Michigan Math. J. 49 (2001)

Lipschitz Estimates for the $\bar{\partial}$-Equation on the Minimal Ball

N. Viêt Anh & E. H. Youssfi

1. Introduction and Statement of the Main Results

The general theory of the $\bar{\partial}$-equation on convex domains in $\mathbb{C}^n$ is still incomplete. It has been studied in several particular cases of smooth convex domains; see, for example, the articles of Range [21], Diederich, Fornæss, and Wiegerinck [7], Bruna and Castillo [3], Bonami and Charpentier [2], Cumenge [5], and Diederich, Fischer, and Fornæss [6]. In these works, the regularity estimates for the $\bar{\partial}$-equation depend intimately on the geometry of the boundary of the domain. For example, if the domain is smooth convex of finite type m then the sharp gain of smoothness is $1/m$ (see [5; 6]). In proving these results, the boundary smoothness is used heavily. The particular case of smooth strictly pseudoconvex domains corresponds to the $\frac{1}{2}$-regularity. This smoothness has been shown to hold even in the case of nonsmooth strictly pseudoconvex domains with, however, a $\mathcal{C}^2$-defining function (see Henkin and Leiterer [12]).

On other hand, Fornæss and Sibony [8] constructed a smoothly bounded pseudoconvex domain that is strictly pseudoconvex except at one boundary point for which (L^p, L^p)-estimates $(p > 2)$ for $\bar{\partial}$ fail.

In the present work we give an example of a convex circular and non–piecewise smooth domain, with a defining function that is not differentiable, for which the $\bar{\partial}$-equation possesses the Lipschitz $\widetilde{\frac{1}{2}}$-estimate. We also give an explicit construction of the $\bar{\partial}$-solving operator. The domain in question is the minimal ball, which is given by

$$\mathbb{B}_* := \{z \in \mathbb{C}^n : \varrho(z) := |z|^2 + |z \bullet z| < 1\},$$

where $z \bullet w := \sum_{j=1}^n z_j w_j$ (see Hahn and Pflug [10]). Then the minimal ball $\mathbb{B}_*$ is just the open unit ball with respect to the norm $N_* := \sqrt{\varrho}$, as featured in several recent works [13; 15; 16; 17; 18; 19; 20; 24; 25]. In particular, it is a non–Lu Qi-Keng domain for $n \geq 4$ and is neither homogeneous nor Reinhardt. In addition, $\mathbb{B}_*$ has a B-regular boundary in the sense of Sibony [23] and Henkin and Iordan [11].

Set $V := \{z \in \mathbb{C}^n \setminus \{0\} : z \bullet z = 0\}$. The singular part of the boundary of $\mathbb{B}_*$ is obviously the set $\partial\mathbb{B}_* \cap V$. The regular part $\partial\mathbb{B}_* \setminus V$ consists of all strictly pseudoconvex points.

Received September 7, 2000. Revision received January 29, 2001.

In order to state our main results, we need some notation. Denote by $\mathcal{A}^{\infty}_{0,1}(\mathbb{B}_*)$ the space of $\bar\partial$-closed $(0,1)$-forms defined on $\mathbb{B}_*$ with L^∞ coefficients, endowed with the sup norm $\|\cdot\|_\infty$. Next, consider the Lipschitz space

$$\Lambda_{\tilde{\frac{1}{2}}}(\mathbb{B}_*) := \left\{ f : \|f\|_\infty + \sup_{\substack{z, z+h \in \mathbb{B}_* \\ 0<|h|<\frac{1}{2}}} \frac{|f(z+h) - f(z)|}{|h|^{\frac{1}{2}}|\log|h||} \equiv \|f\|_{\Lambda_{\tilde{\frac{1}{2}}}} < \infty \right\}.$$

Our main result is as follows.

MAIN THEOREM. *There exist a finite constant C and an explicitly defined linear integral operator*

$$T \ : \mathcal{A}^{\infty}_{0,1}(\mathbb{B}_*) \to \Lambda_{\tilde{\frac{1}{2}}}(\mathbb{B}_*)$$

satisfying $\bar\partial T f = f$ (in the sense of distributions) and $\|Tf\|_{\Lambda_{\tilde{\frac{1}{2}}}} \le C\|f\|_\infty$ for every $f \in \mathcal{A}^{\infty}_{0,1}(\mathbb{B}_*)$.

We should point out that the $\bar\partial$-solving operator T has the form

$$(Tf)(z) = \int_{\mathbb{B}_*} K[f](z,\zeta)\frac{d\zeta \wedge d\bar\zeta}{|\zeta \bullet \zeta|} + \int_{\partial\mathbb{B}_*} S[f](z,\zeta)\frac{d\theta(\zeta)}{|\zeta \bullet \zeta|},$$

where $K[f]$ and $S[f]$ are appropriate kernels associated with T. The measures appearing in this formula are singular near those points where the defining function ϱ is not differentiable.

This paper is organized as follows. We begin Section 2 by introducing an auxiliary complex manifold $\mathbb{M}$ that is a ramified covering of degree 2 of $\mathbb{B}_* \setminus \{0\}$. The corresponding covering map π will allow us to relate the $\bar\partial$-equation of the manifold $\mathbb{M}$ to that on $\mathbb{B}_*$. The remainder of Section 2 is devoted to construction of the kernels that are necessary for our study of the $\bar\partial$-equation on the manifold $\mathbb{M}$. The integral estimates related to the integral operators appearing in this section will be proved in Section 3. The construction of the $\bar\partial$-solving kernel on $\mathbb{M}$ is given by Theorem 4.2 in Section 4, where we also prove (in Theorem 4.7) a nonisotropic Lipschitz estimate for the $\bar\partial$-equation on $\mathbb{M}$. Finally, in Section 5 we apply the results of Section 4 to establish the main theorem.

In a forthcoming paper, we shall present further study of the $\bar\partial$-equation in a more general class of convex domains.

Throughout this paper, the letter C denotes a finite constant, not necessarily the same at each occurrence, that depends only on the dimension n.

2. Integral Formulas on the Complex Manifold $\mathbb{M}$

Let $n \ge 2$ and set

$$\mathbb{H} = \mathbb{H}_n := \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0\}.$$

Let $\mathbb{B} = \mathbb{B}_{n+1}$ be the unit ball of $\mathbb{C}^{n+1}$. The complex manifold $\mathbb{M}$ is defined by

$$\mathbb{M} = \mathbb{M}_n := \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0 \text{ and } |z| < 1\} = \mathbb{H} \cap \mathbb{B}.$$

The manifold $\mathbb{M}$ is not relatively compact in $\mathbb{H}$ owing to the singularity point 0. The compact group $\mathrm{SO}(n+1, \mathbb{R})$ acts transitively on $\partial\mathbb{M} := \{z \in \mathbb{H} : |z| = 1\}$. The Haar measure of this group induces a unique $\mathrm{SO}(n+1, \mathbb{R})$-invariant probability measure σ on $\partial\mathbb{M}$ (see Mengotti and Youssfi [16]). Finally, denote by dV the surface measure on $\mathbb{H}$.

Recall from Lemma 2.2 in [17] that the $(n, 0)$-form on $(\mathbb{C} \setminus \{0\})^{n+1}$,

$$\frac{1}{n+1} \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

induces by restriction an $\mathrm{SO}(n+1, \mathbb{C})$-invariant and holomorphic $(n, 0)$-form α on $\mathbb{H}$.

Proposition 2.1. *For all compactly supported continuous functions f on $\mathbb{H}$, we have*

$$\int_{\mathbb{H}} f(z)\, dV(z) = \left(\frac{i}{2}\right)^n \int_{\mathbb{H}} f(z)|z|^2 \alpha(z) \wedge \overline{\alpha(z)}.$$

Proof. Let $\omega := \left(\frac{i}{2}\right) \sum_{k=1}^{n+1} dz_k \wedge d\bar z_k$. Then the canonical volume form on $\mathbb{H}$ is $\left(\frac{1}{n!}\right)\omega^n|_{\mathbb{H}}$. Using the open chart $\mathcal{U}_j := \{z \in \mathbb{H} : z_j \neq 0\}$, a little computing shows that if $z \in \mathcal{U}_j$ then, on the n-fold tangent to $\mathbb{H}$ at z, we have

$$\begin{aligned}
\frac{1}{n!}\left(\frac{2}{i}\right)^n \omega^n|_{\mathbb{H}} &= dz_1 \wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \widehat{d\bar z_j} \wedge \cdots \wedge dz_{n+1} \wedge d\bar z_{n+1} \\
&\quad + \sum_{k \neq j} dz_1 \wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \widehat{d\bar z_k} \wedge \cdots \wedge dz_{n+1} \wedge d\bar z_{n+1} \\
&= \frac{|z|^2}{|z_j|^2} dz_1 \wedge d\bar z_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \widehat{d\bar z_j} \wedge \cdots \wedge dz_{n+1} \wedge d\bar z_{n+1} \\
&= |z|^2 \alpha(z) \wedge \overline{\alpha(z)}.
\end{aligned}$$

This completes the proof. □

In what follows we shall establish some integral formulas on $\mathbb{M}$. To do so, we shall approximate $\mathbb{M}$ by appropriate regular varieties that are complete intersections. Then we apply to each of these varieties the results of Berndtsson [1].

For $0 < r < 1$, let $\mathbb{B}_r$ be the ball in $\mathbb{C}^{n+1}$ centered at 0 with radius r and set $\mathbb{M}_r := \mathbb{H} \cap \mathbb{B}_r$. Let

$$s := (s_1, \ldots, s_{n+1}) \colon \overline{\mathbb{B} \setminus \mathbb{B}_r} \times \overline{\mathbb{B} \setminus \mathbb{B}_r} \to \mathbb{C}^{n+1}$$

be a $\mathcal{C}^1$ function that satisfies

$$|s(\zeta, z)| \le C|\zeta - z| \quad \text{and} \quad |s \bullet (\zeta - z)| \ge C|\zeta - z|^2 \tag{2.1}$$

uniformly for $\zeta \in \overline{\mathbb{B} \setminus \mathbb{B}_r}$ and for z in any compact subset of $\mathbb{B} \setminus \overline{\mathbb{B}_r}$. We shall use the same symbol s and set $s := \sum_{k=1}^{n+1} s_j d\zeta_j$.

