tau \\
&\quad \le C_1\biggl(\|F_0\|^2 + \left|\int_0^t\int_0^{+\infty} w_1'(\tau)F^2\,dx\,d\tau\right| \\
&\qquad + \left|\int_0^t\int_0^{+\infty} w_1(\tau)F^2\tilde{w}_t\,dx\,d\tau\right| \\
&\qquad + \left|\int_0^t\int_0^{+\infty} w_1(\tau)\tilde{w}_t F(a-a_1)\,dx\,d\tau\right| \\
&\qquad + \left|\int_0^t\int_0^{+\infty} w_1(\tau)(fF^2)_x^2\,dx\,d\tau\right|\biggr).
\end{aligned} \tag{2.25}
$$

Here $\{\cdots\}_x$ denotes the term that does not need to be computed explicitly, since it will disappear by integrating in x. Observe that the following inequality on F,

$$
\|F\|_{L^\infty} \le C\|F_x\|^{2/3}, \tag{2.26}
$$

holds, since

$$
\begin{aligned}
\|F\|_{L^\infty} &\le C\|F\|^{1/2}\|F_x\|^{1/2} \\
&\le C\|F\|_{L^\infty}^{1/4}\|F_x\|^{1/2}\|F\|_{L^1}^{1/4}.
\end{aligned}
$$

We now have the following estimates:

$$
\begin{aligned}
& \int_0^t\int_0^{+\infty}\bigl(|w_1'(\tau)F^2| + |w_1(\tau)\tilde{w}_t(a-a_1)F| + |w_1(\tau)\tilde{w}_t F^2|\bigr)\,dx\,d\tau \\
&\quad \le C\int_0^t (1+\tau)^{-1}w_1(\tau)\|F\|_{L^\infty}\,d\tau \\
&\quad \le C\int_0^t (1+\tau)^{-1}w_1(\tau)\|F_x(\tau)\|^{2/3}\,d\tau \\
&\quad \le C + \frac{1}{2C_1}\int_0^t w_1(\tau)\|F_x(\tau)\|^2\,d\tau;
\end{aligned} \tag{2.27}
$$

$$
\begin{aligned}
& \int_0^t\int_0^{+\infty} w_1(\tau)(fF^2)_x^2\,dx\,d\tau \\
&\quad \le C\varepsilon\int_0^t w_1(\tau)\|F_x(\tau)\|^2\,d\tau + C\varepsilon\int_0^t (1+\tau)^{-1}w_1(\tau)\|F\|_{L^\infty}\,d\tau \\
&\quad \le C\varepsilon + C\varepsilon\int_0^t w_1(\tau)\|F_x(\tau)\|^2\,d\tau.
\end{aligned} \tag{2.28}
$$

Here we have used

$$
|w_i'(t)| \le C(1+t)^{-1}w_i(t) \quad \text{for } i = 1, 2, \ldots.
$$

Hence, by the smallness of ε, we conclude from (2.25)–(2.28) that

$$w_1(t)\|F(\cdot,t)\|^2+\int_0^t w_1(t)\|F_x(\tau)\|^2\,d\tau\le C. \tag{2.29}$$

Multiplying (2.5) by $w_2(t)F_{xx}$, we obtain

$$\begin{aligned}&\left(\tfrac12 w_2(t)F_x^2\right)_t-a\psi(\tilde w)w_2(t)F_{xx}^2-\tfrac12 w_2'(t)F_x^2-\psi_1(\tilde w)FF_{xx}\tilde w_t w_2(t)\\&\quad=-w_2(t)\psi(\tilde w)(a-a_1)(\tilde w^{-\gamma})_{xx}F_{xx}-a\psi(\tilde w)(fF^2)_{xx}F_{xx}w_2(t)+\{\cdots\}_x.\end{aligned}$$

Then one has

$$\begin{aligned}&w_2(t)\|F_x(\cdot,t)\|^2+\int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2\,d\tau\\&\quad\le C+C\biggl(\left|\int_0^t\int_0^{+\infty}\tilde w_t^2w_2(\tau)(a-a_1)^2\,dx\,d\tau\right|\\&\qquad+\left|\int_0^t\int_0^{+\infty}F^2\tilde w_t^2w_2(\tau)\,dx\,d\tau\right|\\&\qquad+\int_0^t\int_0^{+\infty}w_2(\tau)(fF^2)_{xx}^2\,dx\,d\tau\biggr).\end{aligned}\tag{2.30}$$

