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Almost Periodicity and the Remainder in the Ellipsoid Problem

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1. Introduction

Let $\mathfrak{S} \in \mathbb{R}^{m\times m}$ $(m \geq 2)$ be a positive definite real matrix, let $Q[\mathfrak{x}] := {}^t\mathfrak{x}\mathfrak{S}\mathfrak{x}$ be the associated quadratic form, and let $Q^{-1}[\mathfrak{x}] := {}^t\mathfrak{x}\mathfrak{S}^{-1}\mathfrak{x}$. For $\mathfrak{a} \in \mathbb{R}^m$, define

$$N_{\mathfrak{a}}(x) := \#\{\mathfrak{x} \in \mathbb{Z}^m \mid Q[\mathfrak{x} - \mathfrak{a}] \leq x\}, \quad x \geq 1,$$

which is the number of lattice points in the ellipsoid $\mathfrak{a} + \sqrt{x}E$, where $E := \{\mathfrak{x} \in \mathbb{R}^m \mid Q[\mathfrak{x}] \leq 1\}$. A simple lattice point argument shows that

$$\Delta_{\mathfrak{a}}(x) := N_{\mathfrak{a}}(x) - \mathrm{vol}(E)x^{m/2} \ll x^{(m-1)/2},$$

where

$$\mathrm{vol}(E) = \frac{\pi^{m/2}}{(\det \mathfrak{S})^{1/2}\Gamma(m/2+1)}$$

is the Euclidean volume of E. Landau [18] improved this estimate to

$$\Delta_{\mathfrak{a}}(x) \ll x^{m/2-1+1/(m+1)} \quad (m \geq 2)$$

using the functional equation of the Epstein zeta function for Q. Krätzel and Nowak [17] derived (in the more general case of a convex body with smooth boundary of strictly positive Gaussian curvature) the better estimate $\Delta_{\mathfrak{a}}(x) \ll x^{m/2-1+\lambda}$ with

$$\lambda = \frac{5}{6m+2} \quad \text{for } m \geq 8, \qquad \lambda = \frac{12}{14m+8} \quad \text{for } 3 \leq m \leq 7.$$

They used exponential sum estimates. In the special case of a rational ellipsoid (i.e., when there is some $a > 0$ with $a\mathfrak{S} \in \mathbb{Q}^{m\times m}$), Landau [19] proved the estimate

$$\Delta_{\mathfrak{a}}(x) \ll x^{m/2-1} \quad (m \geq 5).$$

In this case the theory of theta series can be applied, giving better results. Recently the same estimate was proved by Bentkus and Götze [1] for an arbitrary real ellipsoid E and $m \geq 9$. For rational ellipsoids, the bound $O(x^{m/2-1})$ is optimal. For irrational ellipsoids and $m \geq 9$, Bentkus and Götze [2] showed that $\Delta_{\mathfrak{a}}(x) = o(x^{m/2-1})$, which has important applications to conjectures of Davenport and Lewis and of Oppenheim. In [2] the authors used techniques from probability theory that they originally invented to obtain optimal rates of convergence in central limit theorems.

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For small values of m (i.e., $m \le 8$), no optimal remainder estimates are known. Hardy's conjecture $\Delta_0(x) \ll_\varepsilon x^{1/4+\varepsilon}$ for $m = 2$ and E the unit circle is still unproved, and the best result in this direction is $\Delta_0(x) \ll_\varepsilon x^{23/73+\varepsilon}$, due to Huxley [15]. For numerical results on the oscillatory behavior of $\Delta_0(x)$, see [12] and [7].

Heath-Brown [14] was the first to ask for limit distributions of error terms. For $m = 2$, $\mathfrak{a} = 0$, and E the unit disc he proved that $F(t) := t^{-1/2}\Delta_0(t^2)$ has a limit distribution ν_0 on the Borel sets of $\mathbb{R}$ in the sense that, for all continuous bounded functions $\phi\colon \mathbb{R} \to \mathbb{C}$,

$$\lim_{X\to\infty} \frac{1}{X}\int_1^X \phi(F(t))\,dt = \int_{\mathbb{R}} \phi(x)\,d\nu_0(x).$$

Here ν_0 is absolutely continuous with respect to the Lebesque measure and decreases at infinity faster than polynomially. A key step is to show that $F(t)$ is $\mathcal{B}^2$-almost periodic in the sense of Besicovitch. A function $f\colon [0,\infty) \to \mathbb{C}$ is called $\mathcal{B}^q$-almost periodic ($1 \le q < \infty$) if, for every $\varepsilon > 0$, there is a trigonometric polynomial $p(x) = \sum_{j=1}^J c_j e^{i\alpha_j x}$ ($c_j \in \mathbb{C}$, $\alpha_j \in \mathbb{R}$, $1 \le j \le J$) such that

$$\|f - p\|_q := \left(\limsup_{X\to\infty} \frac{1}{X}\int_0^X |f(x) - p(x)|^q\,dx\right)^{1/q} \le \varepsilon.$$

For the theory of these functions, see Besicovitch [3] or Maak [20]. Bleher and colleagues [9] extended Heath-Brown's result to general $\mathfrak{a}$ and proved that the density of the limit distribution $\nu_{\mathfrak{a}}$ with respect to the Lebesque measure decreases at infinity roughly as $\exp(-|x|^4)$. Bleher and Dyson [10] showed that the variance of $\nu_{\mathfrak{a}}$ as a function of $\mathfrak{a}$ is continuous but otherwise very irregularly behaved (see also [11]). Bleher [4] generalized parts of these results to the case where the circle is replaced by a smooth convex curve with positive curvature. In [5] he generalized the tail asymptotics of the density function to this general situation under a suitable condition of linear independency of frequencies. If the curvature of the boundary vanishes at isolated points, the asymptotics of the remainder depends on the arithmetical nature of the slope of the boundary at these points. This situation was investigated in [6].

Results about the limit distribution of the error term in lattice point asymptotics are of interest in mathematical physics because the spectral function of certain integrable quantum mechanical systems can be related to lattice point counting problems (for an overview, see [7]).

In the present paper, the function

$$\Delta_{\lambda,\mathfrak{a}}(x) := \Gamma(\lambda+1)^{-1} \sum_{\mathfrak{x}\in\mathbb{Z}^m\colon Q[\mathfrak{x}-\mathfrak{a}]\le x} \left(1 - \frac{Q[\mathfrak{x}-\mathfrak{a}]}{x}\right)^{\lambda} - \frac{\pi^{m/2}x^{m/2}}{(\det\mathfrak{S})^{1/2}\Gamma(m/2+\lambda+1)}, \quad x \ge 1,$$

is investigated, where $\lambda \geq 0$ is an additional parameter. If $\lambda = 0$, then $\Delta_{0,\mathfrak{a}}(x)$ is just the error in the ellipsoid problem. The introduction of the weight $a \mapsto (1-a/x)^\lambda$ smoothes the behavior of $\Delta_{\lambda,\mathfrak{a}}(x)$ so that almost periodicity results can be obtained. Of course, one is interested in taking λ as small as possible.

THEOREM 1.1. *Let E be rational, $m \geq 3$, and $\mathfrak{a}$ arbitrary. Assume $(m-3)/2 < \lambda < (m-1)/2$. Then $F_{\lambda,\mathfrak{a}}(t) := t^{\lambda-(m-1)/2}\Delta_{\lambda,\mathfrak{a}}(t^2)$, $t \geq 1$, is $\mathcal{B}^1$-almost periodic.*

THEOREM 1.2. *Let E and $\mathfrak{a}$ be rational and $m \geq 3$ or let E be real, $\mathfrak{a}$ arbitrary, and $m \geq 8$. Assume $(m-3)/2 < \lambda < (m-1)/2$. Then $F_{\lambda,\mathfrak{a}}$ is $\mathcal{B}^2$-almost periodic.*

In the proof for arbitrary E and $m \geq 8$, the results of [1] play an essential part.

Since almost periodic functions have limit distributions [4, Thm. 4.1], the existence of a limit distribution for $F_{\lambda,\mathfrak{a}}$ follows immediately under the assumptions of Theorem 1.1 or Theorem 1.2.

Let $0 < \mu_1 < \mu_2 < \cdots$ be the values taken by Q^{-1} on $\mathbb{Z}^m \setminus \{0\}$, and let

$$b_n := \sum_{\mathfrak{x}\in\mathbb{Z}^m : Q^{-1}[\mathfrak{x}]=\mu_n} e({}^t\mathfrak{a}\mathfrak{x}), \quad n \in \mathbb{N} \tag{1.1}$$

$(e(x) := e^{2\pi i x})$. The next theorem gives the Fourier coefficients of $F_{\lambda,\mathfrak{a}}$ in a weak sense (whether or not $F_{\lambda,\mathfrak{a}}$ is almost periodic).