Now we write $z \bullet z - \zeta \bullet \zeta = \sum_{k=1}^{n+1} (\zeta_j + z_j)(z_j - \zeta_j)$ and put

$$g := \sum_{k=1}^{n+1} (\zeta_j + z_j) d\zeta_j.$$

For $\varepsilon > 0$, consider the $(n+1, n)$-form

$$K_s^\varepsilon := \frac{s \wedge (\bar\partial s)^{n-1} \wedge \bar\partial Q_\varepsilon}{[s \bullet (\zeta - z)]^n}, \tag{2.2}$$

where Q_ε is the (1, 0)-form given by

$$Q_\varepsilon := \frac{\overline{\zeta \bullet \zeta}}{|\zeta \bullet \zeta|^2 + \varepsilon} g. \tag{2.3}$$

Consider the differential forms

$$\omega_k(\bar\zeta) := (-1)^{k-1} d\bar\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_k} \wedge \cdots \wedge d\bar\zeta_{n+1} \quad \text{for } 1 \le k \le n+1,$$
$$\omega(\zeta) := d\zeta_1 \wedge \cdots \wedge d\zeta_{n+1}.$$

LEMMA 2.2. *Let* $0 < r \le \delta \le 1$.

(1) *If* $u \in \mathcal{C}^1(\overline{\mathbb{B} \setminus \mathbb{B}_r})$ *and* $z \in \mathbb{M} \setminus \overline{\mathbb{M}_r}$, *then*

$$u(z) = C_0 \lim_{\varepsilon \to 0} \left(\int_{\partial(\mathbb{B} \setminus \mathbb{B}_r)} u K_s^\varepsilon - \int_{\mathbb{B} \setminus \mathbb{B}_r} \bar\partial u \wedge K_s^\varepsilon \right).$$

(2) *If* $u \in \mathcal{C}(\overline{\mathbb{B}_\delta \setminus \mathbb{B}_r})$, *then*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{B}_\delta \setminus \mathbb{B}_r} \frac{\varepsilon u(\zeta)}{(|\zeta \bullet \zeta|^2 + \varepsilon)^2} \omega(\bar\zeta) \wedge \omega(\zeta) = C \int_{\mathbb{M}_\delta \setminus \mathbb{M}_r} u(\zeta) \alpha(\zeta) \wedge \overline{\alpha(\zeta)}.$$

(3) *If* $u \in \mathcal{C}(\partial \mathbb{B}_r)$, *then*

$$\lim_{\varepsilon \to 0} \int_{\partial \mathbb{B}_r} \frac{\varepsilon u(\zeta)}{(|\zeta \bullet \zeta|^2 + \varepsilon)^2} \omega_k(\bar\zeta) \wedge \omega(\zeta) = C r^{2n-3} \int_{\partial \mathbb{M}} u(r\zeta) \zeta_k \, d\sigma(\zeta).$$

Proof. Part (1) of the lemma follows from formulas (23) and (26) in the proof of Theorem 1 in [1].

Recall from equality (25) in [1] that

$$\frac{\varepsilon |\zeta|^2}{(|\zeta \bullet \zeta|^2 + \varepsilon)^2} \to C \, dV \quad \text{as } \varepsilon \to 0$$

in the sense of distributions. This, when combined with Proposition 2.1, gives part (2) of the lemma.

To prove part (3), we may assume without loss of generality that $r = 1$. Applying equality (3) in Proposition 16.4.4 of Rudin [22] yields that, for each $\varepsilon > 0$, we have

$$\int_{\partial \mathbb{B}} \frac{\varepsilon u(\zeta)}{(|\zeta \bullet \zeta|^2 + \varepsilon)^2} \omega_k(\bar\zeta) \wedge \omega(\zeta) = C \int_{\partial \mathbb{B}} \frac{\varepsilon u(\zeta) \zeta_k}{(|\zeta \bullet \zeta|^2 + \varepsilon)^2} \cdot [\bar\zeta \lrcorner (\omega(\bar\zeta) \wedge \omega(\zeta))].$$

Note that the $(2n+1)$-forms $\frac{\varepsilon|\zeta|^2}{(|\zeta\bullet\zeta|^2+\varepsilon)^2}[\bar\zeta\lrcorner(\omega(\bar\zeta)\wedge\omega(\zeta))]$ are $\mathrm{SO}(n+1,\mathbb{R})$-invariant. Using local coordinates and Lelong theory [9], we see that these forms converge as $\varepsilon\to 0$ to a $(2n-1)$-form on $\partial\mathbb{M}$ that is clearly $\mathrm{SO}(n+1,\mathbb{R})$-invariant. Hence it induces a measure that is a constant times the measure $d\sigma$. The proof of part (3) is thus complete. □

Now let

$$K_s := \frac{s\wedge(\bar\partial s)^{n-1}\wedge g\wedge\overline{\partial(\zeta\bullet\zeta)}}{[s\bullet(\zeta-z)]^n}. \tag{2.4}$$

In view of (2.2) and (2.3), we see that K_s satisfies

$$K_s^\varepsilon = K_s\frac{\varepsilon}{(|\zeta\bullet\zeta|^2+\varepsilon)^2}. \tag{2.5}$$

We write K_s in the form

$$K_s = (-1)^{\frac{n(n+1)}{2}}\sum_{k=1}^{n+1} h_k(\zeta,z)\omega_k(\bar\zeta)\wedge\omega(\zeta), \tag{2.6}$$

where the h_k are the component functions of K_s with respect to the $(n+1,n)$-forms $\omega_1(\bar\zeta)\wedge\omega(\zeta),\dots,\omega_{n+1}(\bar\zeta)\wedge\omega(\zeta)$.

If $f := \sum_{k=1}^{n+1} f_k d\bar\zeta_k$ is a $(0,1)$-form that is defined in a neighborhood of $\overline{\mathbb{M}}$ $(:=\mathbb{M}\cup\partial\mathbb{M})$ in $\overline{\mathbb{B}}$, then let $f|_{\mathbb{M}}$ denote the pull-back of f under the canonical injection of $\mathbb{M}$ in this neighborhood. Set

$$\|f\|_{\mathbb{M},\infty} := \sup_{\zeta\in\mathbb{M}}\sum_{k=1}^{n+1}|f_k(\zeta)|.$$

Let $\bar\partial_{\mathbb{M}}$ be the $\bar\partial$-operator on $\mathbb{M}$.

Proposition 2.3. *Given a section s satisfying* (2.1)*: Consider a function $u\in\mathcal{C}^1(\overline{\mathbb{M}\setminus\mathbb{M}_r})$ and a continuous* $(0,1)$*-form $f := \sum_{k=1}^{n+1} f_k d\bar\zeta_k$ defined in a neighborhood of $\overline{\mathbb{M}\setminus\mathbb{M}_r}$ that satisfy $\bar\partial_{\mathbb{M}}u = f|_{\mathbb{M}}$ on $\mathbb{M}\setminus\overline{\mathbb{M}_r}$. Let h_k be the functions defined in* (2.6)*. Then, for $z\in\mathbb{M}\setminus\overline{\mathbb{M}_r}$,*

$$\begin{aligned}u(z) &= C_1\int_{\partial\mathbb{M}} u(\zeta)\left(\sum_{k=1}^{n+1}\zeta_k h_k(\zeta,z)\right)d\sigma(\zeta)\\ &\quad - C_1 r^{2n-3}\int_{\partial\mathbb{M}} u(r\zeta)\left(\sum_{k=1}^{n+1}\zeta_k h_k(r\zeta,z)\right)d\sigma(\zeta)\\ &\quad + C_2\int_{\mathbb{M}\setminus\overline{\mathbb{M}_r}}\left(\sum_{k=1}^{n+1} f_k(\zeta)h_k(\zeta,z)\right)\alpha(\zeta)\wedge\overline{\alpha(\zeta)}.\end{aligned}$$

Proof. Consider a $\mathcal{C}^1$ extension of u (which is also denoted by u) on $\overline{\mathbb{B}\setminus\mathbb{B}_r}$ that satisfies $\bar\partial u = f$ on $\mathbb{M}\setminus\overline{\mathbb{M}_r}$. Suppose without loss of generality that $f=\bar\partial u$ on $\overline{\mathbb{B}\setminus\mathbb{B}_r}$. By Lemma 2.2(1), (2.5), and (2.6), we have that

$$Cu(z) = \lim_{\varepsilon\to 0}\left\{ \int_{\partial(\mathbb{B}\setminus\mathbb{B}_r)} \frac{\varepsilon u(\zeta)}{(|\zeta\bullet\zeta|^2+\varepsilon)^2}\left(\sum_{k=1}^{n+1} h_k(\zeta,z)\omega_k(\bar\zeta)\wedge\omega(\zeta)\right)\right.$$
$$\left. - \int_{\mathbb{B}\setminus\mathbb{B}_r} \frac{\varepsilon}{(|\zeta\bullet\zeta|^2+\varepsilon)^2}\left(\sum_{k=1}^{n+1} f_k(\zeta)h_k(\zeta,z)\right)\omega(\bar\zeta)\wedge\omega(\zeta)\right\}$$

for $z\in\mathbb{M}\setminus\overline{\mathbb{M}_r}$. Proposition 2.3 now follows from parts (2) and (3) of Lemma 2.2. □

The following result gives the Martinelli–Bochner formula on $\mathbb{M}$.