Since

$$\begin{aligned}&\left|\int_0^t\int_0^{+\infty}\tilde w_t^2w_2(\tau)(a-a_1)^2\,dx\,d\tau\right|+\left|\int_0^t\int_0^{+\infty}F^2\tilde w_t^2w_2(\tau)\,dx\,d\tau\right|\\&\quad\le C\delta^2\int_0^t(1+\tau)^{-3}w_2(\tau)\,d\tau+C\delta^2\int_0^t(1+\tau)^{-1}w_1(\tau)\|F\|_{L^\infty}\,d\tau\\&\quad\le C\delta^2\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}(fF^2)_{xx}&=(2FF_xf+f_FF^2F_x+f_{\tilde w}\tilde w_xF^2)_x\\&=(2fF+f_FF^2)F_{xx}+(2f+4f_FF+f_{FF}F^2)F_x^2\\&\quad+(4f_{\tilde w}F+2f_{F\tilde w}F^2)F_x\tilde w_x+(f_{\tilde w}\tilde w_{xx}+f_{\tilde w\tilde w}\tilde w_x^2)F^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\left|\int_0^t\int_0^{+\infty}w_2(\tau)(fF^2)_{xx}^2\,dx\,d\tau\right|\\&\quad\le C+C\varepsilon\int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2\,d\tau+C\int_0^t\int_0^{+\infty}F_x^4w_2(\tau)\,dx\,d\tau\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}&\int_0^t\int_0^{+\infty}F_x^4w_2(\tau)\,dx\,d\tau\\&\quad\le C\varepsilon^2\int_0^t w_2(\tau)\|F_{xx}\|^2\,d\tau+C\int_0^t w_2(\tau)\|F_x(\tau)\|^2\|F_x(\tau)\|^2\,d\tau.\end{aligned}\tag{2.33}$$

Owing to the smallness of ε, we deduce from (2.30)–(2.33) that

$$w_2(t)\|F_x(t)\|^2 + \int_0^t w_2(\tau)\|F_{xx}(\tau)\|^2\,d\tau \le C\left(1 + \int_0^t w_2(\tau)\|F_x(\tau)\|^2\|F_x(\tau)\|^2\,d\tau\right). \tag{2.34}$$

Therefore, Gronwall's inequality gives

$$w_2(t)\|F_x(t)\|^2 + \int_0^t w_2(\tau)\|F_{xx}(\tau)\|^2\,d\tau \le C. \tag{2.35}$$

Hence, (2.29) and (2.35) complete the proof of this lemma. □

The following lemma contains the decay rates for the derivatives of F, which will be useful in the next section.

LEMMA 2.8. *The solution F to* (2.5) *in Theorem 2.6 satisfies*

$$\begin{cases} w_3(t)\|F_t(t)\|^2 + \int_0^t w_3(\tau)\|F_{tx}(\tau)\|^2\,d\tau \le C, \\ w_4(t)\|F_{tx}\|^2 + \int_0^t w_4(\tau)\|F_{txx}(\tau)\|^2\,d\tau \le C; \end{cases} \tag{2.36}$$
$$\|F_t\|_{L^\infty} \le C w_3(t)^{-1/4} w_4(t)^{-1/4}.$$

Proof. It is sufficient to prove (2.36), since the estimate for $\|F_t\|_{L^\infty}$ can be derived from (2.36) by using the Sobolev inequality.

Let us differentiate $(2.5)_1$ in t; then

$$\begin{aligned} &F_{tt} + a\psi(\tilde w)F_{txx} + a\psi'(\tilde w)\tilde w_t F_{xx} - [\psi(\tilde w)(a-a_1)(\tilde w^{-\gamma})_{xx}]_t \\ &\quad - (\psi_1(\tilde w)F\tilde w_t)_t - [a\psi(\tilde w)(fF^2)_{xx}]_t = 0. \end{aligned} \tag{2.37}$$

Multiplying (2.37) by $a^{-1}w_3(t)F_t$, we obtain

$$\begin{aligned} &\left(\tfrac12 a^{-1}w_3(t)F_t^2\right)_t - \psi(\tilde w)w_3(t)F_{tx}^2 + \tfrac12 F_t^2\psi(\tilde w)_{xx}w_3(t) - \tfrac12 F_t^2 a^{-1}w_3'(t) \\ &\quad + \psi'(\tilde w)\tilde w_t F_{xx}w_3(t)F_t - a^{-1}[\psi(\tilde w)(a-a_1)(\tilde w^{-\gamma})_{xx}]_t w_3(t)F_t \\ &\quad - a^{-1}(\psi_1(\tilde w)F\tilde w_t)_t w_3(t)F_t - [\psi(\tilde w)(fF^2)_{xx}]_t w_3(t)F_t + \{\cdots\}_x = 0. \end{aligned} \tag{2.38}$$

From the proof of Lemma 2.7 and $(2.5)_1$, it is clear that

$$\int_0^t w_2(\tau)\|F_t(\cdot,\tau)\|^2\,d\tau \le C. \tag{2.39}$$

Moreover, we notice that

$$\begin{cases} a^{-1}(\psi_1(\tilde w)F\tilde w_t)_t w_3(t)F_t = O(1)[\tilde w_t w_3(t)F_t^2 + (\tilde w_t^2 + \tilde w_{tt})w_3(t)FF_t], \\ a^{-1}[\psi(\tilde w)(a-a_1)(\tilde w^{-\gamma})_{xx}]_t w_3(t)F_t = O(1)(a-a_1)(\tilde w_t^2 + \tilde w_{tt})w_3(t)F_t, \\ [\psi(\tilde w)(fF^2)_{xx}]_t w_3(t)F_t \\ \qquad = O(1)\tilde w_t(fF^2)_{xx}w_3(t)F_t - \psi(\tilde w)(fF^2)_{xxt}w_3(t)F_t. \end{cases}$$

Now we can use a similar argument as used in deriving the estimate (2.29) to obtain

$$w_3(t)\|F_t(t)\|^2 + \int_0^t w_3(\tau)\|F_{tx}(\tau)\|^2\,d\tau \le C, \tag{2.40}$$

which is the first part of (2.36).