THEOREM 1.3. *If $m \geq 2$, E and $\mathfrak{a}$ arbitrary, $\phi \in C_c^\infty(\mathbb{R}^+)$, $0 \leq \lambda < (m-1)/2$, and $\gamma \in \mathbb{R}$, then*

$$c(\gamma,\phi) := \lim_{X\to\infty} \frac{1}{X}\int_1^\infty F_{\lambda,\mathfrak{a}}(t)e(\gamma t)\phi\left(\frac{t}{X}\right) dt$$

exists. If $\gamma = \pm\sqrt{\mu_n}$ for some $n \in \mathbb{N}$, then

$$c(\gamma,\phi) = \frac{b_n}{2\pi^{\lambda+1}(\det\mathfrak{S})^{1/2}\mu_n^{(m+1+2\lambda)/4}} e^{\pm\pi i(m+1+2\lambda)/4}\int_0^\infty \phi(x)\,dx.$$

For all other values of γ, $c(\gamma,\phi) = 0$.

For rational E, Theorems 1.2 and 1.3 give a characterization of those values of λ for which $F_{\lambda,0}$ is $\mathcal{B}^2$-almost periodic.

COROLLARY 1.4.

(1) *Let E be rational and $m \geq 3$.*
   (a) *If $0 \leq \lambda \leq (m-3)/2$ then $\|F_{\lambda,0}\|_2 = \infty$. In particular, $F_{\lambda,0}$ is not $\mathcal{B}^2$-almost periodic.*
   (b) *If $(m-3)/2 < \lambda < (m-1)/2$ then $F_{\lambda,0}$ is $\mathcal{B}^2$-almost periodic.*

(2) *Let E be real and $m \geq 3$. If $\lambda \geq (m-1)/2$ then $\Delta_{\lambda,0}(x) \ll \log x$.*

Corollary 1.4(1)(a) hinges upon the fact that, for rational E and $\mathfrak{a} = 0$, $|b_n|^2$ can be sufficiently well estimated in the mean from below such that

$$\sum_{n\geq 1} \left| c\left(\sqrt{\mu_n}, \phi\right)\right|^2 = \infty$$

follows. On the other hand, Bleher and Bourgain [8] announced that they could prove the $\mathcal{B}^1$-almost periodicity of $F_{0,\mathfrak{a}}$ under certain diophantine conditions for $\mathfrak{a}$.

For E rational and $m \geq 3$, F_{00} is not $\mathcal{B}^2$-almost periodic because $\|F_{00}\|_2 = \infty$. One could imagine that multiplying it by a decreasing function ρ could render F_{00} almost periodic. The following result shows that, for reasonably well-behaved functions ρ, the only almost periodic functions ρF_{00} that can be obtained in this way are zero functions.

COROLLARY 1.5. *Let $m \geq 2$, E and $\mathfrak{a}$ arbitrary, and $0 \leq \lambda < (m-1)/2$. Let $\rho\colon [1,\infty) \to \mathbb{R}^+$ be continuously differentiable and decreasing, and let $t|\rho'(t)| \ll \rho(t)$ as $t \to \infty$ and $\lim_{t\to\infty} \rho(t) = 0$. Assume that $\rho F_{\lambda,\mathfrak{a}}$ is $\mathcal{B}^q$-almost periodic. Then $\|\rho F_{\lambda,\mathfrak{a}}\|_q = 0$.*

Examples for ρ are $t^{-\alpha}$ and $(\log t)^{-\alpha}$ $(\alpha > 0)$.

Assume that $\mathfrak{S} \in \mathbb{Z}^{m\times m}$, $m \geq 2$. From Corollary 1.4(1)(a) and the result of [4] it follows that F_{00} is $\mathcal{B}^2$-almost periodic if and only if $m = 2$. One should mention that the opposite is true for the function

$$G(n) := \#\{\mathfrak{x} \in \mathbb{Z}^m \mid Q[\mathfrak{x}] = n\} \cdot n^{1-m/2}, \quad n \in \mathbb{N}.$$

For $m \geq 3$, $G \in \mathcal{D}^q$ for all $q \geq 1$; for $m = 2$, $G \notin \mathcal{D}^q$ for all $q \geq 1$ (see [21]; for the definition of the spaces $\mathcal{D}^q$ of q-limit periodic functions on $\mathbb{N}$, see Schwarz and Spilker [22]). On the other hand, if E is irrational and $m \geq 9$, then it follows from [2] that

$$H(n) := \#\{\mathfrak{x} \in \mathbb{Z}^m \mid Q[\mathfrak{x}] \in [n, n+1)\} \cdot n^{1-m/2}, \quad n \in \mathbb{N},$$

is asymptotically constant, namely,

$$H(n) = (\det \mathfrak{S})^{-1/2} \pi^{m/2} \Gamma\left(\frac{m}{2}\right)^{-1} + o(1)$$

as $n \to \infty$.

Hereafter, $\varepsilon > 0$ will denote an arbitrarily small positive real.

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2. A Voronoi Formula

First a truncated Voronoi formula with Riesz mean is derived. Let $\mathfrak{S} \in \mathbb{R}^{m\times m}$ $(m \geq 3)$ be positive definite and let $\lambda > 0$. The Dirichlet series

$$Z(s) := \sum_{\mathfrak{x}\in\mathbb{Z}^m, \neq \mathfrak{a}} \frac{1}{Q[\mathfrak{x}-\mathfrak{a}]^s} \quad \text{and} \quad \tilde{Z}(s) := \sum_{\mathfrak{x}\in\mathbb{Z}^m\setminus\{0\}} \frac{e({}^t\mathfrak{a}\mathfrak{x})}{Q^{-1}[\mathfrak{x}]^s}$$

are absolutely convergent for $\mathrm{Re}(s) > m/2$ and can be continued to meromorphic functions on $\mathbb{C}$. These Epstein zeta functions have poles at most at $m/2$ that are simple and have residues

$$(\det \mathfrak{S})^{-1/2}\pi^{m/2}\Gamma\left(\frac{m}{2}\right)^{-1} \quad \text{and (resp.)} \quad \delta(\mathfrak{a})(\det \mathfrak{S})^{1/2}\pi^{m/2}\Gamma\left(\frac{m}{2}\right)^{-1},$$

where $\delta(\mathfrak{a}) = 1$ for $\mathfrak{a} \in \mathbb{Z}^m$ and $\delta(\mathfrak{a}) = 0$ otherwise. They are connected by the functional equation

$$Z(s) = (\det \mathfrak{S})^{-1/2}\pi^{2s-m/2}\frac{\Gamma(m/2-s)}{\Gamma(s)}\tilde{Z}(m/2-s) \tag{2.1}$$

(see [13, pp. 625ff.]). Write

$$Z(s) = \sum_{n\geq 1}\frac{a_n}{\lambda_n^s}, \quad \tilde{Z}(s) = \sum_{n\geq 1}\frac{b_n}{\mu_n^s}, \quad \mathrm{Re}(s) > \frac{m}{2}$$

(see (1.1)). Let $c = m/2 + \varepsilon$. A generalization of Perron's formula states that

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} Z(s)\frac{x^s\Gamma(s)}{\Gamma(s+\lambda+1)}\,ds &\\ = \Gamma(\lambda+1)^{-1}\sum_{\lambda_n<x} a_n\left(1-\frac{\lambda_n}{x}\right)^{\lambda} &\\ + O\left(x^cT^{-(\lambda+1)} + T^{-\lambda}\sum_{x/2<\lambda_n<2x}|a_n|\min\left\{\frac{x}{T|\lambda_n-x|},1\right\}\right) & \qquad (2.2)\end{aligned}$$

for $x, T \geq 1$. This can be proved along the same lines as the formula for $\lambda = 0$ (see [16, Apx. A.3], where a formula for $T = \infty$ is proved). Assume $c_1x \leq T \leq c_2x$. The following O-constants may depend on $c_1, c_2 > 0$. From the functional equation (2.1) and Stirling's formula it follows that

$$|Z(s)| \ll_\varepsilon \begin{cases} 1 & \text{for } \mathrm{Re}(s) = c, \\ (1+|\mathrm{Im}(s)|)^{m/2+2\varepsilon} & \text{for } \mathrm{Re}(s) = -\varepsilon. \end{cases}$$

Since $Z(s)$ grows exponentially in $|\mathrm{Im}(s)|$ in vertical strips of finite width, the Phragmen–Lindelöf principle gives

$$|Z(s)| \ll |\mathrm{Im}(s)|^{c-\sigma} \quad \text{for } -\varepsilon \leq \sigma := \mathrm{Re}(s) \leq c,\ |\mathrm{Im}(s)| \geq 1. \tag{2.3}$$