THEOREM 2.4. *Let u be a bounded function in $\mathcal{C}^1(\overline{\mathbb{M}})$ and $f := \sum_{k=1}^{n+1} f_k d\bar\zeta_k$ a continuous $(0,1)$-form defined in a neighborhood of $\overline{\mathbb{M}}$ in $\overline{\mathbb{B}}$ that satisfy $\bar\partial_{\mathbb{M}} u = f|_{\mathbb{M}}$ and $\|f\|_{\mathbb{M},\infty} < \infty$. Then, for $z\in\mathbb{M}$,*

$$u(z) = C_1\int_{\partial\mathbb{M}} \frac{1+z\bullet\bar\zeta-\bar z\bullet\zeta-|z\bullet\bar\zeta|^2+|z\bullet\zeta|^2}{|z-\zeta|^{2n}}u(\zeta)\,d\sigma(\zeta)$$
$$+ C_2\int_{\mathbb{M}}\sum_{k=1}^{n+1}\frac{[(\bar z_k-\bar\zeta_k)(z\bullet\bar\zeta+|\zeta|^2) - (z_k+\zeta_k)\overline{z\bullet(\zeta-z)}]}{|z-\zeta|^{2n}} f_k(\zeta)\alpha(\zeta)\wedge\overline{\alpha(\zeta)}.$$

Proof. Consider the Martinelli–Bochner section $s_b(z,\zeta) := \bar\zeta - \bar z$. By (2.4) we then have that

$$K_b = \frac{1}{|z-\zeta|^{2n}}\left[\sum_{j=1}^{n+1}(-\bar z_j+\bar\zeta_j)d\zeta_j\right]\wedge\left[\sum_{j=1}^{n+1} d\bar\zeta_j\wedge d\zeta_j\right]^{n-1}$$
$$\wedge\left[\sum_{j,k=1}^{n+1}(z_k+\zeta_k)\bar\zeta_j d\zeta_k\wedge d\bar\zeta_j\right]$$

so that, by (2.6) and the facts that $\sum_{j\neq k}\bar\zeta_j^2 = -\bar\zeta_k^2$ and $z\bullet z = 0$, we obtain

$$h_k(\zeta,z) = \frac{(\bar z_k-\bar\zeta_k)(z\bullet\bar\zeta+|\zeta|^2)-(z_k+\zeta_k)\overline{z\bullet(\zeta-z)}}{|z-\zeta|^{2n}}. \tag{2.7}$$

Therefore, a simple computation gives that

$$\sum_{k=1}^{n+1}\zeta_k h_k(\zeta,z) = \frac{-|z\bullet\zeta|^2+|z\bullet\bar\zeta|^2-|\zeta|^2(|\zeta|^2+z\bullet\bar\zeta-\bar z\bullet\zeta)}{|z-\zeta|^{2n}}. \tag{2.8}$$

Combining (2.7) and (2.8) with the hypothesis that u is bounded and $\|f\|_{\mathbb{M},\infty} < \infty$, it is not hard to prove that

$$\lim_{r\to 0} r^{2n-3}\int_{\partial\mathbb{M}}\left|u(r\zeta)\left(\sum_{k=1}^{n+1}\zeta_k h_k(r\zeta,z)\right)\right|d\sigma(\zeta) = 0 \tag{2.9}$$

and

$$\lim_{r\to 0}\int_{\mathbb{M}_r}\left|\sum_{k=1}^{n+1} f_k(\zeta)h_k(\zeta,z)\right|\alpha(\zeta)\wedge\overline{\alpha(\zeta)} \lesssim \lim_{r\to 0}\int_{\mathbb{M}_r}\alpha(\zeta)\wedge\overline{\alpha(\zeta)} = 0, \tag{2.10}$$

where the latter equality holds by an application of [16, Lemma 2.1]. The theorem now follows by combining Proposition 2.3 and (2.7)–(2.10). □

REMARK 2.5. If $u\in\mathcal{C}^1(\mathbb{M})$ is bounded, then Theorem 2.4 holds for the dilated functions $u_r(z) := u(rz)$, $0<r<1$. This shows that Theorem 2.4 remains true if we assume only that $u\in\mathcal{C}^1(\mathbb{M})$ is bounded and

$$\lim_{r\to 1-}\int_{\partial\mathbb{M}}|u(\zeta)-u(r\zeta)|\,d\sigma(\zeta) = 0.$$

We recall from Youssfi's work [25] that the Szegö projection $\mathcal{S}_{\mathbb{M}}$ of $\mathbb{M}$ is given by

$$\mathcal{S}_{\mathbb{M}}[u](z) = \int_{\partial\mathbb{M}}\frac{1+z\bullet\bar\zeta}{(1-z\bullet\bar\zeta)^n}u(\zeta)\,d\sigma(\zeta)\quad\text{for }\ z\in\mathbb{M}. \tag{2.11}$$

Following Charpentier [4], let

$$s_0(\zeta,z) := \bar\zeta(1-\zeta\bullet\bar z)-\bar z(1-|\zeta|^2)\quad\text{and}\quad D(\zeta,z) := s_0(\zeta,z)\bullet(\zeta-z).$$

THEOREM 2.6. *There are polynomials $P_k(\zeta,z)$ and $Q_k(\zeta,z)$, $1\le k\le n+1$, such that*

$$P_k(z,z) = Q_k(z,z) = 0\quad\forall z\in\mathbb{C}^{n+1}$$

with the following property. Given a bounded function $u\in\mathcal{C}^1(\overline{\mathbb{M}})$ and a continuous $(0,1)$-form $f := \sum_{k=1}^{n+1} f_k\,d\bar\zeta_k$ defined in a neighborhood of $\overline{\mathbb{M}}$ in $\overline{\mathbb{B}}$ that satisfy $\bar\partial_{\mathbb{M}}u = f|_{\mathbb{M}}$ and $\|f\|_{\mathbb{M},\infty}<\infty$, for $z\in\mathbb{M}$ we have

$$u(z) = \int_{\mathbb{M}}\sum_{k=1}^{n+1}\frac{(1-\zeta\bullet\bar z)^{n-2}}{D(\zeta,z)^n}[(1-\zeta\bullet\bar z)P_k(\zeta,z) + (1-|\zeta|^2)Q_k(\zeta,z)]\,f_k(\zeta)\alpha(\zeta)\wedge\overline{\alpha(\zeta)} + \mathcal{S}_{\mathbb{M}}[u].$$

Proof. Consider a $\mathcal{C}^1$ extension of u in $\overline{\mathbb{B}}\setminus\{0\}$. By (2.4), the kernel K_0 associated with the section s_0 is

$$\begin{aligned} K_0 := {}&\frac{1}{D(\zeta,z)^n}\sum_{j=1}^{n+1}[\bar\zeta_j(1-\zeta\bullet\bar z)-\bar z_j(1-|\zeta|^2)]d\zeta_j\\ &\wedge\left\{(1-\zeta\bullet\bar z)^{n-1}\left[\sum_{j=1}^{n+1}d\bar\zeta_j\wedge d\zeta_j\right]^{n-1}\right.\\ &\qquad\left.+(n-1)(1-\zeta\bullet\bar z)^{n-2}\left[\sum_{j=1}^{n+1}d\bar\zeta_j\wedge d\zeta_j\right]^{n-2}\wedge\bar\partial|\zeta|^2\wedge\left(\sum_{j=1}^{n+1}\bar z_j d\zeta_j\right)\right\}\\ &\wedge\left[\sum_{j,k=1}^{n+1}(z_k+\zeta_k)\bar\zeta_j\,d\zeta_k\wedge d\bar\zeta_j\right]. \end{aligned}\tag{2.12}$$

In view of (2.5), if we integrate uK_0^ε over $\partial\mathbb{B}$ then all terms containing $\bar\partial|\zeta|^2$ vanish; moreover, we have that $1-|\zeta|^2=0$ and $D(\zeta,z)=|1-z\bullet\bar\zeta|^2$ so that

$$\begin{aligned}\lim_{\varepsilon\to0}&\int_{\partial\mathbb{B}} uK_0^\varepsilon\\ &=\int_{\partial\mathbb{B}}\frac{\varepsilon u(\zeta)}{(|\zeta\bullet\zeta|^2+\varepsilon)^2}\\ &\quad\cdot\left\{\frac{\sum_{j=1}^{n+1}\bar\zeta_j d\zeta_j\wedge\left[\sum_{k=1}^{n+1}d\bar\zeta_k\wedge d\zeta_k\right]^{n-1}\wedge\left[\sum_{j,k=1}^{n+1}(z_k+\zeta_k)\bar\zeta_j d\zeta_k\wedge d\bar\zeta_j\right]}{(1-z\bullet\bar\zeta)^n}\right\}.\end{aligned}$$

Rewriting the differential form in braces as

$$(-1)^{\frac{n(n+1)}{2}}\sum_{k=1}^{n+1}h_k(\zeta,z)\omega_k(\bar\zeta)\wedge\omega(\zeta)$$

and then applying Lemma 2.2(3) to this, we obtain

$$C_0\lim_{\varepsilon\to0}\int_{\partial\mathbb{B}}uK_0^\varepsilon=C_3\int_{\partial\mathbb{M}}\frac{1+z\bullet\bar\zeta}{(1-z\bullet\bar\zeta)^n}u(\zeta)\,d\sigma(\zeta)=C_3\mathcal{S}_{\mathbb{M}}[u],$$

where the latter equality holds by (2.11).