Let us turn to the second part of (2.36). For this purpose, we multiply (2.37) by $w_4(t)F_{txx}$. After similar calculations as before, by virtue of (2.40) we have

$$\begin{aligned} w_4(t)\|F_{tx}(t)\|^2 &+ \int_0^t w_4(\tau)\|F_{txx}(\tau)\|^2\,d\tau \\ &\le C\Big(1 + \int_0^t \int_0^{+\infty} w_4(\tau)(fF^2)_{txx}^2\,dx\,d\tau\Big). \end{aligned} \tag{2.41}$$

Now, a similar argument as used in deriving (2.35) yields

$$\begin{aligned} w_4(t)\|F_{tx}(t)\|^2 &+ \int_0^t w_4(\tau)\|F_{txx}(\tau)\|^2\,d\tau \\ &\le C + C\int_0^t w_4(\tau)\|F_{tx}\|^2\|F_x\|^2\,d\tau. \end{aligned} \tag{2.42}$$

Then the Gronwall inequality implies

$$w_4(t)\|F_{tx}(t)\|^2 + \int_0^t w_4(\tau)\|F_{txx}(\tau)\|^2\,d\tau \le C. \tag{2.43}$$

System (2.36) is then given by (2.40) and (2.43). $\square$

COROLLARY 2.9. *The solution F to* (2.5) *obtained in Theorem 2.6 satisfies*

$$w_3(t)\|F_{xx}\|^2 \le C, \qquad \|F_{xx}\|_{L^\infty} \le C(w_3(t)w_4(t))^{-1/4},$$
$$\|F_x\|_{L^\infty}^2 \le Cw_3(t)^{-1/2}w_2(t)^{-1/2}.$$

Proof. We see from (2.5) that

$$F_{xx} = O(1)\big(F_t + (a-a_1)\tilde{w}_t + F\tilde{w}_t + F_x^2 + FF_x\tilde{w}_x + (\tilde{w}_{xx} + \tilde{w}_x^2)F^2\big). \tag{2.44}$$

Taking the L^2-norm in (2.44), we have

$$\begin{aligned} w_3(t)\|F_{xx}\|^2 &\le Cw_3(t)\big(\|F_t\|^2 + \|(a-a_1)\tilde{w}_t\|^2 + \|F\tilde{w}_t\|^2 + \|F_x^2\|^2 \\ &\qquad + \|FF_x\tilde{w}_x\|^2 + \|(\tilde{w}_{xx} + \tilde{w}_x^2)F^2\|^2\big) \\ &\le C(1 + w_3(t)\|F_x^2\|^2) \\ &\le C(1 + w_3(t)\|F_x\|^2(\|F_x\|^2 + \|F_{xx}\|^2)) \\ &\le C + Cw_3(t)\|F_x\|^2\|F_{xx}\|^2, \end{aligned}$$

which implies that

$$w_3(t)\|F_{xx}\|^2 \le C.$$

Then, we have from the Sobolev inequality that

$$\begin{aligned} \|F_x\|_{L^\infty}^2 &\le C\|F_x\|\|F_{xx}\| \\ &\le Cw_3(t)^{-1/2}w_2(t)^{-1/2}. \end{aligned}$$

Finally, we take the L^∞-norm in (2.44) to obtain

$$\begin{aligned}\|F_{xx}\|_{L^\infty} &\le C\big(\|F_t\|_{L^\infty} + \|(a-a_1)\tilde{w}_t\|_{L^\infty} + \|F_x\|^2_{L^\infty}\\ &\qquad + \|FF_x\tilde{w}_x\|_{L^\infty} + \|F^2(\tilde{w}_{xx}+\tilde{w}_x^2)\|_{L^\infty}\big)\\ &\le C(w_3(t)w_4(t))^{-1/4}.\end{aligned}$$

Theorem 2.2 then follows from Theorem 2.6, Lemmas 2.7–2.8, and Corollary 2.9. □

It is now easy to obtain the solution ϕ of (2.4) and then the unique smooth solution w of (2.2). By defining $\tilde{v} = a^{-1}(x)w$ and $\tilde{u} = -(w^{-\gamma})_x$, we obtain the solution of (1.8). Hence, with the help of (2.26), we have the following theorem.

THEOREM 1. *Assume that $\tilde{v}_0 - v_+ \in L^1$. There exist $\varepsilon_0 > 0$ and δ such that, if $|v_+ - v_-| \le \delta$ and $\|a(x)\tilde{v}_0(x) - \tilde{w}(0,x)\|^2_H \le \varepsilon_0$, then problem* (1.8) *has a unique global classical solution $(\tilde{v}, \tilde{u}, s)$ satisfying*

$$\|\tilde{v} - \bar{v}\|_{L^\infty} \le C(1+t)^{-1/2}(1+\log(1+t))^{\beta_1},$$

$$\|\tilde{u} - \bar{u}\|_{L^\infty} \le C(1+t)^{-1}(1+\log(1+t))^{\beta_2},$$

where $\bar{v} = a^{-1}(x)\tilde{w}$ and $\bar{u} = -(\tilde{w}^{-\gamma})_x$ and where $\beta_1 > \frac{1}{3}$ and $\beta_2 > \frac{1}{2}$ are constants.