Equation (2.2) and the residue theorem give

$$\begin{aligned}\Delta_{\lambda,\mathfrak{a}}(x) = \frac{1}{2\pi i}\left(\int_{c-iT}^{-\varepsilon-iT} + \int_{-\varepsilon-iT}^{-\varepsilon+iT} + \int_{-\varepsilon+iT}^{c+iT}\right)Z(s)\frac{x^s\Gamma(s)}{\Gamma(s+\lambda+1)}\,ds &\\ + O(x^{c-\lambda-1} + 1 + x^{-\lambda}R_1(x)), & \qquad (2.4)\end{aligned}$$

where

$$R_1(x) := \sum_{x/2<\lambda_n<2x}|a_n|\min\left\{\frac{1}{|\lambda_n-x|},1\right\}. \tag{2.5}$$

Now (2.3) and Stirling's formula give

$$\int_{-\varepsilon\pm iT}^{c\pm iT} \ll \int_{-\varepsilon}^{c} T^{c-\sigma} x^{\sigma} T^{-(\lambda+1)}\, d\sigma \ll x^{c-\lambda-1}. \tag{2.6}$$

From the functional equation it follows that

$$\begin{aligned} J &:= \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} Z(s) \frac{x^s \Gamma(s)}{\Gamma(s+\lambda+1)}\, ds \\ &= (\det \mathfrak{S})^{-1/2} \pi^{-m/2} \sum_{n\geq 1} \frac{b_n}{\mu_n^{m/2}} I(\pi^2 \mu_n x, T), \end{aligned} \tag{2.7}$$

where

$$I(y,T) := \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} y^s \frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}\, ds, \quad y > 0,\ T \geq 1.$$

For $-\varepsilon \leq \mathrm{Re}(s) = \sigma \leq m/2 - \varepsilon$ and $|\mathrm{Im}(s)| = |t| \geq 1$, Stirling's formula gives

$$\begin{aligned} &\frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)} \\ &\qquad = e^{-i\pi(m/2+\lambda)(\operatorname{sign} t)/2} |t|^{m/2-2\sigma-\lambda-1} e^{2i(t-t\log|t|)} \left(1 + O\left(\frac{1}{|t|}\right)\right). \end{aligned} \tag{2.8}$$

Let $y > T^2$. Define $F(t) := t\log y + 2t - 2t\log|t|$ and $G(t) := |t|^{m/2+2\varepsilon-\lambda-1}$. Then

$$\begin{aligned} I(y,T) &= \sum_{\pm} \pm \frac{1}{2\pi} e^{\mp i\pi(m/2+\lambda)/2} y^{-\varepsilon} \int_{\pm 1}^{\pm T} G(t) e^{iF(t)}\, dt \\ &\quad + O\left(y^{-\varepsilon} + y^{-\varepsilon} \int_1^T t^{m/2+2\varepsilon-\lambda-2}\, dt\right). \end{aligned}$$

Define

$$H(t) := \frac{F'(t)}{G(t)} = |t|^{\lambda+1-m/2-2\varepsilon} \log\left(\frac{y}{|t|^2}\right).$$

Then

$$H'(t) = (\operatorname{sign} t)|t|^{\lambda-m/2-2\varepsilon} \left(-2 + \left(\lambda + 1 - \frac{m}{2} - 2\varepsilon\right) \log\left(\frac{y}{|t|^2}\right)\right)$$

has at most one zero in $[1, T]$ and $[-T, -1]$. Each of these intervals can therefore be divided into two subintervals on which $H(t)$ is monotonic. Furthermore,

$$H(t) \geq \log\left(\frac{y}{T^2}\right) \min\{1, T^{\lambda+1-m/2-2\varepsilon}\} > 0 \quad \text{for } 1 \leq |t| \leq T.$$

From [23, Lemma 4.3], it follows that

$$\begin{aligned} I(y,T) &\ll y^{-\varepsilon} \log^{-1}\left(\frac{y}{T^2}\right) \max\{1, T^{m/2+2\varepsilon-\lambda-1}\} \\ &\quad + y^{-\varepsilon} \max\{1, T^{m/2+2\varepsilon-\lambda-1} \log T\} \end{aligned}$$

for $y > T^2$. Therefore,

$$\sum_{\mu_n > T^2/(\pi^2 x)+1} \frac{b_n}{\mu_n^{m/2}} I(\pi^2 \mu_n x, T) \\ \ll 1 + x^{m/2+2\varepsilon-\lambda-1} + x^{1-m/2-2\varepsilon}(1 + x^{m/2+3\varepsilon-\lambda-1}) R_2\left(\frac{T^2}{\pi^2 x}\right), \quad (2.9)$$

where

$$R_2(y) := \sum_{y+1<\mu_n\le 2y} \frac{|b_n|}{\mu_n - y}. \quad (2.10)$$

Choose $m/4 < d < m/2$. Let $0 < y < T^2$. From (2.8) it follows that

$$\int_{-\varepsilon\pm iT}^{d\pm iT} y^s \frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}\, ds \ll \int_{-\varepsilon}^{d} \left(\frac{y}{T^2}\right)^{\sigma} T^{m/2-\lambda-1}\, d\sigma \\ \ll \left(\frac{y}{T^2}\right)^{-\varepsilon} T^{m/2-\lambda-1}. \quad (2.11)$$

Define $\tilde{G}(t) := |t|^{m/2-2d-\lambda-1}$ and

$$\tilde{H}(t) := \frac{F'(t)}{\tilde{G}(t)} = |t|^{2d-m/2+\lambda+1} \log\left(\frac{y}{T^2}\right).$$

Then

$$\tilde{H}'(t) = (\operatorname{sign} t)|t|^{2d-m/2+\lambda}\left(-2 + \left(2d - \frac{m}{2} + \lambda + 1\right)\log\left(\frac{y}{T^2}\right)\right)$$

has at most one zero in $(-\infty, -T]$ and $[T, \infty)$. Each of these intervals can therefore be divided into two subintervals on which $\tilde{H}(t)$ is monotonic. Furthermore,

$$\tilde{H}(t) \le -T^{2d-m/2+\lambda+1} \log\left(\frac{T^2}{y}\right) < 0$$

for $|t| \ge T$. From [23, Lemma 4.3] and (2.8), it follows that

$$\int_{d\pm iT}^{d\pm i\infty} y^s \frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}\, ds = i y^d e^{\mp i\pi(m/2+\lambda)/2} \int_{\pm T}^{\pm\infty} \tilde{G}(t) e^{iF(t)}\, dt \\ + O\left(y^d \int_T^\infty t^{m/2-2d-\lambda-2}\, dt\right) \\ \ll y^d T^{m/2-2d-\lambda-1}\left(\log^{-1}\left(\frac{T^2}{y}\right) + 1\right). \quad (2.12)$$

From the integral representation of the Bessel function (see [24, 6·5(7)]), it follows that

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} y^s \frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}\, ds = y^{(m-2\lambda)/4} J_{m/2+\lambda}(2\sqrt{y}), \quad y > 0. \quad (2.13)$$

Using (2.11), (2.12), and Cauchy's theorem, for $0 < y < T^2$ we now have

$$I(y,T) = y^{(m-2\lambda)/4} J_{m/2+\lambda}(2\sqrt{y}) + O(y^{-\varepsilon} T^{m/2+2\varepsilon-\lambda-1}) \\ + O\left(y^d T^{m/2-2d-\lambda-1}\left(1 + \log^{-1}\left(\frac{T^2}{y}\right)\right)\right).$$

This gives

$$\begin{aligned}\sum_{\mu_n<T^2/(\pi^2x)-1}\frac{b_n}{\mu_n^{m/2}}I(\pi^2\mu_nx,T)\\ &=\pi^{m/2-\lambda}x^{(m-2\lambda)/4}\sum_{\mu_n<T^2/(\pi^2x)-1}\frac{b_n}{\mu_n^{(m+2\lambda)/4}}J_{m/2+\lambda}\big(2\pi\sqrt{\mu_nx}\big)\\ &\quad+O\left(x^{m/2+\varepsilon-\lambda-1}+x^{-\lambda}R_3\left(\frac{T^2}{\pi^2x}\right)\right),\end{aligned}\tag{2.14}$$

where

$$R_3(y):=\sum_{y/2<\mu_n<y-1}\frac{|b_n|}{y-\mu_n}.\tag{2.15}$$

Equation (2.8) gives the trivial estimate

$$I(y,T)\ll\int_{-T}^{T}y^{-\varepsilon}(|t|+1)^{m/2+2\varepsilon-\lambda-1}\,dt\ll y^{-\varepsilon}(1+T^{m/2+3\varepsilon-\lambda})$$

and consequently

$$\begin{aligned}\sum_{T^2/(\pi^2x)-1\le\mu_n\le T^2/(\pi^2x)+1}\frac{b_n}{\mu_n^{m/2}}I(\pi^2\mu_nx,T)\\ &\ll(x^{-m/2-2\varepsilon}+x^{\varepsilon-\lambda})R_4\left(\frac{T^2}{\pi^2x}\right),\end{aligned}\tag{2.16}$$

where

$$R_4(y):=\sum_{y-1\le\mu_n\le y+1}|b_n|.\tag{2.17}$$

From [24, 7·21(1)] it follows that

$$\begin{aligned}J_{m/2+\lambda}(y)\\ &=\sqrt{\frac{2}{\pi y}}\cos\left(y-\frac{\pi}{2}\left(\frac{m}{2}+\lambda\right)-\frac{\pi}{4}\right)+O_\delta(y^{-3/2}),\quad y\ge\delta>0.\end{aligned}\tag{2.18}$$