If we set $u\equiv1$ in Lemma 2.2(1), then the last equation implies that $C_3=1$. Thus

$$C_0\lim_{\varepsilon\to0}\int_{\partial\mathbb{B}}uK_0^\varepsilon=\mathcal{S}_{\mathbb{M}}[u]. \tag{2.13}$$

We now write the kernel K_0 in the form of (2.6):

$$K_0=(-1)^{\frac{n(n+1)}{2}}\sum_{k=1}^{n+1}h_k(\zeta,z)\omega_k(\bar\zeta)\wedge\omega(\zeta).$$

We may assume (as in the proof of Proposition 2.3) that $f=\bar\partial u$ on $\mathbb{B}\setminus\{0\}$. Then we set

$$I:=\bar\partial u\wedge K_0=\sum_{k=1}^{n+1}f_k(\zeta)h_k(\zeta,z)\omega(\bar\zeta)\wedge\omega(\zeta). \tag{2.14}$$

Arguing as in the proof of (2.9)–(2.10), we see that

$$\lim_{r\to0}r^{2n-3}\int_{\partial\mathbb{M}}\left|u(r\zeta)\left(\sum_{k=1}^{n+1}\zeta_k h_k(r\zeta,z)\right)\right|d\sigma(\zeta)=0 \tag{2.15}$$

and

$$\lim_{r\to0}\int_{\mathbb{M}_r}\left|\sum_{k=1}^{n+1}f_k(\zeta)h_k(\zeta,z)\right|\alpha(\zeta)\wedge\overline{\alpha(\zeta)}=0. \tag{2.16}$$

To finish the proof of the theorem, it suffices to prove the following lemma.

LEMMA 2.7. *The functions h_k in formula* (2.14) *can be written in the form*

$$h_k(\zeta,z)=\frac{(1-\zeta\bullet\bar z)^{n-2}}{D(\zeta,z)^n}[(1-\zeta\bullet\bar z)P_k(\zeta,z)+(1-|\zeta|^2)Q_k(\zeta,z)], \tag{2.17}$$

where $P_k(\zeta, z)$ and $Q_k(\zeta, z)$ are polynomials such that

$$P_k(z, z) = Q_k(z, z) = 0 \quad \forall z \in \mathbb{C}^{n+1}.$$

The proof of this lemma will be given shortly.

Proof of Theorem 2.6 (cont.). Suppose that Lemma 2.7 has been proved. Combining (2.13)–(2.16), we may then deduce from Proposition 2.3 that

$$u(z) = \mathcal{S}_{\mathbb{M}}[u] + \int_{\mathbb{M}} \sum_{k=1}^{n+1} f_k(\zeta) h_k(\zeta, z)\alpha(\zeta) \wedge \overline{\alpha(\zeta)}.$$

Applying Lemma 2.7 to the last equation, the theorem follows. □

Proof of Lemma 2.7. In view of (2.12) and (2.14), we can write $I = I_1 + I_2$, where

$$\begin{aligned} I_1 := C\bigg(\sum_{l=1}^{n+1} f_l(\zeta) d\overline{\zeta_l}\bigg) & \\ \wedge \bigg\{ & \frac{(1-\zeta\bullet\bar z)^{n-1}}{D(\zeta,z)^n} \sum_{j=1}^{n+1} [\bar\zeta_j(1-\zeta\bullet\bar z) - \bar z_j(1-|\zeta|^2)] d\zeta_j \\ & \wedge \bigg[\sum_{p,k=1}^{n+1} (z_k+\zeta_k)\bar\zeta_p d\zeta_k \wedge d\bar\zeta_p\bigg] \wedge \bigg[\sum_{q=1}^{n+1} d\bar\zeta_q \wedge d\zeta_q\bigg]^{n-1}\bigg\} \end{aligned}$$

and

$$\begin{aligned} I_2 := C\bigg(\sum_{l=1}^{n+1} f_l(\zeta) d\overline{\zeta_l}\bigg) & \\ \wedge \bigg\{ & \frac{(1-\zeta\bullet\bar z)^{n-2}}{D(\zeta,z)^n} \sum_{j=1}^{n+1} [\bar\zeta_j(1-\zeta\bullet\bar z) - \bar z_j(1-|\zeta|^2)] d\zeta_j \\ & \wedge \bar\partial|\zeta|^2 \wedge \bigg[\sum_{r=1}^{n+1} \bar z_r d\zeta_r\bigg] \wedge \bigg[\sum_{p,k=1}^{n+1} (z_k+\zeta_k)\bar\zeta_p d\zeta_k \wedge d\bar\zeta_p\bigg] \\ & \wedge \bigg[\sum_{q=1}^{n+1} d\bar\zeta_q \wedge d\zeta_q\bigg]^{n-2}\bigg\}. \end{aligned}$$

A simple computation shows that

$$\begin{aligned} I_1 = C \cdot \frac{(1-\zeta\bullet\bar z)^{n-1}}{D(\zeta,z)^n} \sum_{k=1}^{n+1} f_k(\zeta)\{[-\bar\zeta_k(1-\zeta\bullet\bar z) + \bar z_k(1-|\zeta|^2)](z\bullet\bar\zeta + |\zeta|^2) \\ - (1-|\zeta|^2)(z_k+\zeta_k)\overline{z\bullet(\zeta-z)}\}\omega(\bar\zeta)\wedge\omega(\zeta). \end{aligned}$$

Therefore, the functions h_k associated to I_1 (in the same way as the ones associated to I in (2.14)) are in the form (2.17).

We now consider I_2. Since the differential form in braces of I_2 is $\mathrm{SO}(n+1, \mathbb{R})$-invariant with respect to (z, ζ), we may suppose without loss of generality that

$$\zeta := \left(\frac{t}{\sqrt{2}}, \frac{it}{\sqrt{2}}, 0, \ldots, 0\right) \in \mathbb{M}, \quad \text{where } 0 < t < 1.$$

We divide I_2 into two pieces I_{21} and I_{22}. Here

$$\begin{aligned} I_{21} := C\Bigg(\sum_{l=1}^{n+1} f_l(\zeta)d\overline{\zeta_l}\Bigg) \\ \wedge \Bigg\{ \frac{(1-\zeta\bullet\bar{z})^{n-2}}{D(\zeta,z)^n} \sum_{j=1}^{n+1} [\bar{\zeta}_j(1-\zeta\bullet\bar{z}) - \bar{z}_j(1-|\zeta|^2)]d\zeta_j \\ \wedge \zeta_2 d\bar{\zeta}_2 \wedge \Bigg[\sum_{r=1}^{n+1} \bar{z}_r d\zeta_r\Bigg] \wedge \Bigg[\sum_{p=1}^{n+1} (z_p+\zeta_p)\bar{\zeta}_1 d\zeta_p \wedge d\bar{\zeta}_1\Bigg] \\ \wedge \Bigg[\sum_{q=1}^{n+1} d\bar{\zeta}_q \wedge d\zeta_q\Bigg]^{n-2}\Bigg\}. \end{aligned}$$

To obtain I_{22}, it suffices to interchange ζ_1 and ζ_2 in I_{21}.

Observe that I_{21} is the $\mathbb{C}^{n+1}$-canonical volume form multiplied by a function of the form

$$\frac{(1-\zeta\bullet\bar{z})^{n-2}}{D(\zeta,z)^n} \sum_{k=1}^{n+1} R_k(\zeta,z) f_k(\zeta),$$

where the $R_k(\zeta, z)$ are polynomials that we shall examine next. In what follows, $O(|z-\zeta|)$ denotes any polynomial $R(\zeta, z)$ such that $R(z, z) = 0$.

If $k \in \{1, 2\}$, then it is easy to see that $R_k(\zeta, z) = 0$.

If $k \notin \{1, 2\}$, then $\zeta_k = 0$. In addition, $R_k(\zeta, z)$ has three components corresponding to the cases $j = k$, $r = k$, and $p = k$.

Case $j = k$. In this case we already have $\bar{\zeta}_j = 0$. The component corresponding to this case equals

$$O[\bar{\zeta}_j(1-\zeta\bullet\bar{z}) - \bar{z}_j(1-|\zeta|^2)] = (1-|\zeta|^2)O(|z-\zeta|).$$

Case $r = k$. Then $\bar{\zeta}_r = 0$. The presence of $\bar{z}_r$ implies that the component corresponding to this case equals

$$\begin{aligned} &O[(\bar{z}_r - \bar{\zeta}_r)(\bar{\zeta}_j(1-\zeta\bullet\bar{z}) - \bar{z}_j(1-|\zeta|^2))] \\ &\qquad = (1-\zeta\bullet\bar{z})O(|z-\zeta|) + (1-|\zeta|^2)O(|z-\zeta|). \end{aligned}$$

Case $p = k$. Then $\zeta_p = 0$. Because of the factor $z_p + \zeta_p$, it follows (as in the previous case) that the corresponding component is of the form

$$(1-\zeta\bullet\bar{z})O(|z-\zeta|) + (1-|\zeta|^2)O(|z-\zeta|).$$

We conclude that

$$R_k(\zeta,z) = (1-\zeta\bullet\bar{z})O(|z-\zeta|) + (1-|\zeta|^2)O(|z-\zeta|).$$

Therefore, the functions h_k associated to I_{21} (in the same way as the ones associated to I in (2.14)) are in the form (2.17).

Analogous argument shows that the same conclusion holds also for I_{22}. Since $I = I_1 + I_{21} + I_{22}$, the proof of the lemma is complete. □

3. Integral Estimates

In this section we prove some estimates for integrals that are needed in the next section.

