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Rigidity of Connected Limit Sets of Conformal IFSs

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1. Introduction and Preliminaries

In this paper we explore the structure of limit sets J of infinite conformal iterated function systems whose closure is a continuum (compact connected set). Under a natural easily verifiable technical condition (always satisfied if the system is finite), we demonstrate the following dichotomy. Either the Hausdorff dimension of J exceeds 1 or else $\bar{J}$ is a proper compact segment of either a geometric circle or a straight line if $d \geq 3$ or an analytic interval if $d = 2$ (see Theorem 1.3). From the viewpoint of conformal dynamics, this result can be thought of as a far-reaching generalization of results originated in [S] and [B], which are formulated in the plane case. The proofs contained there use the Riemann mapping theorem and can be carried out only in the plane. The proof presented in our paper is different and holds in any dimension. The reader is also encouraged to notice an analogy between our result and a series of other papers (see e.g. [B; FU; MU1; Ma; P; R; S; U1; UV; Z1; Z2]), which are aimed toward establishing a similar dichotomy. However, to our knowledge, all these results—just as those in [B] and [S]—were formulated in the plane and used the Riemann mapping theorem, except those in [MU1]. The current result is, however, much stronger than that in [MU1]; in particular, with our present approach the main result of [MU1] can be strengthened as described at the end of this section. Another corollary of our result is the following: If a continuum C in $\mathbb{R}^d$ is the self-conformal set generated by finitely many conformal mappings satisfying the open set condition, if the Hausdorff 1-measure of C is finite, and if one of the mappings is a similarity, then the continuum is a line segment. This holds in particular if all the maps are similarities, a result obtained early on by Mattila [Ma].

To start the preliminaries, let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an *iterated function system* (IFS). We are especially interested in the properties of the

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limit set defined by such a system. We define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\geq 1} I^n$, the space of finite words, and for $\tau \in I^n$ $(n \geq 1)$ let $\phi_\tau = \phi_{\tau_1} \circ \phi_{\tau_2} \circ \cdots \circ \phi_{\tau_n}$. Let $I^\infty = \{\{\tau_n\}_{n=1}^\infty\}$ be the set of all infinite sequences of elements of I. If $\tau \in I^* \cup I^\infty$ and if $n \geq 1$ does not exceed the length of τ, then we denote by $\tau|_n$ the word $\tau_1\tau_2\ldots\tau_n$. Since the diameters of the compact sets $\phi_{\tau|_n}(X)$ $(\tau \in I^\infty, n \geq 1)$ converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\tau|_n}(X)$$

is a singleton; therefore, denoting its only element by $\pi(\tau)$, we define the coding map

$$\pi : I^\infty \to X.$$

The main object in the theory of iterated function systems is the limit set defined as

$$J = \pi(I^\infty) = \bigcup_{\tau \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X).$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite then J is compact and this property fails for infinite systems. Let $S(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X)$, $i \in I'$, where I' ranges over all infinite subsets of I. The following was proved in [MU2].

PROPOSITION 1.1. *If* $\lim_{i\in I} \operatorname{diam}(\phi_i(X)) = 0$, *then* $\bar{J} = J \cup \bigcup_{\omega \in I^*} \phi_\omega(S(\infty))$.

An iterated function system S is said to be *conformal* if $X \subset \mathbb{R}^d$ for some $d \geq 1$ and the following conditions are satisfied.

(1a) Open set condition (OSC): $\phi_i(\operatorname{Int} X) \cap \phi_j(\operatorname{Int} X) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.

(1b) There exists an open connected set V such that $X \subset V \subset \mathbb{R}^d$ and such that all maps ϕ_i $(i \in I)$ extend to C^1 conformal diffeomorphisms of V into V. (Note: for $d = 1$, this just means that all the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms; for $d = 2$, the words *conformal* mean holomorphic or antiholomorphic; for $d \geq 3$, the maps ϕ_i, $i \in I$, are Möbius transformations. The proof of this last claim can be found e.g. in [BP], where it is called Liouville's theorem.)