Putting together (2.4), (2.6), (2.7), (2.9), (2.14), and (2.16) now yields, for $T,x\ge 1$ and $c_1x\le T\le c_2x$,

$$\begin{aligned}\Delta_{\lambda,\mathfrak{a}}(x)\\ &=(\det\mathfrak{S})^{-1/2}\pi^{-\lambda-1}x^{(m-2\lambda-1)/4}\\ &\quad\times\sum_{\mu_n\le T^2/(\pi^2x)}\frac{b_n}{\mu_n^{(m+2\lambda+1)/4}}\cos\left(2\pi\sqrt{\mu_nx}-\pi\frac{m+2\lambda+1}{4}\right)\\ &\quad+O\bigg(x^{m/2-\lambda-1+2\varepsilon}+1+x^{-\lambda}R_1(x)+(x^{1-m/2-2\varepsilon}+x^{\varepsilon-\lambda})R_2\left(\frac{T^2}{\pi^2x}\right)\\ &\quad+x^{-\lambda}R_3\left(\frac{T^2}{\pi^2x}\right)+(x^{-m/2-2\varepsilon}+x^{\varepsilon-\lambda})R_4\left(\frac{T^2}{\pi^2x}\right)+x^{(m-2\lambda-3)/4}\bigg).\end{aligned}\tag{2.19}$$

In order to estimate the $R_j(y)$, we need an estimate for

$$r_1(y) := \sum_{y \le \lambda_n \le y+1} |a_n|, \quad r_2(y) := \sum_{y \le \mu_n \le y+1} |b_n|, \quad y \ge 1.$$

Since $r_1(y) = \#\{\mathfrak{x} \ne \mathfrak{a} \mid y \le Q[\mathfrak{x} - \mathfrak{a}] \le y+1\}$, a simple geometric argument gives $r_1(y) \ll y^{(m-1)/2}$. The same holds for $r_2(y)$. But these estimates are too weak for our purposes; we need $r_j(y) \ll_\varepsilon y^{m/2-1+\varepsilon}$. For $m \ge 5$ and E and $\mathfrak{a}$ rational, this follows from the circle method or the theory of modular forms. For $m = 3, 4$ and E and $\mathfrak{a}$ rational, it is much more costly to prove this estimate; fortunately it is only needed in the mean. For $m \ge 8$ and E and $\mathfrak{a}$ arbitrary, the estimate follows from [1].

From (2.5) it follows that

$$R_1(x) \ll r_1(x) + \sum_{1 \le l \le x} \frac{r_1(x+l)}{l} + r_1(x-1) + \sum_{1 \le l \le x/2} \frac{r_1(x-l-1)}{l}. \tag{2.20}$$

The Cauchy–Schwarz inequality gives, for $X \gg 1$,

$$\begin{aligned} &\int_X^{2X} |R_1(x)|^2 \, dx \\ &\quad \ll \int_X^{2X} \Bigg[|r_1(x)|^2 + \Bigg(\sum_{1 \le l \le x} \frac{1}{l} \Bigg) \Bigg(\sum_{1 \le l \le x} \frac{|r_1(x+l)|^2}{l} \Bigg) + |r_1(x-1)|^2 \\ &\qquad + \Bigg(\sum_{2 \le l \le x/2+1} \frac{1}{l-1} \Bigg) \Bigg(\sum_{2 \le l \le x/2+1} \frac{|r_1(x-l)|^2}{l-1} \Bigg) \Bigg] \, dx \\ &\quad \ll \log^2 X \int_{X/2-1}^{4X} |r_1(x)|^2 \, dx. \end{aligned} \tag{2.21}$$

It follows analogously from (2.10), (2.15), and (2.17) that, for $X \gg 1$,

$$\int_X^{2X} |R_j(x)|^2 \, dx \ll \log^2 X \int_{X/2-1}^{4X} |r_2(x)|^2 \, dx, \quad j = 2, 3, 4. \tag{2.22}$$

Furthermore,

$$\begin{aligned} &\int_X^{2X} |R_1(x)| \, dx \ll \log X \int_{X/2-1}^{4X} |r_1(x)| \, dx, \\ &\int_X^{2X} |R_j(x)| \, dx \ll \log X \int_{X/2-1}^{4X} |r_2(x)| \, dx, \quad j = 2, 3, 4. \end{aligned} \tag{2.23}$$

3. Almost Periodicity of $F_{\lambda,\mathfrak{a}}$

The estimate of $R_j(x)$ in the 1-norm is connected with no cost at all. For E real and $m \ge 3$ (which will be called condition I in the sequel),

$$\begin{aligned}\int_X^{2X} |r_1(x)|\,dx &= \int_X^{2X}\Bigg(\sum_{\mathfrak{x}\neq\mathfrak{a}:\,x\leq Q[\mathfrak{x}-\mathfrak{a}]\leq x+1} 1\Bigg)\,dx\\ &= \sum_{\mathfrak{x}\neq\mathfrak{a}:\,X\leq Q[\mathfrak{x}-\mathfrak{a}]\leq 2X+1} \int_{[X,2X]\cap[Q[\mathfrak{x}-\mathfrak{a}]-1,\,Q[\mathfrak{x}-\mathfrak{a}]]} dx\\ &\leq \sum_{\mathfrak{x}\neq\mathfrak{a}:\,Q[\mathfrak{x}-\mathfrak{a}]\leq 2X+1} 1 \ll X^{m/2}. \end{aligned}\tag{3.1}$$

Analogously,

$$\int_X^{2X} |r_2(x)|\,dx \ll X^{m/2}. \tag{3.2}$$

If E and $\mathfrak{a}$ are rational and $m \geq 3$ (which will be called condition II), then there is some $a > 0$ with $a\mathfrak{S} \in \mathbb{Z}^{m\times m}$ and $b \in \mathbb{N}$ with $b\mathfrak{a} \in \mathbb{Z}^m$. Let

$$r(n) := \#\{\mathfrak{x} \in \mathbb{Z}^m \mid aQ[\mathfrak{x}] = n\}, \quad n \in \mathbb{N}.$$

Then

$$r_1(x) = \#\{\mathfrak{x} \neq \mathfrak{a} \mid ab^2x \leq aQ[b\mathfrak{x} - b\mathfrak{a}] \leq ab^2(x+1)\} \leq \sum_{ab^2x\leq n\leq ab^2(x+1)} r(n).$$

For all x, the sum on the right-hand side has at most $ab^2 + 1$ terms. Therefore,

$$\begin{aligned}\int_X^{2X} |r_1(x)|^2\,dx &\ll_{a,b} \int_X^{2X}\Bigg(\sum_{ab^2x\leq n\leq ab^2(x+1)} |r(n)|^2\Bigg)\,dx\\ &\ll \sum_{ab^2X\leq n\leq ab^2(2X+1)} |r(n)|^2 \int_{[X,2X]\cap[n/(ab^2)-1,\,n/(ab^2)]} dx\\ &\ll \sum_{n\leq ab^2(2X+1)} |r(n)|^2.\end{aligned}$$

The classical circle method can be applied to show that the function $n \mapsto n^{1-m/2}r(n)$ is almost periodic on $\mathbb{N}$ (see [21] for the most interesting case $m = 3$). In particular, its square has a mean value and therefore

$$\sum_{n\leq y} |r(n)|^2 \ll y^{m-1}.$$

Consequently,

$$\int_X^{2X} |r_1(x)|^2\,dx \ll X^{m-1}. \tag{3.3}$$

The same estimate holds for r_2 (here $\mathfrak{a} \in \mathbb{Q}^m$ is not needed):

$$\int_X^{2X} |r_2(x)|^2\,dx \ll X^{m-1}. \tag{3.4}$$

If E is real and $m \geq 8$ (which will be called condition III), it follows from [1] that

$$\sum_{\mathfrak{x}:Q[\mathfrak{x}-\mathfrak{a}]\le y} 1 = \mathrm{vol}(E)y^{m/2} + O(y^{m/2-1}\log^2 y).$$