LEMMA 3.1. *There exists a constant C such that, for all $w \in \partial\mathbb{M}$ and $0 < r < 1$,*

$$\int_{\partial\mathbb{M}} \frac{|\zeta - w|^{\frac{1}{2}}}{|\zeta - rw|^{2n}} \, d\sigma(\zeta) < C(1-r)^{-\frac{1}{2}}.$$

Proof. Since the group $\mathrm{SO}(n+1,\mathbb{R})$ acts transitively on $\partial\mathbb{M}$, we may suppose without loss of generality that

$$w = w_0 := \left(\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{i}{\sqrt{2}}\right).$$

Let $\partial\mathbb{B}_n$ be the unit sphere of $\mathbb{C}^n$. Consider the map $F\colon \mathbb{H} \to \mathbb{C}^n$ defined by

$$F(w) = \frac{|w|(w_1, \ldots, w_n)}{|(w_1, \ldots, w_n)|}.$$

Observe that F is locally diffeomorphic at w_0 and that $F(\partial\mathbb{M}) \subset \partial\mathbb{B}_n$. In addition, $F(rw) = rF(w)$ for $r \in \mathbb{R}^+$. Using the map F, the desired estimate is an easy consequence of the following one (see [22, pp. 360–361]):

$$\int_{\partial\mathbb{B}_n} \frac{|\zeta - w|^{\frac{1}{2}}}{|\zeta - rw|^{2n}} \, d\sigma(\zeta) < C(1-r)^{-\frac{1}{2}} \quad \text{for } w \in \partial\mathbb{B}_n. \qquad \square$$

Let

$$\Delta(z, w, \zeta) := (L_1(z,\zeta) - L_1(w,\zeta), \ldots, L_{n+1}(z,\zeta) - L_{n+1}(w,\zeta)),$$

where

$$L_k(z,\zeta) := \frac{(\bar z_k - \bar\zeta_k)(z \bullet \bar\zeta + |\zeta|^2) - (z_k + \zeta_k)\overline{z \bullet (\zeta - z)}}{|z - \zeta|^{2n}}.$$

Then Δ has the following invariant property:

$$\Delta(Az, Aw, A\zeta) = A\Delta(z, w, \zeta) \quad \forall A \in \mathrm{SO}(n+1,\mathbb{R});$$

$$\Delta(tz, tw, t\zeta) = t^{-(2n-3)}\Delta(z, w, \zeta) \quad \forall t \in \mathbb{R}^+.$$

LEMMA 3.2. *Given $t \in \mathbb{R}^+$, $A \in \mathrm{SO}(n+1,\mathbb{R})$, and a domain $\Omega \subset \mathbb{H}$, for $z, w \in \mathbb{H}$ we then have*

$$\int_{tA(\Omega)} \|\Delta(tAz, tAw, \zeta)\| \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \le Ct \int_{\Omega} \|\Delta(z, w, \zeta)\| \alpha(\zeta) \wedge \overline{\alpha(\zeta)}.$$

Proof. The change of variable $\zeta = tA\tilde\zeta$ and the invariant properties stated previously give that

$$\|\Delta(tAz, tAw, \zeta)\|\alpha(\zeta) \wedge \overline{\alpha(\zeta)} = t\|\Delta(Az, Aw, A\tilde{\zeta})\|\alpha(\tilde{\zeta}) \wedge \overline{\alpha(\tilde{\zeta})}$$
$$\leq t\|A\| \cdot \|\Delta(z, w, \tilde{\zeta})\|\alpha(\tilde{\zeta}) \wedge \overline{\alpha(\tilde{\zeta})}.$$

Since $A \to \|A\|$ is bounded, the lemma follows. □

LEMMA 3.3. *Fix a point $w \in \partial\mathbb{M}$. Then there exists a constant C such that, for all $t \in (0,1)$ and all $z \in \mathbb{H} \cup \{0\}$,*

$$\int_{\zeta\in\mathbb{H}:|\zeta|<1/t} \|\Delta(z, w, \zeta)\|\alpha(\zeta) \wedge \overline{\alpha(\zeta)} < C|z-w|(|\ln t|z-w|| + 1).$$

Proof. We distinguish three cases.

Case 1: $z = 0$. Applying [16, Lemma 2.1], we see that

$$\int_{|\zeta|\leq\sqrt{2}/2} \|\Delta(0, w, \zeta)\|\alpha(\zeta) \wedge \overline{\alpha(\zeta)} \leq C. \tag{3.1}$$

It can be checked that

$$\int_{|\zeta-w|\leq\sqrt{2}/2} \|\Delta(0, w, \zeta)\|\alpha(\zeta) \wedge \overline{\alpha(\zeta)} \leq C. \tag{3.2}$$

If $\zeta \in \mathbb{H}$ satisfies $|\zeta| > \sqrt{2}/2$ and $|\zeta - w| > \sqrt{2}/2$, then we have $|\zeta| \approx |sw - \zeta|$ for $0 \leq s \leq 1$. Therefore, applying the mean value theorem to the functions $L_k(\cdot, \zeta)$ yields

$$\|\Delta(0, w, \zeta)\| \leq \frac{C}{|\zeta|^{2n-2}}.$$

Hence, by [16, Lemma 2.1],

$$\int_{|\zeta|>\sqrt{2}/2,|\zeta-w|>\sqrt{2}/2,|\zeta|\leq1/t} \|\Delta(0, w, \zeta)\|\alpha(\zeta) \wedge \overline{\alpha(\zeta)} \lesssim \int_{\sqrt{2}/2\leq|\zeta|\leq1/t} \frac{\alpha(\zeta) \wedge \overline{\alpha(\zeta)}}{|\zeta|^{2n-2}}$$
$$\leq C(|\log|t|| + 1). \tag{3.3}$$

The lemma now follows from (3.1), (3.2), and (3.3).

Let $\delta > 0$ be sufficiently small so that any $z \in \mathbb{H} : |z - w| < \delta$ can be connected to w by a smooth curve $\gamma_z : [0, 1] \to \mathbb{H}$ with $\gamma_z(0) = z$, $\gamma_z(1) = w$, and $|\gamma_z'(t)| \leq \frac{3}{2}|z - w|$.

Case 2: $|z - w| \geq \delta$. Choose some $A \in SO(n + 1, \mathbb{R})$ such that $z = |z|Aw$. Then, applying Lemma 3.2 together with the result of case 1, we see that

$$\int_{\zeta\in\mathbb{H}:|\zeta|<1/t} (\|\Delta(0, z, \zeta)\| + \|\Delta(0, w, \zeta)\|)\alpha(\zeta) \wedge \overline{\alpha(\zeta)}$$
$$\lesssim |z|(|\log t|z|| + 1) + |\log t| + 1$$
$$< C|z - w|(|\ln t|z - w|| + 1).$$

Since

$$\|\Delta(z, w, \zeta)\| \leq \|\Delta(0, z, \zeta)\| + \|\Delta(0, w, \zeta)\|,$$

the lemma follows in this case, too.

Case 3: $|z-w| \le \delta$. It can be checked that

$$\begin{aligned}\int_{|\zeta-w|\le 2|z-w|} &\|\Delta(z,w,\zeta)\| \alpha(\zeta)\wedge\overline{\alpha(\zeta)} \\ &\lesssim \int_{|\zeta-z|\le 3|z-w|} \frac{\alpha(\zeta)\wedge\overline{\alpha(\zeta)}}{|z-\zeta|^{2n-1}} \\ &\quad + \int_{|\zeta-w|\le 2|z-w|} \frac{\alpha(\zeta)\wedge\overline{\alpha(\zeta)}}{|\zeta-w|^{2n-1}} \le C|z-w|. \end{aligned} \tag{3.4}$$

If $\zeta \in \mathbb{H}$ satisfies $|\zeta-w| > 2|z-w|$, then $|\gamma_z(s)-\zeta| \approx |\zeta-w|$ for $0 \le s \le 1$. This, combined with the mean value theorem, implies that

$$\|\Delta(z,w,\zeta)\| \le C|z-w| \frac{1+|\zeta|^2}{|\zeta-w|^{2n}}. \tag{3.5}$$

Now consider the following subsets of $\mathbb{H}$:

$$\begin{aligned} E_1 &:= \{2\delta > |\zeta-w| > 2|z-w|\}; \\ E_2 &:= \{|\zeta-w| > 2\delta, |\zeta| \le 2\}; \\ E_3 &:= \{|\zeta-w| > 2|z-w|, 2 < |\zeta| \le 1/t\}. \end{aligned}$$

Then

$$\begin{aligned}\int_{|\zeta-w|>2|z-w|, |\zeta|\le 1/t} &\|\Delta(z,w,\zeta)\| \alpha(\zeta)\wedge\overline{\alpha(\zeta)} \\ &\le \sum_{k=1}^{3} \int_{E_k} \|\Delta(z,w,\zeta)\| \alpha(\zeta)\wedge\overline{\alpha(\zeta)}. \end{aligned} \tag{3.6}$$

The integral over E_2 is clearly majorized by $C|z-w|$.

The estimate (3.5), combined with Lemma 2.1 in [16], shows that

$$\begin{aligned}\int_{E_1} \|\Delta(z,w,\zeta)\| \alpha(\zeta)\wedge\overline{\alpha(\zeta)} &\lesssim |z-w| \int_{E_1} \frac{\alpha(\zeta)\wedge\overline{\alpha(\zeta)}}{|\zeta-w|^{2n}} \\ &< C|z-w|(|\log|z-w|| + 1), \end{aligned}$$

and

$$\begin{aligned}\int_{E_3} \|\Delta(z,w,\zeta)\| \alpha(\zeta)\wedge\overline{\alpha(\zeta)} &\lesssim |z-w| \int_{E_3} \frac{\alpha(\zeta)\wedge\overline{\alpha(\zeta)}}{|\zeta|^{2n-2}} \\ &< C|z-w|(|\log|t|| + 1). \end{aligned}$$

Using the hypothesis that $0 < t < 1$ and putting the estimates just displayed together with (3.4) and (3.6), the lemma follows also in this last case. □

The next lemma gives a final integral estimate for $\|\Delta(z,w,\zeta)\|$.