(1c) Cone condition: There exist $\alpha, l > 0$ such that, for every $x \in \partial X \subset \mathbb{R}^d$, there exists an open cone $\operatorname{Con}(x, u, \alpha) \subset \operatorname{Int}(X)$ with vertex x, where the symmetry axis is determined by a vector $u \in \mathbb{R}^d$ of length l and a central angle of Lebesgue measure α. Here $\operatorname{Con}(x, u, \alpha) = \{y : 0 < (y - x, u) \leq \cos\alpha \|y - x\| \leq l\}$.

(1d) Bounded distortion property (BDP): There exists $K \geq 1$ such that

$$|\phi_\tau'(y)| \leq K|\phi_\tau'(x)|$$

for every $\tau \in I^*$ and every pair of points $x, y \in V$, where $|\phi_\tau'(x)|$ denotes the norm of the derivative.

Under these assumptions, it was shown in [MU2] that the hypothesis of Proposition 1.1 holds and we can change the order of the union and intersection operations to obtain

$$J = \pi(I^\infty) = \bigcap_{n\ge 1} \bigcup_{|\tau|=n} \phi_\tau(X).$$

In fact, throughout the whole paper we will need one additional condition, which (cf. [MU2]) can be considered as a strengthening of the BDP.

(1e) There are two constants $L \ge 1$ and $\alpha > 0$ such that

$$\big||\phi_i'(y)| - |\phi_i'(x)|\big| \le L\|\phi_i'\||y-x|^\alpha$$

for every $i \in I$ and every pair of points $x, y \in V$.

We remark that, in the case $d \ge 3$, conditions (1d) and (1e) are always satisfied—the latter with $\alpha = 1$.

Let us first collect some geometric consequences of the BDP. For all words $\tau \in I^*$ and all convex subsets C of V we have

$$\operatorname{diam}(\phi_\tau(C)) \le \|\phi_\tau'\| \operatorname{diam}(C) \tag{1.1}$$

and

$$\operatorname{diam}(\phi_\tau(V)) \le D\|\phi_\tau'\|, \tag{1.2}$$

where the norm $\|\cdot\|$ is the supremum norm taken over V and where $D \ge 1$ is a universal constant. Moreover,

$$\operatorname{diam}(\phi_\tau(J)) \ge D^{-1}\|\phi_\tau'\| \tag{1.3}$$

and

$$\phi_\tau(B(x,r)) \supset B(\phi_\tau(x), K^{-1}\|\phi_\tau'\|r) \tag{1.4}$$

for every $x \in X$, every $0 < r \le \operatorname{dist}(X, \partial V)$, and every word $\tau \in I^*$.

Let us state now an important geometrical feature of conformal systems that is related to the bounded distortion property. A detailed proof of this fact can be obtained by a slight improvement of Lemma 6 in [MU1].

LEMMA 1.2. *For every $\beta > 0$ and every $0 < \alpha < \beta$ there exists an $\eta > 0$ such that, for every $x \in X$, every $u \in \mathbb{R}^d$ with $\|u\| \le \eta$, and every $\omega \in I^*$, we have*

$$\phi_\omega(\operatorname{Con}(x,u,\alpha)) \subset \operatorname{Con}(\phi_\omega(x), 2\phi_\omega'(x)u, \beta).$$

Let us now recall from [MU2] that a Borel probability measure m is said to be t-*conformal* if $m(J) = 1$ and if, for every Borel set $A \subset X$ and every $i \in I$,

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0$$

for every pair $i, j \in I$, $i \ne j$. It was proved in [MU2] that if a t-conformal measure exists then $t = h$, the Hausdorff dimension of the limit set J_S of S, and that this measure is unique. The system S is called *regular* if a conformal measure exists. The main result of our paper is the following.

THEOREM 1.3. *If* $d \geq 3$, $S = \{\phi_i\}_{i \in I}$ *is a conformal IFS,* $\bar{J}$ *is a* (*compact connected*) *continuum, and* $\dim_H(S(\infty)) < \dim_H(J)$, *then*: *either*

(a) $\dim_H(J) > 1$; *or*
(b) $\bar{J}$ *is a proper compact segment of either a geometric circle or a straight line.*

In addition, if any one of the maps ϕ_i *is a similarity mapping then* $\bar{J}$ *is a line segment*.