Consequently, $r_1(y) \ll y^{m/2-1+\varepsilon}$ and

$$\int_X^{2X} |r_1(x)|^2\,dx \ll X^{m-1+2\varepsilon}; \tag{3.5}$$

analogously,

$$\int_X^{2X} |r_2(x)|^2\,dx \ll X^{m-1+2\varepsilon}. \tag{3.6}$$

Assume that $(m-3)/2 < \lambda < (m-1)/2$. Let M be sufficiently large. From (2.23) and (3.2), it follows that

$$\sum_{j=2}^{4}\int_{M^2}^{4M^2} |R_j(x)|\,dx \ll M^{m+\varepsilon}.$$

Choose $M \le M^* \le 2M$ with

$$\sum_{j=2}^{4} |R_j((M^*)^2)| \ll M^{m-2+\varepsilon}.$$

Let $M \le t \le 2M$. Choose $T := \pi M^* t$ and $x = t^2$. Then $\pi/2 \le T/x \le 2\pi$, and from (2.19) it follows that $F_{\lambda,\mathfrak{a}}(t) = S(t) + R(t)$, where

$$S(t) := (\det\mathfrak{S})^{-1/2}\pi^{-\lambda-1}\sum_{\mu_n\le(M^*)^2}\frac{b_n}{\mu_n^{(m+2\lambda+1)/4}}\cos\left(2\pi\sqrt{\mu_n}t - \frac{\pi}{4}(m+2\lambda+1)\right)$$

and

$$R(t) \ll M^{(m-3)/2-\lambda+4\varepsilon} + M^{\lambda-(m-1)/2} + t^{-(m-1)/2-\lambda}R_1(t^2).$$

Under condition I, it follows from (2.23) and (3.1) that

$$\int_M^{2M} |R(t)|\,dt \ll M(M^{(m-3)/2-\lambda+4\varepsilon} + M^{\lambda-(m-1)/2}). \tag{3.7}$$

Under condition II or III, it follows from (2.21) and (3.3) or (3.5) that

$$\int_M^{2M} |R(t)|^2\,dt \ll M(M^{m-3-2\lambda+8\varepsilon} + M^{2\lambda-m+1}). \tag{3.8}$$

For $N \ge 1$, define

$$p_N(t) := (\det\mathfrak{S})^{-1/2}\pi^{-\lambda-1}\sum_{\mu_n\le N}\frac{b_n}{\mu_n^{(m+2\lambda+1)/4}}\cos\left(2\pi\sqrt{\mu_n}t - \frac{\pi}{4}(m+2\lambda+1)\right).$$

Let $1 \le N \le M^2$. Then $|S(t) - p_N(t)|^2$ is expanded into a double sum and integrated from M to $2M$. Using the estimate

$$\int_M^{2M} \cos(\alpha_1 t + \beta)\cos(\alpha_2 t + \beta)\,dt \ll \min\left\{\big||\alpha_1| - |\alpha_2|\big|^{-1}, M\right\}$$

gives

$$\begin{aligned}
&\int_M^{2M} |S(t) - p_N(t)|^2\,dt \\
&\qquad \ll M \sum_{N<\mu_n \le (M^*)^2} \frac{|b_n|^2}{\mu_n^{(m+2\lambda+1)/2}} \\
&\qquad\quad + \sum_{N<\mu_{n_1}<\mu_{n_2}\le (M^*)^2} \frac{|b_{n_1} b_{n_2}]}{(\mu_{n_1}\mu_{n_2})^{(m+2\lambda+1)/4}} \min\left\{M, \frac{1}{\sqrt{\mu_{n_2}} - \sqrt{\mu_{n_1}}}\right\} \\
&\qquad =: MS_1 + S_2. \qquad (3.9)
\end{aligned}$$

For $l \in \mathbb{N}$,

$$\sum_{\mu_n \in [l,l+1]} |b_n|^2 \le \Bigg(\sum_{\mu_n \in [l,l+1]} |b_n|\Bigg)^2 = r_2(l)^2.$$

Therefore

$$S_1 \ll \sum_{l \ge N-1} \frac{r_2(l)^2}{l^{(m+2\lambda+1)/2}}. \qquad (3.10)$$

In the second sum, distinguish between $\mu_{n_2} \le 2\mu_{n_1}$ and $\mu_{n_2} > 2\mu_{n_1}$. Then $S_2 = S_{21} + S_{22}$, where

$$\begin{aligned}
S_{21} &\ll M \sum_{N<\mu_{n_1}\le (M^*)^2} \frac{|b_{n_1}|}{\mu_{n_1}^{(m+2\lambda+1)/2}} \sum_{\mu_{n_1}<\mu_{n_2}\le 2\mu_{n_1}} |b_{n_2}| \min\left\{1, \frac{1}{\mu_{n_2} - \mu_{n_1}}\right\} \\
&=: MS_{21}^*, \qquad (3.11)
\end{aligned}$$

$$S_{22} \ll \sum_{N<\mu_{n_1}\le (M^*)^2} \frac{|b_{n_1}|}{\mu_{n_1}^{(m+2\lambda+1)/4}} \sum_{2\mu_{n_1}<\mu_{n_2}\le (M^*)^2} \frac{|b_{n_2}|}{\mu_{n_2}^{(m+2\lambda+3)/4}}. \qquad (3.12)$$

Since

$$\sum_{\mu_n \le t} |b_n| \le \#\{\mathfrak{x} \mid Q^{-1}[\mathfrak{x}] \le t\} \ll t^{m/2},$$

the inner sum in (3.12) is $O(M^{(m-1)/2-\lambda}\mu_{n_1}^{-1/2})$. Therefore,

$$S_{22} \ll M^{(m-1)/2-\lambda} \sum_{\mu_n > N} \frac{|b_n|}{\mu_n^{(m+2\lambda+3)/4}} \ll M \cdot N^{((m-3)/2-\lambda)/2}. \qquad (3.13)$$

The inner sum in (3.11) is $O\big(\sum_{0\le l\le \mu_{n_1}} \min\{1, l^{-1}\} r_2(\mu_{n_1} + l)\big)$. Hence

$$\begin{aligned}
S_{21}^* &\ll \sum_{N-1\le k\le (M^*)^2} \frac{1}{k^{(m+2\lambda+1)/2}} \sum_{\mu_n \in [k,k+1]} |b_n| \\
&\qquad \times \sum_{0\le l\le k+1} \min\left\{1, \frac{1}{l}\right\} (r_2(k+l) + r_2(k+l+1)) \\
&\ll \sum_{N-1\le k\le (M^*)^2} \frac{r_2(k)}{k^{(m+2\lambda+1)/2}} \sum_{0\le l\le k+2} \frac{r_2(k+l)}{l+1}.
\end{aligned}$$

Since $2r_2(k)r_2(k+l) \le r_2(k)^2 + r_2(k+l)^2$, it follows that

$$S_{21}^* \ll \sum_{k\ge N-1} \frac{r_2(k)^2 \log k}{k^{(m+2\lambda+1)/2}} + \sum_{k\ge N-1} \frac{1}{k^{(m+2\lambda+1)/2}} \sum_{0\le l\le k+2} \frac{r_2(k+l)^2}{l+1}.$$

The second sum is

$$\sum_{h\ge N-1} r_2(h)^2 \sum_{k\ge N-1,\, 0\le l\le k+2:k+l=h} \frac{1}{l+1}\frac{1}{k^{(m+2\lambda+1)/2}}.$$

Since $k \le h \le 2k+2$, the inner sum is $O(h^{-(m+2\lambda+1)/2}\log h)$. Therefore

$$S_{21}^* \ll \sum_{k\ge N-1} \frac{r_2(k)^2 \log k}{k^{(m+2\lambda+1)/2}}. \tag{3.14}$$

Collecting (3.9), (3.10), (3.13), and (3.14) yields

$$\frac{1}{M}\int_M^{2M} |S(t) - p_N(t)|^2\, dt \ll \sum_{l\ge N-1} \frac{r_2(l)^2 \log l}{l^{(m+2\lambda+1)/2}} + N^{((m-3)/2-\lambda)/2}.$$

If E is rational, $m \ge 3$, and $\mathfrak{a}$ arbitrary, then $\sum_{l\le y} |r_2(l)|^2 \ll y^{m-1}$ (see proof of (3.4)). If E is real and $m \ge 8$, then [1] gives $\sum_{l\le y} |r_2(l)|^2 \ll y^{m-1+\varepsilon}$ (see proof of (3.6)). In both cases, it follows that

$$\frac{1}{M}\int_M^{2M} |S(t) - p_N(t)|^2\, dt \ll N^{((m-3)/2-\lambda)/2}. \tag{3.15}$$

If the assumptions of Theorem 1.1 are fulfilled then, by (3.7), (3.15), and the Cauchy–Schwarz inequality, we have for $M \ge \sqrt{N}$ that

$$\frac{1}{M}\int_M^{2M} |F_{\lambda,\mathfrak{a}}(t) - p_N(t)|\, dt \ll N^{((m-3)/2-\lambda)/4+\varepsilon} + N^{(\lambda-(m-1)/2)/2} \ll N^{-\delta(m,\lambda)+\varepsilon},$$

where $\delta(m,\lambda) > 0$ depends only on λ and m. If follows that

$$\|F_{\lambda,\mathfrak{a}} - p_N\|_1 \ll N^{-\delta(m,\lambda)+\varepsilon}, \quad N \ge 1, \tag{3.16}$$

and Theorem 1.1 is proved.