3. Convergence to Similarity Solutions

In this section we shall prove the global existence and large time behavior for solutions of the problem (P1). Since the result for $s(x,t)$ is clear, we deal only with $(v,u)(x,t)$ in this section.

Let $(\tilde{v}, \tilde{u}, s(x))$ be the solution of (1.8) obtained in Theorem 1. As in [24], we introduce the auxiliary function $(\hat{v}, \hat{u})(x,t)$ as follows:

$$\begin{cases} \hat{v}(x,t) = (u_0(0_-u_+)m_0(x)e^{-t}, \\ \hat{u}(x,t) = \big[(u_0(0) - u_+)\int_x^{+\infty} m_0(\xi)\,d\xi + u_+\big]e^{-t}, \end{cases} \tag{3.1}$$

where m_0 is a smooth function satisfying

$$\int_0^{+\infty} m_0(x)\,dx = 1, \quad \operatorname{supp} m_0(x) \subset \mathbf{R}^+.$$

It is easy to see that $(\hat{v}, \hat{u})$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\hat{u}, \\ (\hat{u}, \hat{v}, \hat{u}_x)(0,t) = (u_0(0)e^{-t}, 0, 0), \\ (\hat{v}, \hat{u})(+\infty, t) = (0, u_+e^{-t}). \end{cases} \tag{3.2}$$

Take $v_e = v - \tilde{v} - \hat{v}$ and $u_e = u - \tilde{u} - \hat{u}$. It follows from (1.2) and (1.8) that

$$\begin{cases} v_{et} - u_{ex} = 0, \\ u_{et} + [p(\tilde{v} + \hat{v} + v_e, s) - p(\tilde{v}, s)]_x = -u_e + p(\tilde{v}, s)_{xt}. \end{cases} \tag{3.3}$$

Define

$$y = -\int_x^\infty v_e(\xi)\,d\xi, \tag{3.4}$$

which satisfies the nonlinear wave equation

$$\begin{cases} y_{tt} + [p(y_x + \tilde{v} + \hat{v}, s) - p(\tilde{v}, s)]_x + y_t = p(\tilde{v}, s)_{xt}, \quad x \in \mathbf{R}^+, \\ y_x(0,t) = 0, \\ y(x,0) = y_0(x) = -\int_x^\infty (v_0(\xi) - \tilde{v}_0(\xi, 0) - \hat{v}(\xi, 0))\,d\xi, \\ y_t(x,0) = y_1(x) = u_0(x) - \tilde{u}(x,0) - \hat{u}(x,0), \end{cases} \tag{3.5}$$

since $y_x = v_e$ and $y_t = u_e$. The main result of this section is the following.

THEOREM 3.1. *There exists some $\delta_0 > 0$ such that, if $0 < \delta < \delta_0$ and $\|y_0\|_{H^3} + \|y_1\|_{H^2} + |v_+ - v_-| \le \delta$, then* (3.5) *has a unique smooth solution y satisfying*

$$\|y(t)\|_{H^3}^2 + \|y_t(t)\|_{H^2}^2 + \int_0^t \|(y_x, y_t)(\tau)\|_{H^2}\,d\tau \le C\delta^2.$$

Moreover,

$$(1+t)\|y_x(\cdot,t)\|^2 + (1+t)^2\|y_t(\cdot,t)\|^2 \le C \tag{3.6}$$

and

$$\|y_x(\cdot,t)\|_{L^\infty} \le C(1+t)^{-3/4}, \qquad \|y_t(\cdot,t)\|_{L^\infty} \le C(1+t)^{-5/4}. \tag{3.7}$$

The combination of Theorem 3.1 and Theorem 2.2 gives the following theorem.

THEOREM 2. *Let y be the solution of* (3.5) *in Theorem 3.1 and let $(\tilde{v}, \tilde{u}, s)$ be the solution of* (1.8) *obtained in Theorem 1. Then, by defining $(v,u,s)(x,t) = (\tilde{v} + \hat{v} + y_x, \tilde{u} + \hat{u} + y_t, s)$, we obtain the unique global classical solution (v,u,s) of* (P1) *that satisfies*

$$\|v - \tilde{v}\|_{L^\infty} \le C(1+t)^{-3/4} \quad \textit{and} \quad \|u - \tilde{u}\|_{L^\infty} \le C(1+t)^{-5/4}$$

as well as

$$\|v - \bar{v}\|_{L^\infty} \le C(1+t)^{-1/2}(1 + \log(1+t))^{\beta_1},$$
$$\|u - \bar{u}\|_{L^\infty} \le C(1+t)^{-1}(1 + \log(1+t))^{\beta_2}.$$

Here $\bar{v}$, $\bar{u}$, β_1, β_2 are the same as in Theorem 1.

We now prove Theorem 3.1. First of all, we have the following.

THEOREM 3.2. *There exists a $\delta_0 > 0$ such that, if $0 < \delta < \delta_0$ and $\|y_0\|_{H^3} + \|y_1\|_{H^2} + |v_+ - v_-| \le \delta$, then* (3.5) *has a unique global smooth solution satisfying*

$$\|y(t)\|_{H^3}^2 + \|y_t(t)\|_{H^2}^2 + \int_0^t \|(y_x, y_t)(\tau)\|_{H^2}\,d\tau \le C\delta^2 \tag{3.8}$$

and

$$\lim_{t\to\infty} (\|y(\cdot,t)\|_{L^\infty} + \|(y_t, y_x)(\cdot,t)\|_{H^1}) = 0.$$

Proof. It is sufficient to prove the uniform estimates (3.8) under the a priori assumption

$$\|y(t)\|_{H^3}^2 + \|y_t\|_{H^2}^2 \le \varepsilon$$

for $\varepsilon > 0$ suitably small.