We note that the technical condition in Theorem 1.3 is necessary. Example 5.2 of [MU2] shows that the dichotomy of Theorem 1.3 fails in general if $\dim_H(S(\infty)) \geq \dim_H(J)$. We also mention that, once the first part of this theorem is proved, the "in addition" part follows immediately from the proof of Lemma 2.5.

We would also like to remark that, in the case $d = 2$, for every $i \in I$ we have that ϕ_{ii} is a holomorphic map that is biholomorphically conjugate with the linear map $\psi(z) = x_{ii} + \phi'(x_{ii})(z - x_{ii})$ on some neighborhood W of x_{ii}. Proceeding then similarly as in the proof of Theorem 1.3, we could demonstrate the same statement with the segment of the line or the circle replaced by an analytic arc.

Because the set $S(\infty)$ is empty in the finite case, from Theorem 1.3 we may immediately deduce the following.

COROLLARY 1.4. *If* $d \geq 2$, $S = \{\phi_i\}_{i \in I}$ *is a finite conformal IFS, and* $\bar{J}$ *is a continuum, then*: *either*

(a) $\dim_H(J) > 1$; *or*
(b) $\bar{J}$ *is a proper compact segment of either a geometric circle or a straight line.*

In addition, if any one of the maps ϕ_i *is a similarity mapping then* $\bar{J}$ *is a line segment*.

We note that the methods of this paper can be used to strengthen the theorem [MU1, p. 88], which concerns conformal repellers, by replacing the words "smooth Jordan curve" by "geometric circle" if $d \geq 3$ or by "a real-analytic Jordan curve" if $d = 2$.

2. Proof of Theorem 1.3

The proof of this theorem will consist of several steps. First of all we assume in the sequel that the assumptions of Theorem 1.3 are satisfied and $\dim_H(J) = 1$. Our goal is to show that then item (b) is satisfied. Since $\dim_H(S(\infty)) < \dim_H(J) = 1$ and $\bar{J}$ is a continuum, we conclude using Proposition 1.1 that $\mathcal{H}^1(J) > 0$. It therefore follows from [MU2, Thm. 4.16] that the system S is regular. Let m be the corresponding 1-conformal measure. By [MU2, Lemma 4.2] and since $\dim_H(S(\infty)) < \dim_H(J) = 1$, the 1-dimensional Hausdorff measure $\mathcal{H}^1$ on $\bar{J}$ is absolutely continuous with respect to m and $d\mathcal{H}^1/dm$ is uniformly bounded away from infinity. Hence, $\bar{J}$ is a continuum whose $\mathcal{H}^1$ measure is finite. The following fact then follows from [EH] and [W].

LEMMA 2.1. *$\bar{J}$ is a locally arcwise connected continuum.*

Given $x \in \mathbb{R}^d$, $\theta \in \mathbb{PR}^d$, and $\gamma > 0$, we put

$$\mathrm{Con}(x, \theta, \gamma) = \mathrm{Con}(x, \eta, \gamma) \cup \mathrm{Con}(x, -\eta, \gamma),$$

where $\eta \in \mathbb{R}^d$ is a representative of $\theta \in \mathbb{PR}^d$. We recall that a set Y has a tangent in the direction $\theta \in \mathbb{PR}^d$ at a point $x \in Y$ if, for every $\gamma > 0$,

$$\lim_{r \to 0} \frac{\mathcal{H}^1\big(Y \cap (B(x,r) \setminus \mathrm{Con}(x,\theta,\gamma))\big)}{r} = 0.$$

We will consider only tangents of 1-sets (the preceding set $\bar{J}$; this definition coincides with the definition given in [Fa, p. 31]). Following [MU1], we say that a set Y has a *strong tangent* in the direction $\theta \in \mathbb{PR}^d$ at a point x provided that, for each $0 < \beta \le 1$, there is some $r > 0$ such that $Y \cap B(x,r) \subset \mathrm{Con}(x,\theta,\beta)$. In [MU1] we proved the following.