Similarly, under the assumptions of Theorem 1.2, (3.8) and (3.15) give

$$\|F_{\lambda,\mathfrak{a}} - p_N\|_2 \ll N^{-\delta(m,\lambda)+\varepsilon}, \quad N \ge 1, \tag{3.17}$$

thus proving Theorem 1.2.

4. The Fourier Coefficients of $F_{\lambda,\mathfrak{a}}$

Let $m \ge 2$, E and $\mathfrak{a}$ real, $0 \le \lambda < (m-1)/2$, $\phi\colon \mathbb{R}^+ \to \mathbb{C}$ infinitely differentiable with compact support, and $\gamma \in \mathbb{R}$. From (3.16) and (3.17), the Fourier coefficients of $F_{\lambda,\mathfrak{a}}$ can be immediately deduced if the assumptions of Theorem 1.1 or Theorem 1.2 are fulfilled. The problem now is to do so under weaker assumptions. For this we employ a method from [14], which consists of proving some sort of

Voronoi formula not for $F_{\lambda,\mathfrak{a}}$ but for an integral thereof. This formula has far better convergence properties because the function $K_U(s)\Gamma(s)\Gamma(s+\lambda+1)^{-1}$ against which $Z(s)$ is integrated is decreasing much faster than $U^s\Gamma(s)\Gamma(s+\lambda+1)^{-1}$ (see (4.3)).

Let $\operatorname{supp}\phi \subseteq [a,b] \subseteq (0,\infty)$ and $X \geq 1/a$. If follows from Perron's formula and the residue theorem that for $x \in [(aX)^2, (bX)^2]$, $x \neq \lambda_n$ for all $n \in \mathbb{N}$, and $T \geq 1$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathcal{K}(T)} Z(s)\frac{x^s\Gamma(s)}{\Gamma(s+\lambda+1)}\,ds \\ &= \Delta_{\lambda,\mathfrak{a}}(x) - \delta(\mathfrak{a})\Gamma(\lambda+1)^{-1} - \Gamma(\lambda+1)^{-1}\sum_{0\leq n<m/2}\frac{(-1)^n Z(-n)}{\Gamma(-n+\lambda+1)n!}x^{-n} \\ &\quad + O_{a,b}(\min\{c_1(x)T^{-(\lambda+1)}, c_2(X)\})\end{aligned}$$

(see (2.2)). Here $c_1(x) > 0$ depends only on x, $C_2(X) > 0$ depends only on X, and $\mathcal{K}(T)$ consists of straight lines joining the points $m/2+1-iT$, $-m/2-iT$, $-m/2+iT$, and $m/2+1+iT$. Therefore,

$$\frac{1}{2\pi i}\int_{\mathcal{K}(T)} Z(s)\frac{x^s\Gamma(s)}{\Gamma(s+\lambda+1)}\,ds$$

converges boundedly for almost all $x \in [(aX)^2, (bX)^2]$ to

$$\Delta_{\lambda,\mathfrak{a}}(x) - \delta(\mathfrak{a})\Gamma(\lambda+1)^{-1} - \Gamma(\lambda+1)^{-1}\sum_{0\leq n<m/2}\frac{(-1)^n Z(-n)}{\Gamma(-n+\lambda+1)n!}x^{-n}$$

as $T \to \infty$. From Lebesque's dominated convergence theorem it follows that

$$\begin{aligned}I(\gamma, X) &:= \int_1^\infty f_{\lambda,\mathfrak{a}}(t)e(\gamma t)\phi\left(\frac{t}{X}\right)dt \\ &= \frac{1}{2}\int_{(aX)^2}^{(bX)^2} x^{\lambda/2-(m+1)/4}\Delta_{\lambda,\mathfrak{a}}(x)e\big(\gamma\sqrt{x}\big)\omega\left(\frac{x}{U}\right)dx \\ &= \lim_{T\to\infty}\frac{1}{4\pi i}\int_{\mathcal{K}(T)} Z(s)\frac{\Gamma(s)}{\Gamma(s+\lambda+1)}K_U(s)\,ds \\ &\quad + O_{\varepsilon,\phi}(1+X^{\lambda-(m-3)/2+\varepsilon}),\end{aligned} \tag{4.1}$$

where $\omega(t) := \phi\big(\sqrt{t}\big) \in C_c^\infty(\mathbb{R}^+)$, $U := X^2$, and

$$K_U(s) := \int_0^\infty x^{\lambda/2-(m+1)/4+s}e\big(\gamma\sqrt{x}\big)\omega\left(\frac{x}{U}\right)dx. \tag{4.2}$$

From this representation it follows that $K_U(s)$ is entire with respect to s. Integration by parts gives

$$K_U(s) \ll_{\phi,k,\sigma_1,\sigma_2,\gamma,\lambda} (1+|t|)^{-k}U^{\sigma+(\lambda+k)/2-(m-3)/4} \tag{4.3}$$

for $s = \sigma+it$, $\sigma_1 \leq \sigma \leq \sigma_2$, $k \in \mathbb{N}$, and $U \geq 1$.

The functional equation and the Phragmen–Lindelöf principle show that there is some $A \in \mathbb{N}$ such that, for $-m/2 \le \sigma \le m/2+1$ and $|t| \ge 1$, we have $Z(s) \ll |t|^A$. Inequality (4.3) with $k = A$ and Stirling's formula yield

$$\int_{-m/2\pm iT}^{m/2+1\pm iT} Z(s)\frac{\Gamma(s)}{\Gamma(s+\lambda+1)}K_U(s)\,ds \ll_{U,A} T^{-(\lambda+1)}, \quad T \ge 1.$$

From (4.1) it follows that

$$I(\gamma, X) = \frac{1}{4\pi i}\int_{-m/2-i\infty}^{-m/2+i\infty} Z(s)\frac{\Gamma(s)}{\Gamma(s+\lambda+1)}K_U(s)\,ds + O_{\varepsilon,\phi}(1+X^{\lambda-(m-3)/2+\varepsilon}).$$

With the functional equation (2.1), this can be written as

$$I(\gamma, X) = \frac{1}{2}(\det\mathfrak{S})^{-1/2}\pi^{-m/2}\sum_{n\ge 1}\frac{b_n}{\mu_n^{m/2}}I(\pi^2\mu_n, \gamma, X) + O(1+X^{\lambda-(m-3)/2+\varepsilon}), \tag{4.4}$$

where

$$I(y, \gamma, X) := \frac{1}{2\pi i}\int_{-m/2-i\infty}^{-m/2+i\infty} K_U(s)\frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}y^s\,ds, \quad y > 0. \tag{4.5}$$

For $\mathrm{Re}(s) = -m/2$, Stirling's formula yields

$$\frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)} \ll (1+|t|)^{-\lambda-1}, \quad t \in \mathbb{R}.$$

Choosing $k = 0$ and $k > 3m/2$ in (4.3), for $y > 0$ we have

$$\begin{aligned} I(y, \gamma, X) \ll{}& \int_{|t|\le\sqrt{U}} U^{(\lambda-m)/2-(m-3)/4}(1+|t|)^{3m/2-\lambda-1}y^{-m/2}\,dt \\ &+ \int_{|t|\ge\sqrt{U}}(1+|t|)^{3m/2-k-\lambda-1}U^{(\lambda+k-m)/2-(m-3)/4}y^{-m/2}\,dt \\ \ll{}& U^{3/4}y^{-m/2}. \end{aligned}$$

From (4.4) it follows that

$$I(\gamma, X) = \frac{1}{2}(\det\mathfrak{S})^{-1/2}\pi^{-m/2}\sum_{\mu_n\le U^{1/m}}\frac{b_n}{\mu_n^{m/2}}I(\pi^2\mu_n, \gamma, X) + O(1+X^{\lambda-(m-3)/2+\varepsilon}+U^{1/4}). \tag{4.6}$$

Choose $m/4 < d < m/2$. Since the integrand in (4.5) decreases faster than polynomially as $|\mathrm{Im}(s)| \to \infty$ in $\{-m/2 \le \mathrm{Re}(s) \le d\}$, for $y > 0$ Cauchy's theorem gives

$$\begin{aligned}
&I(y,\gamma,X)\\
&\quad=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}K_U(s)\frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}y^s\,ds\\
&\quad=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}y^s\int_{a^2U}^{b^2U}x^{\lambda/2-(m+1)/4+s}e\big(\gamma\sqrt{x}\big)\omega\Big(\frac{x}{U}\Big)\,dx\,ds.
\end{aligned}$$