Multiplying $(3.5)_1$ by $y + 2y_t$, we have

$$\begin{aligned}&\left[y_t^2 + \tfrac{1}{2}y^2 + yy_t + 2q\right]_t + y_t^2 - p_v(\tilde{v} + \theta(y_x + \hat{v}), s)(y_x^2 + \hat{v}y_x)\\ &\quad = O(1)\big(\tilde{v}_t(y_x^2 + \hat{v})^2 + \hat{v}_t(|y_x| + |\hat{v}|)\big) + p(\tilde{v}, s)_{xt}(y + 2y_t) + \{\cdots\}_x, \qquad (3.9)\end{aligned}$$

where $\theta \in [0, 1]$ and $q = -\int_0^{y_x+\hat{v}}[p(\tilde{v} + \xi, s) - p(\tilde{v}, s)]\, d\xi$.

We now observe the following estimates:

$$\begin{cases} q = O(1)(y_x + \hat{v})^2, \quad |\hat{v}y_x| \le C(\alpha_1)\hat{v}^2 + \alpha_1 y_x^2, \\ \int_0^t \int_0^\infty \hat{v}^2 + \int_0^\infty \hat{v}^2\, dx \le C\delta^2; \end{cases} \qquad (3.10)$$

$$\begin{aligned}&\int_0^t \int_0^\infty \big(\tilde{v}_t(y_x + \hat{v})^2 + \hat{v}_t(|y_x| + |\hat{v}|)\big)\, dx\, d\tau\\ &\qquad \le C\delta^2 + C\delta \int_0^t \int_0^\infty y_x^2\, dx\, d\tau, \qquad (3.11)\end{aligned}$$

$$\begin{aligned}&\left|\int_0^t \int_0^\infty p(\tilde{v}, s)_{xt}(y + 2y_t)\, dx\, d\tau\right|\\ &\qquad \le \alpha_1 \int_0^t \int_0^\infty (y_x^2 + y_t^2)\, dx\, d\tau + C(\alpha_1)\delta^2. \qquad (3.12)\end{aligned}$$

By choosing α_1, ε, and δ suitably small, we may integrate (3.9) over $[0, t] \times [0, \infty)$ and thus obtain

$$\|(y, y_t, y_x)(t)\|^2 + \int_0^t \|(y_x, y_t)(\tau)\|^2\, d\tau \le C\delta^2. \qquad (3.13)$$

Here, the boundary conditions at $x = 0$ are given as

$$0 = y_x = p(\tilde{v}, s)_t = p(\tilde{v} + \hat{v} + y_x, s) - p(\tilde{v}, s) = y_{tx} = y_{ttx} = \cdots. \qquad (3.14)$$

We now differentiate (3.5) in x and arrive at

$$y_{ttx} + [p(\tilde{v} + \hat{v} + y_x, s) - p(\tilde{v}, s)]_{xx} + y_{tx} = p(\tilde{v}, s)_{txx}. \qquad (3.15)$$

Then, we multiply (3.15) by $y_x + 2y_{tx}$; after a long but routine computation (see [19] or our previous calculation), we have

$$\|(y_x, y_{tx}, y_{xx})(t)\|^2 + \int_0^t \|(y_{tx}, y_{xx})(\tau)\|^2\, d\tau \le C\delta^2. \qquad (3.16)$$

Repeating this procedure, we can easily obtain the third-order estimates and so complete the proof of Theorem 3.2. □

We now investigate the problem of the decay rate. We will follow the approach introduced in [21] for the isentropic case. However, since the entropy $s(x)$ is not constant here, some modifications are needed.

LEMMA 3.3. *Let y be the solution of* (3.5) *in Theorem 3.2 and let $V = p_v(\tilde{v}, s)y_x$. Then*

$$\sum_{k=0}^{1}[(1+t)^{k+1}\|\partial_x^k V(\cdot,t)\|^2 + (1+t)^{k+2}\|\partial_x^k y_t(\cdot,t)\|^2] \le C.$$

Proof. We first multiply $(3.5)_1$ by $(1+t)y_t$; after some calculations, this yields

$$\begin{aligned}\Big[(1+t)\Big(\frac{1}{2}y_t^2+q\Big)\Big]_t &+ (1+t)y_t^2 - q + (1+t)\hat{v}_t[p(\tilde{v}+\hat{v}+y_x,s)-p(\tilde{v},s)]\\ &+ (1+t)\tilde{v}_t\int_0^{y_x+\hat{v}}[p_v(\tilde{v}+\xi,s)-p_v(\tilde{v},s)]\,d\xi - \frac{1}{2}y_t^2\\ &= (1+t)y_t p(\tilde{v},s)_{xt} + \{\cdots\}_x. \qquad (3.17)\end{aligned}$$