THEOREM 2.2. *If Y is locally arcwise connected at a point x and if Y has a tangent θ at x, then Y has strong tangent θ at x.*

We call a point $\tau \in I^\infty$ *transitive* if $\omega(\tau) = I^\infty$, where $\omega(\tau)$ is the ω-limit set of τ under the shift transformation $\sigma\colon I^\infty \to I^\infty$. We denote the set of these points by I_t^∞ and put

$$J_t = \pi(I_t^\infty).$$

We call the J_t the set of transitive points of J and notice that, for every $\tau \in I_t^\infty$, the set $\{\pi(\sigma^n\tau) : n \ge 0\}$ is dense in J (or in $\bar{J}$, if this is the space under consideration).

LEMMA 2.3. *If $\bar{J}$ has a strong tangent at a point $x = \pi(\tau)$, $\tau \in I^\infty$, then $\bar{J}$ has a strong tangent at every point $\overline{\pi(\omega(\tau))}$.*

Proof. Suppose on the contrary that $\bar{J}$ does not have a strong tangent at some point $y \in \overline{\pi(\omega(\tau))}$. Let $\theta \in \mathbb{PR}^d$ be the tangent direction of $\bar{J}$ at x and let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers such that $\lim_{k\to\infty} \pi(\sigma^{n_k}\tau) = y$. Passing to a subsequence, we may assume that

$$\lim_{k \to \infty} \frac{\big(\phi^{-1}_{\omega|_{n_k}}\big)'(x)}{\big|\big(\phi^{-1}_{\omega|_{n_k}}\big)'(x)\big|}\theta = \xi$$

for some $\xi \in \mathbb{PR}^d$. Since $\bar{J}$ does not have a strong tangent at y, it follows that there exists $0 < \beta \le 1$ such that, for every $r > 0$,

$$\bar{J} \cap B(y,r) \setminus \bar{J} \cap \mathrm{Con}(y,\xi,\beta) \ne \emptyset.$$

Then

$$\bar{J} \cap B(\pi(\sigma^{n_k}\tau), r) \setminus \bar{J} \cap \mathrm{Con}(\pi(\sigma^{n_k}\tau), \xi_k, \beta/2) \neq \emptyset \tag{2.1}$$

for all k large enough, where

$$\xi_k = \frac{(\phi_{\omega|_{n_k}}^{-1})'(x)}{|(\phi_{\omega|_{n_k}}^{-1})'(x)|}\theta.$$

But in view of Lemma 1.2 applied to $\phi_{\omega|_{n_k}}^{-1}$, we see that

$$\begin{aligned}
&\phi_{\omega|_{n_k}}\big(B(\pi(\sigma^{n_k}\tau), r) \setminus \mathrm{Con}(\pi(\tau), \xi_k, \beta/2)\big) \\
&\quad \subset B\big(x, r\|\phi'_{\omega|_{n_k}}\|\big) \setminus \mathrm{Con}\left(x, \frac{\phi'_{\omega|_{n_k}}(\pi(\sigma^{n_k}\tau))}{|\phi'_{\omega|_{n_k}}(\pi(\sigma^{n_k}\tau))|}\xi_k, \frac{\beta}{4}\right) \\
&\quad = B\big(x, r\|\phi'_{\omega|_{n_k}}\|\big) \setminus \mathrm{Con}(x, \theta, \beta/4)
\end{aligned}$$

holds for all $r > 0$ small enough. In view of (2.1), $\bar{J} \cap \phi_{\omega|_{n_k}}\big(B(\pi(\sigma^{n_k}\tau), r) \setminus \mathrm{Con}(\pi(\sigma^{n_k}\tau), \xi_k, \beta/2)\big) \neq \emptyset$ and so we conclude that, for every k large enough, $\bar{J} \cap \big(B\big(x, r\|\phi'_{\omega|_{n_k}}\|\big) \setminus \mathrm{Con}(x, \theta, \beta/4)\big) \neq \emptyset$. Since $\lim_{k\to\infty}\|\phi'_{\omega|_{n_k}}\| = 0$, this implies that θ is not the strong density direction of $\bar{J}$ at x. This contradiction finishes the proof. □