LEMMA 3.4. *There exists a constant C such that, for all $z, w \in \overline{\mathbb{M}}$,*

$$\int_{\mathbb{M}} \|\Delta(z,w,\zeta)\|\alpha(\zeta)\wedge\overline{\alpha(\zeta)} \le C|z-w|(|\log|z-w||+1).$$

Proof. Fix $w_0 \in \partial\mathbb{M}$. Choose some $A \in \mathrm{SO}(n+1,\mathbb{R})$ such that $w = |w|Aw_0$. We first apply Lemma 3.2 and then Lemma 3.3 to obtain

$$\begin{aligned}\int_{\mathbb{M}} &\|\Delta(z,w,\zeta)\|\alpha(\zeta)\wedge\overline{\alpha(\zeta)} \\ &\lesssim |w| \int_{\zeta\in\mathbb{H},\,|\zeta|\le 1/|w|} \left\|\Delta\left(\frac{A^{-1}z}{|w|}, w_0, \zeta\right)\right\| \alpha(\zeta)\wedge\overline{\alpha(\zeta)} \\ &\le C|z-w|(|\log|z-w||+1).\end{aligned}$$

□

LEMMA 3.5. *Let f satisfy the hypothesis of Theorem 2.6. For $z \in \overline{\mathbb{M}}$, define*

$$\begin{aligned}u_f(z) := \int_{\mathbb{M}} \sum_{k=1}^{n+1} \frac{(1-\zeta\bullet\bar z)^{n-2}}{D(\zeta,z)^n}&[(1-\zeta\bullet\bar z)P_k(\zeta,z) \\ &+ (1-|\zeta|^2)Q_k(\zeta,z)]f_k(\zeta)\alpha(\zeta)\wedge\overline{\alpha(\zeta)}.\end{aligned}$$

Then the dilated functions $u_f(rz)$ converge uniformly to $u_f(z)$ on $\partial\mathbb{M}$ as $r \to 1^-$.

Proof. If $\Gamma(\zeta,z)$ denotes either of the kernels

$$\frac{(1-\zeta\bullet\bar z)^{n-1}P_k(\zeta,z)}{D(\zeta,z)^n} \quad \text{or} \quad \frac{(1-\zeta\bullet\bar z)^{n-2}(1-|\zeta|^2)Q_k(\zeta,z)}{D(\zeta,z)^n},$$

then it is enough to prove that

$$\int_{\mathbb{M}} |\Gamma(\zeta,rz)-\Gamma(\zeta,z)|\alpha(\zeta)\wedge\overline{\alpha(\zeta)} \xrightarrow[r\to 1^-]{} 0 \quad \forall z \in \partial\mathbb{M}. \tag{3.7}$$

For $w \in \mathbb{H}$, let T_w be the complex tangent space to $\mathbb{H}$ at w and let π_w be the orthogonal projection of $\mathbb{C}^{n+1}$ onto T_w.

From the equality $D(\zeta,w) = (1-|w|^2)|\zeta-w|^2 + |\bar w\bullet(\zeta-w)|^2$, it follows that there is a neighborhood $\mathcal{U}$ of w in $\mathbb{H}$ such that $\pi_w|_{\mathcal{U}}$ is biholomorphic and

$$D(\zeta,w) \approx D(\pi_w(\zeta),\pi_w(w)) \quad \forall\zeta\in\mathcal{U}. \tag{3.8}$$

Moreover, it is easy to see that

$$\begin{aligned} 1-\zeta\bullet\bar w &= 1-\pi_w(\zeta)\bullet\overline{\pi_w(w)}, \\ (1-|w|^2)^{\frac12}|\zeta-w| &\lesssim \sqrt{D(\zeta,w)}, \\ |\zeta-w| &\lesssim \sqrt{|1-\zeta\bullet\bar w|}, \\ D(\zeta,w) &= |1-\zeta\bullet\bar w|^2-(1-|\zeta|^2)(1-|w|^2). \end{aligned} \tag{3.9}$$

The following estimate can be proved by the same method as in [2, Lemma I.5]:

$$\int_{\zeta\in\mathbb{B}_n,\,D(\zeta,w)\le\delta} |1-\zeta\bullet\bar w|^{n-\frac32}[D(\zeta,w)]^{-n+\frac12}\,d\zeta\wedge d\bar\zeta \le C\delta^{\frac14}, \quad w\in\mathbb{B}_n. \tag{3.10}$$

Here $\delta = C(1-|w|)^2$ and $d\zeta \wedge d\bar\zeta$ denotes the Lebesgue measure on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$.

Since $D(\zeta, w)$ is a quasi-metric on $\overline{\mathbb{M}}$ (see [2]), we can choose a constant C that satisfies $D(\zeta, w) \approx D(\zeta, z)$ if $D(\zeta, z) \geq CD(z, w)$.

Now fix $z \in \partial\mathbb{M}$. For $0 < r < 1$, put $\delta := C(1-r)^2$ and $w := rz$. If $D(\zeta, z) \geq CD(z, w) = \delta$ then $D(\zeta, w) \approx D(\zeta, z)$, which in turn gives that

$$|1 - \zeta \bullet \bar{w}| \approx |1 - \zeta \bullet \bar{z}|. \tag{3.11}$$

Thus, by (3.9),

$$|D(\zeta, w) - D(\zeta, z)| = |z - w| O(|1 - \zeta \bullet \bar{z}|). \tag{3.12}$$

Estimate (3.7) will follow by combining the following estimates:

$$\mathrm{I} := \int_{\zeta \in \mathbb{M},\, D(\zeta, w) \leq C\delta} |\Gamma(\zeta, w)| \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \lesssim \delta^{\frac{1}{4}},$$

$$\mathrm{II} := \int_{\zeta \in \mathbb{M},\, D(\zeta, z) \leq \delta} |\Gamma(\zeta, z)| \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \lesssim \delta^{\frac{1}{8}},$$

$$\mathrm{III} := \int_{\zeta \in \mathbb{M},\, D(\zeta, z) \geq \delta} |\Gamma(\zeta, z) - \Gamma(\zeta, w)| \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \lesssim \delta^{\frac{1}{8}}.$$

We now prove these estimates.

By Theorem 2.6 we have that

$$P_k(\zeta, w) = O(|\zeta - w|) \quad \text{and} \quad Q_k(\zeta, w) = O(|\zeta - w|).$$

Observe that if $\zeta \in \mathbb{M}$ and $D(\zeta, w) \leq \delta$, then the estimates given in [2, p. 68] show that $|1 - \zeta \bullet \bar{w}| \approx 1 - |w|^2$. This, combined with estimates (3.8)–(3.10), yields

$$\begin{aligned}
\mathrm{I} &\lesssim \int_{\zeta \in \mathbb{M},\, D(\zeta, w) \leq C\delta} |1 - \zeta \bullet \bar{w}|^{n - \frac{3}{2}} [D(\zeta, w)]^{-n + \frac{1}{2}} \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \\
&\lesssim \int_{\zeta \in \pi_w(\mathbb{M}),\, D(\pi_w(\zeta), \pi_w(w)) \leq C\delta} |1 - \pi_w(\zeta) \bullet \overline{\pi_w(w)}|^{n - \frac{3}{2}} \\
&\quad \cdot [D(\pi_w(\zeta), \pi_w(w))]^{-n + \frac{1}{2}} \, dV_n(\zeta) \\
&\lesssim \delta^{\frac{1}{4}}.
\end{aligned}$$

Since $D(\zeta, z) = |1 - \zeta \bullet \bar{z}|^2$, it follows that

$$\begin{aligned}
\mathrm{II} &\lesssim \int_{|1 - \zeta \bullet \bar{z}| \leq \delta^{1/2}} |1 - \zeta \bullet \bar{z}|^{-n - \frac{1}{2}} \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \\
&\lesssim \delta^{\frac{1}{8}} \int_{\mathbb{M}} |1 - \zeta \bullet \bar{z}|^{-n - \frac{3}{4}} \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \lesssim \delta^{\frac{1}{8}},
\end{aligned}$$

where the latter inequality holds by an application of [16, Lemma 5.1].

It now remains to majorize III. Using (3.11) and (3.12), we see that

$$\begin{aligned}\mathrm{III} &\lesssim |z-w|\biggl(\int_{|1-\zeta\bullet\bar z|\geq\delta^{1/2}}\frac{|1-\zeta\bullet\bar z|^{n+\frac12}}{D(\zeta,z)^{n+1}}\alpha(\zeta)\wedge\overline{\alpha(\zeta)}\biggr)\\ &\lesssim \delta^{\frac18}\int_{\mathbb M}|1-\zeta\bullet\bar z|^{-n-\frac34}\alpha(\zeta)\wedge\overline{\alpha(\zeta)}\lesssim\delta^{\frac18}.\end{aligned}$$

The proof of the lemma is complete. □

4. Lipschitz Estimates on the Complex Manifold $\mathbb M$

The following Hardy–Littlewood-type lemma is needed.

LEMMA 4.1. *For every* $0<\alpha\leq 1$, *there exists a constant* $C=C(\alpha)$ *with the following property. Suppose* u *is a differentiable function defined in a neighborhood of* $\mathbb M$ *in* $\mathbb B$, *and let* K *be some finite constant such that*

$$|(\operatorname{grad}u)(z)|\leq K(1-|z|)^{\alpha-1}\quad\textit{for }\ z\in\mathbb M.$$

Then $|u(z)-u(w)|\leq CK|z-w|^{\alpha}$ *for* $z,w\in\mathbb M$.

Proof. Let $a,b\in\mathbb M$ with $0<|a|\leq|b|<1$, and set $\delta:=|a-b|$ and $c:=(|a|/|b|)b\in\partial\mathbb M_{|a|}$. Clearly $|b-c|\leq\delta$ and $|a-c|\leq\delta$. We now distinguish the cases $\delta\leq 1-|b|$ and $\delta>1-|b|$.