Integrating (3.17) over $[0,t]\times(0,+\infty)$, with the help of (3.8) we obtain

$$\begin{aligned}(1+t)\|(y_x,y_t)(t)\|^2 &+ \int_0^t(1+\tau)\|y_t(\tau)\|^2\,d\tau\\ &\le C\delta^2 + C\int_0^t\|p(\tilde{v},s)_{xt}\|^2(1+\tau)\,d\tau\\ &\le C\delta^2. \qquad (3.18)\end{aligned}$$

Then we note that

$$p(\tilde{v}+\hat{v}+y_x)-p(\tilde{v},s) = p_v(\tilde{v},s)(y_x+\hat{v}) + F_1(y_x,\hat{v},\tilde{v},s)(y_x+\hat{v})^2. \qquad (3.19)$$

Differentiating $(3.5)_1$ in t, we have

$$y_{ttt} + (p_v(\tilde{v},s)y_x)_{xt} + y_{tt} = p(\tilde{v},s)_{xtt} - (F_1(y_x+\hat{v})^2)_{xt} - (p_v(\tilde{v},s)\hat{v})_{xt}. \qquad (3.20)$$

Let us multiply (3.20) by $(1+t)y_t$ and $(1+t)y_{tt}$. Then we deduce (respectively)

$$\begin{aligned}\big[(1+t)\big(y_t y_{tt} + \tfrac{1}{2}y_t^2\big)\big]_t &- p_v(\tilde{v},s)(1+t)y_{tx}^2\\ &- (1+t)y_{tt}^2 - \tfrac{1}{2}y_t^2 - y_t y_{tt} - p_{vv}\tilde{v}_t(1+t)y_x y_{tx}\\ &= (1+t)y_t\big(p(\tilde{v},s)_{xtt} - (F_1(y_x+\hat{v}^2)_{xt} - p_v(\tilde{v},s)\hat{v})_{xt}\big) + \{\cdots\} \qquad (3.21)\end{aligned}$$

and

$$\begin{aligned}\big[\tfrac{1}{2}(1+t)(y_{tt}^2 - p_v y_{tx}^2)\big]_t &+ (1+t)y_{tt}^2 - \tfrac{1}{2}y_{tt}^2 + \tfrac{1}{2}p_v y_{tx}^2\\ &+ \tfrac{1}{2}(1+t)p_{vv}\tilde{v}_t y_{tx}^2 + (1+t)y_{tt}(y_x p_{vv}\tilde{v}_t)_x\\ &= (1+t)y_{tt}\big(p(\tilde{v},s)_{xtt} - (F_1(y_x+\hat{v}^2)_{xt} - p_v(\tilde{v},s)\hat{v})_{xt}\big) + \{\cdots\}. \qquad (3.22)\end{aligned}$$

Using (3.8) and (3.18) and integrating $8\times(3.22)+(3.21)$, one has

$$\begin{aligned}(1+t)\|(y_{tt},y_{tx})\|^2 &+ \int_0^t(1+\tau)\|(y_{tt},y_{tx})(\tau)\|^2\,d\tau\\ &\le C\delta^2 + C\left|\int_0^t\int_0^{+\infty}(1+\tau)(y_{tx}+y_{ttx})(F_1 y_x^2)_t\,dx\,d\tau\right|. \qquad (3.23)\end{aligned}$$

We see that

$$
\begin{aligned}
&\left|\int_0^t\int_0^{+\infty}(1+\tau)y_{tx}(F_1y_x^2)_t\,dx\,d\tau\right| \\
&\qquad\le C\int_0^t\int_0^{+\infty}(1+\tau)(|y_x|y_{tx}^2+|\tilde v_t y_x^2 y_{tx}|)\,dx\,d\tau \\
&\qquad\le C\delta^2+C\delta\int_0^t\int_0^{+\infty}(1+\tau)y_{tx}^2\,dx\,d\tau
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
&\left|\int_0^t\int_0^{+\infty}(1+\tau)y_{ttx}(F_1y_x^2)_t\,dx\,d\tau\right| \\
&\qquad\le C\delta^2+C\delta\biggl((1+t)\|y_{tx}(t)\|^2+\int_0^t(1+\tau)\|y_{tx}\|^2\,d\tau\biggr).
\end{aligned}
\tag{3.25}
$$

In view of the smallness of δ, from (3.23)–(3.25) we conclude that

$$
(1+t)\|(y_t,y_{tt},y_{tx})(t)\|^2+\int_0^t(1+\tau)\|(y_{tt},y_{tx})(\tau)\|^2\,d\tau\le C. \tag{3.26}
$$

Now we multiply (3.20) by $(1+t)^2y_t$ and $(1+t)^2y_{tt}$ and then repeat the previous calculations; this yields

$$
(1+t)^2\|(y_t,y_{tt},y_{tx})(t)\|^2+\int_0^t(1+\tau)^2\|(y_{tt},y_{tx})(\tau)\|^2\,d\tau\le C. \tag{3.27}
$$

We turn now to estimating $V=p_v(\tilde v,s)y_x$ and the higher-order derivatives. It is easy to see from the preceding estimates and equations $(3.5)_1$ and (3.19) that

$$
(1+t)^2\|(V_t,V_x)(t)\|^2+\int_0^t(1+\tau)\|(V_t,V_x)(\tau)\|^2\,d\tau\le C. \tag{3.28}
$$