For $\mathrm{Re}(s)=d$ and $a^2U\le x\le b^2U$, the integrand is $O_{y,U,\phi}(|t|^{m/2-2d-\lambda-1})$. The exponent in this last estimate is less than -1 and thus the order of the integrations may be interchanged:

$$\begin{aligned}
&I(y,\gamma,X)\\
&\quad=\int_{a^2U}^{b^2U}x^{\lambda/2-(m+1)/4}e\big(\gamma\sqrt{x}\big)\omega\Big(\frac{x}{U}\Big)\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma(m/2-s)}{\Gamma(s+\lambda+1)}(yx)^s\,ds\,dx.
\end{aligned}$$

From (2.13) and (2.18), it follows that for $y\ge\delta>0$ we have

$$\begin{aligned}
I(y,\gamma,X)&=\pi^{-1/2}y^{(m-2\lambda-1)/4}\\
&\qquad\times\int_{a^2U}^{b^2U}x^{-1/2}e\big(\gamma\sqrt{x}\big)\omega\Big(\frac{x}{U}\Big)\cos\Big(2\sqrt{xy}-\frac{\pi}{2}\Big(\frac{m+1}{2}+\lambda\Big)\Big)\,dx\\
&\qquad+O(y^{(m-2\lambda-3)/4}).
\end{aligned}\tag{4.7}$$

Denote the integral on the right-hand side by $I^*(y,\gamma,X)$. If $\gamma=\pm\sqrt{\mu_n}$, then

$$\begin{aligned}
I^*(\pi^2\mu_n,\gamma,X)&=\int_0^\infty\phi\Big(\frac{t}{X}\Big)\,dt\,e^{\pm\pi i((m+1)/2+\lambda)/2}\\
&\qquad+\int_{aX}^{bX}e^{\pm4\pi i\sqrt{\mu_n}t}\phi\Big(\frac{t}{X}\Big)\,dt\,e^{\mp\pi i((m+1)/2+\lambda)/2}.
\end{aligned}\tag{4.8}$$

Integration by parts shows that the second integral is $O(\mu_n^{-1/2})=O_\gamma(1)$. If $\gamma\notin\{\sqrt{\mu_n},-\sqrt{\mu_n}\}$, then

$$\begin{aligned}
I^*(\pi^2\mu_n,\gamma,X)&=\sum_\pm\int_{aX}^{bX}\phi\Big(\frac{t}{X}\Big)e^{2\pi it(\pm\sqrt{\mu_n}+\gamma)}\,dt\,e^{\mp\pi i((m+1)/2+\lambda)/2}\\
&\ll\frac{1}{\big|\sqrt{\mu_n}-|\gamma|\big|}\ll_\gamma 1.
\end{aligned}\tag{4.9}$$

If $\gamma=\pm\sqrt{\mu_n}$, define

$$C(\gamma):=\tfrac{1}{2}(\det\mathfrak{S})^{-1/2}\pi^{-\lambda-1}b_n\mu_n^{-(m+2\lambda+1)/4}e^{\pm\pi i((m+1)/2+\lambda)/2}.$$

Otherwise, define $C(\gamma):=0$. Then (4.6), (4.7), (4.8), and (4.9) give, for $X\ge\max\{a^{-1},|\gamma|^m\}$,

$$I(\gamma,X)=C(\gamma)\int_0^\infty\phi\Big(\frac{t}{X}\Big)\,dt+O_\gamma(X^{1/2}+X^{\lambda-(m-3)/2+\varepsilon}).$$

Therefore

$$\lim_{X\to\infty}\frac{1}{X}I(\gamma,X)=C(\gamma)\int_0^\infty \phi(x)\,dx,$$

which proves Theorem 1.3.

5. Proof of Corollary 1.4

Part (1)(a) is proved by contradiction. Let $0\le\lambda\le(m-3)/2$ and assume that $\|F_{\lambda 0}\|_2<\infty$. Choose $0\ne\phi\in C_c^\infty(\mathbb{R}^+)$ with $\phi\ge 0$. Then, for $X\ge 1$ and $N\ge 1$,

$$\begin{aligned}
0 &\le \frac{1}{X}\int_1^\infty \Big|F_{\lambda 0}(t)-\sum_{n\le N}C\big(\sqrt{\mu_n}\big)e\big(-\sqrt{\mu_n}t\big)\Big|^2\phi\Big(\frac{t}{X}\Big)\,dt\\
&\le \frac{1}{X}\int_1^\infty |F_{\lambda 0}(t)|^2\phi\Big(\frac{t}{X}\Big)\,dt\\
&\quad -\sum_{n\le N}2\,\mathrm{Re}\Big(\overline{C\big(\sqrt{\mu_n}\big)}\frac{1}{X}\int_1^\infty F_{\lambda 0}(t)e\big(\sqrt{\mu_n}t\big)\phi\Big(\frac{t}{X}\Big)\,dt\Big)\\
&\quad +\frac{1}{X}\sum_{n\le N}\big|C\big(\sqrt{\mu_n}\big)\big|^2\int_1^\infty \phi\Big(\frac{t}{X}\Big)\,dt\\
&\quad +\frac{1}{X}\sum_{n_1,n_2\le N:n_1\ne n_2}C\big(\sqrt{\mu_{n_1}}\big)\overline{C\big(\sqrt{\mu_{n_2}}\big)}\\
&\quad \times\frac{1}{X}\int_1^\infty e\big(\big(\sqrt{\mu_{n_2}}-\sqrt{\mu_{n_1}}\big)t\big)\phi\Big(\frac{t}{X}\Big)\,dt.
\end{aligned}$$

From Theorem 1.3 it follows that the integrals in the second term, after division by X, tend to $C\big(\sqrt{\mu_n},\phi\big)$ as $X\to\infty$. Integration by parts shows that the integrals in the last term are $O_{n_1,n_2,\phi}(1)$ as $X\to\infty$. Letting $X\to\infty$ therefore gives

$$\sum_{n\le N}\big|C\big(\sqrt{\mu_n}\big)\big|^2\int_0^\infty\phi(x)\,dx\ll_\phi \|F_{\lambda 0}\|_2^2$$

uniformly in N. Consequently,

$$\sum_{n\ge 1}\frac{|b_n|^2}{\mu_n^{(m+2\lambda+1)/2}}\ll\sum_{n\ge 1}\big|C\big(\sqrt{\mu_n}\big)\big|^2<\infty.$$

In particular,

$$\sum_{x\le\mu_n\le 2x}|b_n|^2=o(x^{(m+2\lambda+1)/2})$$

as $x\to\infty$. Let $a>0$ with $a\mathfrak{S}^{-1}\in\mathbb{Z}^{m\times m}$. Then we have $a\mu_n\in\mathbb{N}$ for $n\in\mathbb{N}$ and $\sum_{x\le\mu_n\le 2x}1\ll x$. The Cauchy–Schwarz inequality gives

$$\sum_{x\le\mu_n\le 2x}b_n\le\Big(\sum_{x\le\mu_n\le 2x}1\Big)^{1/2}\Big(\sum_{x\le\mu_n\le 2x}|b_n|^2\Big)^{1/2}=o(x^{(m+2\lambda+3)/4})=o(x^{m/2})$$

as $x \to \infty$. On the other hand, a trivial lattice point estimate gives

$$\sum_{x \le \mu_n \le 2x} b_n = \sum_{\mathfrak{x}: x \le Q^{-1}[\mathfrak{x}] \le 2x} 1 = cx^{m/2} + O(x^{(m-1)/2})$$

with some $c > 0$, which gives a contradiction. Thus, part (1)(a) is proved.

Part (1)(b) follows from Theorem 1.2.

In order to prove part (2), assume that $\lambda \ge (m-1)/2$ and E is real. Let $x \ge 2$. From (2.23) and (3.2) it follows that

$$\sum_{j=2}^{4} \int_x^{2x} |R_j(t)|\, dt \ll x^{m/2} \log x.$$

Choose $x \le x^* \le 2x$ with

$$\sum_{j=2}^{4} |R_j(x^*)| \ll x^{m/2-1} \log x. \tag{5.1}$$

Now use (2.19) with $T = \pi\sqrt{x^* x}$. Then $\pi \le T/x \le \sqrt{2}\pi$ and

$$\Delta_{\lambda,\mathfrak{a}}(x) \ll \sum_{\mu \le x^*} \frac{b_n}{\mu_n^{m/2}} + 1 + x^{-(m-1)/2} R_1(x) + x^{1-m/2} \sum_{j=2}^{4} R_j(x^*). \tag{5.2}$$

The trivial estimates $r_1(y) \ll y^{(m-1)/2}$ and $\sum_{\mu_n \le y} b_n \ll y^{m/2}$, together with (2.20), yield

$$R_1(x) \ll x^{(m-1)/2} \log x, \qquad \sum_{\mu_n \le x^*} \frac{b_n}{\mu_n^{m/2}} \ll \log x. \tag{5.3}$$

Now (5.1), (5.2), and (5.3) give $\Delta_{\lambda,\mathfrak{a}}(x) \ll \log x$.