In the first case, using that the group $\mathrm{SO}(n+1,\mathbb R)$ acts transitively on $\partial\mathbb M$, we see that there is a smooth curve $\gamma(t)$ on $\partial\mathbb M_{|a|}$ that satisfies $\gamma(0)=a$, $\gamma(1)=c$, and $|\gamma'(t)|\leq C|a-c|$.

The hypothesis implies that $|(\operatorname{grad}u)(z)|\leq K\delta^{\alpha-1}$ on the line from b to c and on the curve $\gamma(t)$. Hence

$$|u(b)-u(c)|+|u(a)-u(c)|\leq CK\delta^{\alpha},$$

showing that

$$|u(a)-u(b)|\leq CK|a-b|^{\alpha}.$$

The case $\delta>1-|b|$ can be checked using the same argument as in [22, Lemma 6.4.8]. □

Consider the Lipschitz space

$$\Lambda_{\frac12}(\mathbb M):=\biggl\{f\in L^{\infty}(\mathbb M):\|f\|_{\infty}+\sup_{z,z+h\in\mathbb M}\frac{|f(z+h)-f(z)|}{|h|^{\frac12}}\equiv\|f\|_{\Lambda_{\frac12}}<\infty\biggr\}.$$

THEOREM 4.2. *Suppose that* $u\in\mathcal C^1(\overline{\mathbb M})$ *is bounded and that* $f:=\sum_{k=1}^{n+1}f_k\,d\bar w_k$ *is a continuous* $(0,1)$*-form defined in a neighborhood of* $\overline{\mathbb M}$ *in* $\overline{\mathbb B}$ *such that* $\|f\|_{\mathbb M,\infty}<\infty$ *and* $\bar\partial_{\mathbb M}u=f|_{\mathbb M}$. *Define* Tf *on* $\partial\mathbb M$ *by*

$$\begin{aligned}&(Tf)(\zeta)\\ &:=\int_{\mathbb M}\sum_{k=1}^{n+1}\biggl[\frac{(1-w\bullet\bar\zeta)P_k(w,\zeta)+(1-|w|^2)Q_k(w,\zeta)}{(1-\bar w\bullet\zeta)^n(1-w\bullet\bar\zeta)^2}\biggr]f_k(w)\alpha(w)\wedge\overline{\alpha(w)}.\end{aligned}$$

Then the definition of Tf *can be extended to* $\mathbb M$ *by setting*

$$(Tf)(z) := J_1(z) + J_2(z), \tag{4.1}$$

where

$$J_1(z) := C_1 \int_{\partial\mathbb{M}} \frac{1 + z \bullet \bar\zeta - \bar z \bullet \zeta - |z \bullet \bar\zeta|^2 + |z \bullet \zeta|^2}{|z-\zeta|^{2n}} (Tf)(\zeta)\, d\sigma(\zeta),$$

$$J_2(z) := C_2 \int_{\mathbb{M}} \sum_{k=1}^{n+1} \frac{[(\bar z_k - \bar w_k)(z \bullet \bar w + |w|^2) - (z_k + w_k)\overline{z \bullet (w-z)}]}{|z-w|^{2n}}$$
$$\cdot\, f_k(w) \alpha(w) \wedge \overline{\alpha(w)}.$$

Moreover, the operator Tf satisfies:

(i) $\bar\partial_{\mathbb{M}} Tf = f|_{\mathbb{M}}$;
(ii) $Tf \in \Lambda_{\frac{1}{2}}(\mathbb{M})$ *and* $\|Tf\|_{\Lambda_{\frac{1}{2}}} \le C\|f\|_{\mathbb{M},\infty}$.

Proof. Define

$$g(z) := \int_{\mathbb{M}} \sum_{k=1}^{n+1} \left[\frac{(1 - w \bullet \bar z) P_k(w,z) + (1-|w|^2) Q_k(w,z)}{(1 - \bar w \bullet z)^n (1 - w \bullet \bar z)^2} \right] f_k(w) \alpha(w) \wedge \overline{\alpha(w)}$$

for z in a neighborhood of $\overline{\mathbb{M}}$ in $\bar{\mathbb{B}}$.

Let u_f be the function defined in Lemma 3.5. Then, applying Theorem 2.6 to the function u gives that

$$u_f(z) := u(z) - \mathcal{S}_{\mathbb{M}}[u](z) \quad \forall z \in \mathbb{M}. \tag{4.2}$$

Note that $u_f(\zeta) = (Tf)(\zeta) = g(\zeta)$ for $\zeta \in \partial\mathbb{M}$.

By Lemma 3.5, we have that $\lim_{r\to 1^-} \int_{\partial\mathbb{M}} |u_f(\zeta) - u_f(r\zeta)|\, d\sigma(\zeta) = 0$. Therefore, in view of Remark 2.5, we can apply Theorem 2.4 to the function u_f.

Observe that (4.1) is just the Martinelli–Bochner formula. Hence, by virtue of (4.2) and the hypothesis, we obtain

$$Tf = u_f \quad \text{and} \quad \bar\partial_{\mathbb{M}} Tf = \bar\partial_{\mathbb{M}} u = f|_{\mathbb{M}}. \tag{4.3}$$

In the proof of the theorem, the following lemmas will be needed.

LEMMA 4.3.

$$\|g\|_\infty \le C\|f\|_{\mathbb{M},\infty}; \tag{4.4}$$

$$|g(z) - g(w)| \le C\|f\|_{\mathbb{M},\infty} |z-w|^{\frac{1}{2}} \quad \textit{for } z, w \in \overline{\mathbb{M}}. \tag{4.5}$$

Proof. We remark that $|z-w|^2 \le 2|1 - z \bullet \bar w|$ and that, by Theorem 2.6, we have

$$P_k(w,z) = O(|w-z|) \quad \text{and} \quad Q_k(w,z) = O(|w-z|) \quad \text{for } z, w \in \bar{\mathbb{B}}.$$

Therefore,

$$|g(z)| \le \|f\|_{\mathbb{M},\infty} \int_{\mathbb{M}} \left[\frac{O(|w-z|)}{|1 - w \bullet \bar z|^{n+1}} + \frac{(1-|w|^2) O(|w-z|)}{|1 - w \bullet \bar z|^{n+2}} \right] \alpha(w) \wedge \overline{\alpha(w)}$$

$$\lesssim \|f\|_{\mathbb{M},\infty} \int_{\mathbb{M}} \left[\frac{1}{|1 - w \bullet \bar z|^{n+\frac{1}{2}}} + \frac{(1-|w|^2)}{|1 - w \bullet \bar z|^{n+\frac{3}{2}}} \right] \alpha(w) \wedge \overline{\alpha(w)}$$
$$\text{for } z \in \mathbb{M}.$$

An application of [16, Lemma 5.1] shows that the latter integral is bounded. This proves (4.4).

Next, we see that

$$
\begin{aligned}
|\operatorname{grad} g(z)| \lesssim \|f\|_{\mathbb{M},\infty} \sum_{k=1}^{n+1} \int_{\mathbb{M}} \alpha(w) \wedge \overline{\alpha(w)} \\
\cdot \Bigg[\frac{|1-w\bullet\bar{z}||P_k(w,z)| + (1-|w|^2)|Q_k(w,z)|}{|1-w\bullet\bar{z}|^{n+3}} \\
+ \frac{\begin{array}{r}|P_k(w,z)| + |1-w\bullet\bar{z}||\operatorname{grad}_z P_k(w,z)| \\ + (1-|w|^2)|\operatorname{grad}_z Q_k(w,z)|\end{array}}{|1-w\bullet\bar{z}|^{n+2}} \Bigg].
\end{aligned}
$$

Bearing in mind the preceding remark, an application of [16, Lemma 5.1] shows that

$$|(\operatorname{grad} g)(z)| \le C\|f\|_{\mathbb{M},\infty}(1-|z|)^{-\frac{1}{2}} \quad \text{for } z \in \mathbb{M}.$$

This fact, combined with Lemma 4.1, proves (4.5) and thereby completes the proof of the lemma. □

LEMMA 4.4.

$$|J_1(z) - J_1(w)| \le C\|f\|_{\mathbb{M},\infty}|z-w|^{\frac{1}{2}} \quad \textit{for } z, w \in \mathbb{M}.$$

Proof. Observe that for $\zeta \in \partial\mathbb{M}$ we have

$$\left|1 + z\bullet\bar{\zeta} - \bar{z}\bullet\zeta - |z\bullet\bar{\zeta}|^2 + |z\bullet\zeta|^2\right| \le C|z-\zeta|.$$

Thus

$$\left|\operatorname{grad}_z\left(\frac{1 + z\bullet\bar{\zeta} - \bar{z}\bullet\zeta - |z\bullet\bar{\zeta}|^2 + |z\bullet\zeta|^2}{|z-\zeta|^{2n}}\right)\right| \le C|z-\zeta|^{-2n}. \tag{4.6}$$

In addition, if we set $u \equiv 1$ in Theorem 2.4 then

$$C_1 \int_{\partial\mathbb{M}} \frac{1 + z\bullet\bar{\zeta} - \bar{z}\bullet\zeta - |z\bullet\bar{\zeta}|^2 + |z\bullet\zeta|^2}{|z-\zeta|^{2n}}\, d\sigma(\zeta) = 1.$$

Setting $z := rw$ for $w \in \partial\mathbb{M}$, this implies that

$$
\begin{aligned}
|(\operatorname{grad} J_1)(z)| \lesssim \int_{\partial\mathbb{M}} \left|\operatorname{grad}_z\left(\frac{1 + z\bullet\bar{\zeta} - \bar{z}\bullet\zeta - |z\bullet\bar{\zeta}|^2 + |z\bullet\zeta|^2}{|z-\zeta|^{2n}}\right)\right| \\
\cdot |(Tf)(\zeta) - (Tf)(w)|\, d\sigma(\zeta).
\end{aligned}
$$

Since $Tf = g$ on $\partial\mathbb{M}$, it follows from Lemma 4.3 that

$$|(Tf)(\zeta) - (Tf)(w)| \le C\|f\|_{\mathbb{M},\infty}|\zeta - w|^{\frac{1}{2}}.$$

By (4.6), we now have

$$|(\operatorname{grad} J_1)(z)| \le C\|f\|_{\mathbb{M},\infty} \int_{\partial\mathbb{M}} \frac{|\zeta - w|^{\frac{1}{2}}}{|\zeta - rw|^{2n}}\, d\sigma(\zeta),$$

so that Lemma 3.1 yields

$$|(\operatorname{grad} J_1)(z)| \leq C\|f\|_{\mathbb{M},\infty}(1-|z|)^{-\frac{1}{2}}. \tag{4.7}$$

This inequality, combined with Lemma 4.1, completes the proof of the lemma. □

A combination of Lemma 3.4 and Lemma 4.4 with formula (4.1) gives that

$$|J_2(z) - J_2(w)| \leq C\|f\|_{\mathbb{M},\infty}|z-w|(|\ln|z-w|| + 1) \tag{4.8}$$

and

$$|(Tf)(z) - (Tf)(w)| \leq C\|f\|_{\mathbb{M},\infty}|z-w|^{\frac{1}{2}} \quad \text{for } z, w \in \mathbb{M}. \tag{4.9}$$

It now remains to prove the following.