Now differentiate (3.20) with respect to x and repeat the previous arguments to obtain

$$
(1+t)^2\|(y_{ttx},y_{txx})(t)\|^2+\int_0^t(1+\tau)^2\|(y_{ttx},y_{txx})(\tau)\|^2\,d\tau\le C. \tag{3.29}
$$

Finally, multiply (3.20) by $(1+t)^3y_{tt}$; this yields

$$
\begin{aligned}
&(1+t)^3\|(y_{tt},y_{tx})(t)\|^2+\int_0^t(1+\tau)^3\|y_{tt}(\tau)\|^2\,d\tau \\
&\qquad\le C+C\left|\int_0^t\int_0^{+\infty}(1+\tau)^3y_{ttx}(F_1y_x^2)_t\,dx\,d\tau\right|.
\end{aligned}
\tag{3.30}
$$

The last term in (3.30) can be estimated as follows:

$$\begin{aligned}
&\left|\int_0^t \int_0^{+\infty} (1+\tau)^3 y_{ttx} (F_1 y_x^2)_t \, dx \, d\tau\right| \\
&\quad \le C\left|\int_0^t \int_0^{+\infty} [O(1)(1+\tau)^3 y_{tx}^2 y_x]_t \, dx \, d\tau\right| \\
&\qquad + C\int_0^t \int_0^{+\infty} ((1+t)^2 y_{tx}^2 + (1+\tau)^3 |y_{tx}|^3) \, dx \, d\tau \\
&\qquad + C\left|\int_0^t \int_0^{+\infty} (1+\tau)^3 y_{tt} (F_3 \tilde{v}_t V^2)_x \, dx \, d\tau\right| \\
&\quad \le C(\alpha_1) + C\int_0^t \int_0^{+\infty} (1+\tau)^3 |y_{tx}|^3 \, dx \, d\tau \\
&\qquad + C\delta(1+t)^3 \|y_{tx}(t)\|^2 + \alpha_1 \int_0^t (1+\tau)^3 \|y_{tt}\|^2 \, d\tau,
\end{aligned} \tag{3.31}$$

where $F_3 V^2 = F_{1\tilde{v}} y_x^2$. By choosing α_1 suitably small, we conclude from (3.30)–(3.31) that

$$(1+t)^3 \|(y_{tt}, y_{tx})(t)\|^2 + \int_0^t (1+\tau)^3 \|y_{tt}\|^2 \, d\tau \le C. \tag{3.32}$$

Here we have used the following estimates:

$$\begin{aligned}
\int_0^t \int_0^{+\infty} (1+\tau)^3 |y_{tx}|^3 \, dx \, d\tau &\le \int_0^t \int_0^{+\infty} (1+\tau)^4 y_{tx}^4 + (1+\tau)^2 y_{tx}^2) \, dx \, d\tau \\
&\le C + C\int_0^t (1+\tau)^4 \|y_{tx}\|^2 (\|y_{tx}\|^2 + \|y_{txx}\|^2) \, d\tau \\
&\le C.
\end{aligned}$$

Lemma 3.3 follows from (3.18), (3.27), (3.28), and (3.29). □

Lemma 3.3 and the interpolation inequality together imply that

$$\|y_x(\cdot, t)\|_{L^\infty} \le C(1+t)^{-3/4}, \qquad \|y_t(\cdot, t)\|_{L^\infty} \le C(1+t)^{-5/4}, \tag{3.33}$$

where

$$\begin{aligned}
\|y_x(\cdot, t)\|_{L^\infty} &\le C\|V(\cdot, t)\|_{L^\infty} \\
&\le C(1+t)^{-3/4}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

4. Convergence to Stationary Solutions

The aim of this section is to apply essentially the same technique used in Sections 2 and 3 to deal with the hyperbolic problem (P2) and the diffusive problem (1.9).

Similarly to Section 2, the problem (1.9) is equivalent to

$$\begin{cases} w_t + a(x)(w^{-\gamma})_{xx} = 0, \quad x \in \mathbf{R}^+, \\ w_x(0,t) = 0, \\ w(x,0) = w_0(x) \to w_+ \text{ as } x \to \infty. \end{cases} \tag{4.1}$$

The definition of w and $a(x)$ can be found in Section 2. Setting $\phi = w - w_+$, we have

$$\begin{cases} \phi_t + a(x)\phi_{xx} + a(x)(f_1(\phi)\phi^2)_{xx} = 0, \quad x \in \mathbf{R}^+, \\ \phi_x(0,t) = 0, \\ \phi(x,0) = \phi_0(x) = w_0(x) - w_+, \end{cases} \tag{4.2}$$

where $b = \gamma w_+^{-(\gamma+1)}$ and $f_1\phi^2 = (w_+ + \phi)^{-\gamma} = w_+^{-\gamma} - b\phi$. Observe that if $\phi_0(x) \in L^1$ then we can use the same argument as in Lemma 2.3 to prove

$$\|\phi(\cdot,t)\|_{L^1} \le C\|\phi_0\|_{L^1}. \tag{4.3}$$

The same approach as used in Section 2 gives the following results.