6. Proof of Corollary 1.5

Let the assumptions of Corollary 1.5 be fulfilled. It will be shown that all the Fourier coefficients of $\rho F_{\lambda,\mathfrak{a}}$ vanish. Since $\rho F_{\lambda,\mathfrak{a}}$ is supposed to be $\mathcal{B}^q$-almost periodic, it will then follow from the general theory that $\|\rho F_{\lambda,\mathfrak{a}}\|_q = 0$.

Let $\gamma \in \mathbb{R}$ and let $0 < \varepsilon < 1/2$. All estimates will be uniform in ε. Choose $\phi \in C_c^\infty(\mathbb{R}^+)$ such that $0 \le \phi \le 1$, $\phi(t) = 1$ for $\varepsilon \le t \le 1-\varepsilon$, $\phi'(t) \ge 0$ for $0 \le t \le \varepsilon$, $\phi'(t) \le 0$ for $1-\varepsilon \le t \le 1$, and $\phi(t) = 0$ for $t \ge 1$ and $t \le 0$. Define $G(t) := F_{\lambda,\mathfrak{a}}(t) e(\gamma t)$. From Theorem 1.3 it follows that, for $u \ge 1$,

$$\left| \int_1^\infty G(t)\phi\left(\frac{t}{u}\right) dt \right| \le d(\gamma,\phi) u, \tag{6.1}$$

where $d(\gamma,\phi) > 0$ is a constant depending only on γ and ϕ. Choose $T_0(\varepsilon) \ge 1$ such that $d(\gamma,\phi)|\rho(T)| \le \varepsilon$ for $T \ge T_0(\varepsilon)$. Then, for $T \ge T_0(\varepsilon)$,

$$\left|\frac{1}{T}\int_{T_0}^{T}\int_1^{\infty} G(t)\phi\left(\frac{t}{u}\right)dt\,\rho'(u)\,du\right| \le \frac{d(\gamma,\phi)}{T}\int_{T_0}^{T} u|\rho'(u)|\,du$$
$$\ll \frac{d(\gamma,\phi)}{T}\int_{T_0}^{T}\rho(u)\,du \ll \varepsilon. \qquad (6.2)$$

Integration by parts shows that, for $t \ge 1$,

$$\int_{T_0}^{T}\phi\left(\frac{t}{u}\right)\rho'(u)\,du = \phi\left(\frac{t}{T}\right)\rho(T) - \phi\left(\frac{t}{T_0}\right)\rho(T_0) + \int_{T_0}^{T}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)\rho(u)\,du.$$

From (6.1) and (6.2) it follows that, for $T \ge T_1(\varepsilon) \ge T_0(\varepsilon)$,

$$\frac{1}{T}\int_1^{\infty} G(t)\int_{T_0}^{T}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)\rho(u)\,du\,dt$$
$$\ll \varepsilon + \rho(T)d(\gamma,\phi) + \rho(T_0)\frac{T_0}{T}d(\gamma,\phi) \ll \varepsilon. \qquad (6.3)$$

Call the inner integral on the left-hand side $I(t,T)$. Let $t \ge 1$ and $T \ge T_1(\varepsilon)$. Then

$$I(t,T) = \int_{[T_0,T]\cap[t,t/(1-\varepsilon)]}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)\rho(u)\,du + \int_{[T_0,T]\cap[t/\varepsilon,\infty)}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)\rho(u)\,du$$
$$= I_1(t,T) + I_2(t,T). \qquad (6.4)$$

If $T > \varepsilon T$ then $I_2(t,T) = 0$. If $t \le \varepsilon T$,

$$|I_2(t,T)| = \rho\left(\frac{t}{\varepsilon}\right)\int_{t/\varepsilon}^{T}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)du = \rho\left(\frac{t}{\varepsilon}\right)\left(\phi(\varepsilon) - \phi\left(\frac{t}{T}\right)\right) \le \rho(t). \qquad (6.5)$$

If $t \le u \le t/(1-\varepsilon)$ then the mean value theorem gives, with some $t \le \zeta \le u$,

$$|\rho(u) - \rho(t)| = |u-t|\cdot|\rho'(\zeta)| \le \frac{t\varepsilon}{1-\varepsilon}\,\frac{\rho(\zeta)}{\zeta} \ll \varepsilon\rho(t).$$

Therefore,

$$I_1(t,T) = \rho(t)\int_{[T_0,T]\cap[t,t/(1-\varepsilon)]}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)du$$
$$+ O\left(\varepsilon\rho(t)\int_{[T_0,T]\cap[t,t/(1-\varepsilon)]}\frac{t}{u^2}\left|\phi'\left(\frac{t}{u}\right)\right|du\right). \qquad (6.6)$$

If $t > T$ then the second integral is 0. If $t \le T$, it is

$$\le -\int_t^{t/(1-\varepsilon)}\frac{t}{u^2}\phi'\left(\frac{t}{u}\right)du = \phi(1-\varepsilon) - \phi(1) = 1. \qquad (6.7)$$

If $t > T$ or $t < (1-\varepsilon)T_0$, then the first integral in (6.6) vanishes. If $(1-\varepsilon)T_0 \le t \le T$, it equals $-\phi(\max\{t/T, 1-\varepsilon\}) + \phi(t/T_0)$. Define

$$\psi(\tau) := \phi(\max\{\tau, 1-\varepsilon\}).$$

By (6.4)–(6.7) it now follows that, for $t \ge 1$ and $T \ge T_1(\varepsilon)$,

$$I(t,T) = \rho(t)\left(-\psi\left(\frac{t}{T}\right) + \phi\left(\frac{t}{T_0}\right)\right) I_{[(1-\varepsilon)T_0,T]}(t) + O(\rho(t) I_{[0,\varepsilon T]}(t)) + O(\varepsilon\rho(t) I_{[0,T]}(t)).$$

Hence, for $T \geq T_1(\varepsilon)$,

$$\begin{aligned}\int_1^\infty & G(t)I(t,T)\,dt \\ &= -\int_{(1-\varepsilon)T_0}^{T} G(t)\rho(t)\psi\left(\frac{t}{T}\right) dt + O\left(\int_1^\infty |F_{\lambda,\mathfrak{a}}(t)|\rho(t)\phi\left(\frac{t}{T_0}\right) dt\right) \\ &\quad + O\left(\int_1^{\varepsilon T} |F_{\lambda,\mathfrak{a}}(t)|\rho(t)\,dt\right) + O\left(\varepsilon \int_1^{T} |F_{\lambda,\mathfrak{a}}(t)|\rho(t)\,dt\right). \end{aligned} \tag{6.8}$$

Because $\rho F_{\lambda,\mathfrak{a}}$ is assumed to be $\mathcal{B}^q$-almost periodic, it is also $\mathcal{B}^1$-almost periodic and thus $\|\rho F_{\lambda,\mathfrak{a}}\|_1 < \infty$. Hence, the last two terms in (6.8) are $O(\varepsilon T)$. This, together with (6.3), yields

$$\int_1^\infty G(t)\rho(t)\psi\left(\frac{t}{T}\right) dt = O(\varepsilon T) \tag{6.9}$$

for $T \geq T_2(\varepsilon) \geq T_1(\varepsilon)$. Since $\rho F_{\lambda,\mathfrak{a}}$ is almost periodic, integration by parts gives

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T G(t)\rho(t)\psi\left(\frac{t}{T}\right) dt = fc(\rho F_{\lambda,\mathfrak{a}},\gamma)\int_0^\infty \psi(t)\,dt,$$

where

$$fc(\rho F_{\lambda,\mathfrak{a}},\gamma) := \lim_{T\to\infty}\frac{1}{T}\int_1^T \rho(t)F_{\lambda,\mathfrak{a}}(t)e(\gamma t)\,dt$$

is the Fourier coefficient of $\rho F_{\lambda,\mathfrak{a}}$ corresponding to γ. It therefore follows from (6.9) that

$$fc(\rho F_{\lambda,\mathfrak{a}},\gamma)\int_0^\infty \psi(t)\,dt = O(\varepsilon).$$

Since $\int_0^\infty \psi(t)\,dt \geq 1-\varepsilon$, this means that $fc(\rho F_{\lambda,\mathfrak{a}},\gamma) = O(\varepsilon)$ uniformly in ε. Consequently, $fc(\rho F_{\lambda,\mathfrak{a}},\gamma) = 0$. According to the introductory remark, this proves the corollary.

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