Corollary 2.4. *The continuum $\bar{J}$ has a strong tangent at every point.*

Proof. Since $\mathcal{H}^1(\bar{J}) < \infty$, in view of [Fa, Cor. 3.15] we see that $\bar{J}$ has a tangent at $\mathcal{H}^1$-a.e. point in $\bar{J}$ and hence at a set of points of positive m measure. Since $m(J_t) = 1$, there must exist at least one transitive point x in J having a tangent of J. By Theorem 2.2 and Lemma 2.1, $\bar{J}$ has a strong tangent at x; it then follows from Lemma 2.3 that $\bar{J}$ has a strong tangent at every point. The proof is complete. □

Now, the following lemma finishes the proof.

Lemma 2.5. *Suppose that $\phi\colon \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$, $d \geq 3$, is a conformal diffeomorphism that has an attracting fixed point a ($\phi(a) = a$, $|\phi'(a)| < 1$). Suppose that a compact connected set M has a strong tangent at a, that $\phi(M) \subset M$, and that $\lim_{n\to\infty}\phi^n(x) = a$ for all $x \in M$. Then M is a segment of a ϕ-invariant line or circle. If ϕ is affine ($\phi(\infty) = \infty$), then the former possibility holds.*

Proof. Since a is an attracting fixed point of ϕ, there exists a radius $r > 0$ so small that $\phi^{-1}(\overline{\mathbb{R}^d} \setminus B(a, r)) \subset \overline{\mathbb{R}^d} \setminus B(a, r)$, where $\overline{\mathbb{R}^d}$ is the Alexandrov compactification of $\mathbb{R}^d$ achieved by adding the point at infinity. Since $\overline{\mathbb{R}^d} \setminus B(a, r)$ is a topological closed ball, it follows (in view of the Brouwer fixed point theorem) that there exists a fixed point b of ϕ^{-1} in $\overline{\mathbb{R}^d} \setminus B(a, r)$. Hence b is also a fixed point of ϕ and $b \neq a$. Then the map

$$\psi = i_{b,1} \circ \phi \circ i_{b,1}$$

($i_{b,1}$ equals identity if $b = \infty$) fixes ∞, which means that this map is affine, and $w = i_{b,1}(a)$ is an attracting fixed point of ψ. In addition $\psi(\tilde{M}) \subset \tilde{M}$, where $\tilde{M} = i_{b,1}(M)$, $w \in \tilde{M}$, and $\tilde{M}$ has a strong tangent at w. Let l be the line through w determined by the strongly tangent direction of $\tilde{M}$ at w. Since $\psi(w) = w$, since $\psi(l)$ is a straight line through w, and since $\psi(\tilde{M}) \subset \tilde{M}$, we conclude that $\psi(l) = l$. Suppose now that $\tilde{M}$ is not contained in l. Consider $x \in \tilde{M} \setminus l$. Then, for every $n \geq 0$,

$$\psi^n(x) \in \psi(\tilde{M}) \setminus \psi(l) \subset \tilde{M} \setminus l;$$

since the map ψ is conformal and affine, we have

$$\angle(\psi^n(x) - w, l) = \angle(\psi^n(x - w), \psi^n(l)) = \angle(x - w, l).$$

Since $\lim_{n\to\infty} \psi^n(x) = w$, we therefore conclude that l is not a strongly tangent line of $\tilde{M}$ at w. This contradiction shows that $\tilde{M} \subset l$. Since $\tilde{M}$ is also a continuum, it is a segment of l. We are done. □

Indeed, to conclude the proof of Theorem 1.3 it suffices to pick an arbitrary index $i \in I$ (affine if it exists) and to put $\phi = \phi_i$, $M = \bar{J}$, and $a = x_i$, the only attracting fixed point of ϕ_i belonging to J.

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