THEOREM 4.1. *Suppose that $\phi_0(x)$ and $s_0(x)$ are smooth functions such that $\phi_0 \in H^2 \cap L^1$. There exists some δ_0 such that, if $0 < \delta \le \delta_0$ and $\|\phi_0\|_{H^2} \le \delta$, then* (4.2) *has a unique global smooth solution $\phi(x,t)$ satisfying*

$$\sum_{j=0}^{2} w_{j+1}(t)\|\partial_x^j \phi(\cdot,t)\|^2 + \int_0^t \sum_{j=1}^{3} w_j(\tau)\|\partial_x^j \phi(\cdot,\tau)\|^2 \, d\tau \le C.$$

By defining $v_1(x,t) = a^{-1}w(x,t) = a^{-1}(w_+ + \phi(x,t))$, $u_1(x,t) = -p(v_1,s)_x$, and $v_* = a^{-1}w_+$, we obtain the unique global smooth solution of problem (1.9), *(v_1,u_1,s), satisfying*

$$\|v_1 - v_*\|_{L^\infty} \le C(1+t)^{-1/2}(1+\log(1+t))^{\beta_1},$$
$$\|u_1\|_{L^\infty} \le C(1+t)^{-1}(1+\log(1+t))^{\beta_2}.$$

Here the constants β_1 and β_2 are the same as in Theorem 1.

Define the auxiliary function $(\hat v_1, \hat u_1)$ by

$$(\hat v_1, \hat u_1)(x,t) = \left(u_+ m_0 e^{-t}, u_+ e^{-t} \int_0^x m_0(\xi)\, d\xi \right), \tag{4.4}$$

where $m_0(x)$ is the same as in Section 3. Setting $\tilde y = -\int_x^\infty (v - v_1 - \hat v_1)(\xi,t)\, d\xi$ then yields

$$\begin{cases} \tilde y_{tt} + [p(v_1 + \hat v_1 + \tilde y_x, s) - p(v_2,s)]_x + \tilde y_t = p(v_1,s)_{xt}, \quad x \in \mathbf{R}^+, \\ \tilde y(0,t) = 0, \\ \tilde y(x,0) = \tilde y_0(x) = -\int_x^\infty (v_0(\xi) - v_1(\xi,0) - \hat v_1(\xi,0))\, d\xi, \\ \tilde y_t(x,0) = \tilde y_1(x) = u_0(x) - u_1(x,0) - \hat u(x,0). \end{cases} \tag{4.5}$$

Thus, similarly to Section 3, one can easily prove the following theorem.

THEOREM 4.2. *There exists a δ_0 such that, if $0 < \delta \le \delta_0$ and $\|\tilde{y}_0\|_3^2 + \|\tilde{y}_1\|_2^2 \le \delta^2$, then* (4.5) *has a unique smooth solution $\tilde{y}$ satisfying*

$$\sum_{k=0}^{1}[(1+t)^{k+1}\|\partial_x^k V_1(\cdot,t)\|^2 + (1+t)^{k+2}\|\partial_x^k \tilde{y}_t(\cdot,t)\|^2] \le C, \tag{4.6}$$

with $V_1 = p_v(v_1, s)\tilde{y}_x$ and

$$\|\tilde{y}_x(\cdot,t)\|_{L^\infty} \le C(1+t)^{-3/4}, \qquad \|\tilde{y}_t(\cdot,t)\|_{L^\infty} \le C(1+t)^{-5/4}. \tag{4.7}$$

With the aid of Theorems 4.1 and 4.2, it is now easy to obtain the following results.

THEOREM 4.3. *Let $\tilde{y}$ be the solution of* (4.7) *in Theorem 4.2 and let (v_1, u_1, s) be the solution of* (1.9) *obtained in Theorem 4.1. Then, by defining $(v, u, s)(x,t) = (v_1 + \hat{v}_1 + \tilde{y}_x, u_1 + \hat{u}_1 + \tilde{y}_t, s)$, we obtain the unique global classical solution (v, u, s) of* (P2) *that satisfies*

$$\|v - v_1\|_{L^\infty} \le C(1+t)^{-3/4} \quad \text{and} \quad \|u - u_1\|_{L^\infty} \le C(1+t)^{-5/4}$$

as well as

$$\|v - \bar{v}\|_{L^\infty} \le C(1+t)^{-1/2}(1+\log(1+t))^{\beta_1},$$
$$\|u\|_{L^\infty} \le C(1+t)^{-1}(1+\log(1+t))^{\beta_2}.$$

REMARK. We note that $w = w_+$ is a stationary solution of the equation $(4.1)_1$, and thus $(\bar{v}, \bar{u}, s) = (a^{-1}w_+, 0, s(x))$ is the stationary solution of system (1.2). Hence, by choosing suitable initial data (v_0, u_0, s_0), one can derive the solution of (P2), (v, u, s), converging to (v_2, u_2, s) with faster rates. More precisely, we have the following theorem.

THEOREM 4.4. *Assume that $v_0 - v_+ \in L^1$. Let $z(x,t) = -\int_x^\infty (v - \bar{v} - \hat{v}_1)(\xi, t)\,d\xi$, $z_0 = (x, 0)$, and $z_1 = u_0 - \hat{u}_1(x, 0)$. There exists some $\delta_0 > 0$ such that, if $0 < \delta \le \delta_0$ and $\|z_0\|_{H^3} + \|z_1\|_{H^2} \le \delta$, then* (P2) *has a unique global smooth solution (v, u, s) such that*

$$\|v - \bar{v}\|_{L^\infty} \le C(1+t)^{-3/4}, \qquad \|u\|_{L^\infty} \le C(1+t)^{-5/4}.$$

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