LEMMA 4.5.

$$\|Tf\|_\infty \leq C\|f\|_{\mathbb{M},\infty}.$$

Proof. On the one hand, by virtue of the formula for J_1 and in view of (4.4), we have that

$$|J_1(0)| \leq C_1 \int_{\partial\mathbb{M}} |(Tf)(\zeta)|\, d\sigma(\zeta) \leq C\|f\|_{\mathbb{M},\infty}.$$

On the other hand, by virtue of the formula for J_2 we have

$$|J_2(0)| \leq (n+1)C_2\left(\int_{\mathbb{M}} \frac{\alpha(w)\wedge\overline{\alpha(w)}}{|w|^{2n-3}}\right)\|f\|_{\mathbb{M},\infty} \leq C\|f\|_{\mathbb{M},\infty},$$

where the latter inequality holds by an application of [16, Lemma 2.1]. Therefore,

$$|(Tf)(0)| \leq |J_1(0)| + |J_2(0)| \leq C\|f\|_{\mathbb{M},\infty}.$$

This, combined with (4.9), implies that

$$|(Tf)(z)| \leq |(Tf)(0)| + |(Tf)(z) - (Tf)(0)| \leq C\|f\|_{\mathbb{M},\infty}.$$

The proof of Lemma 4.5 is therefore finished. □

Finally, the proof of Theorem 4.2 follows from (4.3), (4.9) and Lemma 4.5. □

REMARK 4.6. The solution Tf is the one characterized by $\bar{\partial}Tf = f$ with Tf orthogonal to holomorphic functions, where the orthogonality is in terms of integration taken on $\partial\mathbb{M}$. In other words, Tf is the solution of $\bar{\partial}u = f$ that has smallest $L^2(\sigma)$-norm.

Using the notation of Krantz [14], we denote by $\mathcal{C}_1^2(\mathbb{B})$ the family of $\mathcal{C}^2$-admissible curves in $\mathbb{B}$ (with respect to the radial projection from $\mathbb{B}\setminus\{0\}$ onto $\partial\mathbb{B}$). Let $\mathcal{C}_1^2(\mathbb{M})$ be those curves of $\mathcal{C}_1^2(\mathbb{B})$ that lie in $\mathbb{M}$. Let $\Gamma_{\frac{1}{2},\tilde{1}}$ be the following nonisotropic Lipschitz space of functions on $\mathbb{M}$:

$$\Gamma_{\frac{1}{2},\tilde{1}} := \left\{f : \|f\|_{\Lambda_{\frac{1}{2}}(\mathbb{M})} + \sup_{\gamma\in\mathcal{C}_1^2(\mathbb{M})} \|f\circ\gamma\|_{\Lambda_{\tilde{1}}([0,1])} \equiv \|f\|_{\Gamma_{\frac{1}{2},\tilde{1}}} < \infty\right\}.$$

THEOREM 4.7. *There exists a finite constant C such that, for all $(0,1)$-forms f satisfying the hypothesis of Theorem 4.2,*

$$\|Tf\|_{\Gamma_{\frac{1}{2},\tilde{1}}} \le C\|f\|_{\mathbb{M},\infty}.$$

Proof. For $w \in \partial\mathbb{M}$ and $0 < r_0 < 1$, let $B(w, r_0)$ denote the ball centered at $(1 - r_0)w$ with radius r_0. Note that $B(w, r_0)$ is contained in $\mathbb{B}$ and is internally tangent (to first order) to $\partial\mathbb{M}$ at w.

We shall use the notation in the proof of Lemma 3.5. Because the group $\mathrm{SO}(n+1, \mathbb{R})$ acts transitively on $\partial\mathbb{M}$, there is a small enough r_0 such that, for all $w \in \partial\mathbb{M}$, π_w is biholomorphic from a neighborhood of $B(w, r_0) \cap \mathbb{M}$ in $\mathbb{H}$ onto its image in T_w. Put $D_w := \pi_w(B(w, r_0) \cap \mathbb{M})$.

Let $\gamma \in \mathcal{C}_1^2(\mathbb{M})$ be close to $\partial\mathbb{M}$; say, $\mathrm{dist}(\gamma, \partial\mathbb{M}) < r_0/2$. We shall see that, for the proof of the main theorem, it is sufficient to check such γ.

Let $w := \gamma(0)/|\gamma(0)| \in \partial\mathbb{M}$. Note that $\pi_w(\gamma(0)) = \gamma(0)$. Then there is a curve $\tilde{\gamma} \in \mathcal{C}_1^2(D_w)$ with $\gamma(0) = \tilde{\gamma}(0)$ and $|\gamma(t) - \tilde{\gamma}(t)| \le Ct^2$ $(0 \le t \le 1)$. This assertion follows from the fact that the complex tangent spaces to ∂D_w and $\partial B(w, r_0) \cap \mathbb{H}$ at w are the same.

We estimate

$$\begin{aligned} |(Tf)(\gamma(h)) - (Tf)(\gamma(0))| &\le |(Tf)(\gamma(h)) - (Tf \circ \pi_w^{-1})(\tilde{\gamma}(h))| \\ &\quad + |(Tf \circ \pi_w^{-1})(\tilde{\gamma}(h)) - (Tf \circ \pi_w^{-1})(\tilde{\gamma}(0))| \\ &\equiv T_1 + T_2. \end{aligned} \tag{4.10}$$

Since

$$|\gamma(h) - (\pi_w^{-1})(\tilde{\gamma}(h))| \le C|\gamma(h) - \tilde{\gamma}(h)| \le Ch^2,$$

applying Theorem 4.2 yields

$$T_1 \le C\|f\|_{\mathbb{M},\infty} h. \tag{4.11}$$

On the one hand, since π_w is an orthogonal projection, it is not hard to show that

$$\|(\pi_w)_*(f|_{\mathbb{M}})\|_{L^\infty(D_w)} \le C\|f\|_{\mathbb{M},\infty} < \infty. \tag{4.12}$$

On the other hand, by Theorem 4.2 we have

$$\bar{\partial}(Tf \circ \pi_w^{-1}) = (\pi_w)_*(f|_{\mathbb{M}}) \text{ on } D_w \tag{4.13}$$

and

$$\|Tf \circ \pi_w^{-1}\|_{\Lambda_{\frac{1}{2}}(D_w)} \le C\|Tf\|_{\Lambda_{\frac{1}{2}}(\mathbb{M})} \le C\|f\|_{\mathbb{M},\infty}. \tag{4.14}$$

Observe that D_w is a smooth, strictly pseudoconvex Euclidian domain and that $D_{Aw} = A(D_w)$ for all $A \in \mathrm{SO}(n+1, \mathbb{R})$; we may thus apply Theorem 8.2 of [14]. It then follows from (4.13) that there exists a constant C independant of w such that

$$\|Tf \circ \pi_w^{-1}\|_{\Gamma_{\frac{1}{2},\tilde{1}}(D_w)} \le C(\|Tf \circ \pi_w^{-1}\|_{\Lambda_{\frac{1}{2}}(D_w)} + \|(\pi_w)_*(f|_{\mathbb{M}})\|_{L^\infty(D_w)}).$$

This, combined with (4.12) and (4.14), implies that

$$T_2 \le \|Tf \circ \pi_w^{-1}\|_{\Gamma_{\frac{1}{2},\tilde{1}}(D_w)} |h||\log|h|| \le C\|f\|_{\mathbb{M},\infty}|h||\log|h||.$$

Combining estimates (4.10) and (4.11) with the one just displayed, the proof of Theorem 4.7 is complete. □

5. Proof of the Main Theorem

Consider the proper map $\pi : \overline{\mathbb{M}} \to \overline{\mathbb{B}_*} \setminus \{0\}$ defined by

$$\pi(z_1, \dots, z_n, z_{n+1}) := (z_1, \dots, z_n).$$

Let θ be the Lebesgue surface measure on $\partial\mathbb{B}_* \setminus V$ and denote by $dz \wedge d\bar{z}$ the canonical volume form of $\mathbb{C}^n$.

We recall from [24] and [17] that

$$d\sigma(\zeta, \zeta_{n+1}) = C\pi^*\left(\frac{d\theta(\zeta)}{|\zeta \bullet \zeta|}\right) \quad \text{for } (\zeta, \zeta_{n+1}) \in \partial\mathbb{M} \tag{5.1}$$

and

$$[\alpha(z) \wedge \overline{\alpha(z)}](z, z_{n+1}) = C\pi^*\left(\frac{dz \wedge d\bar{z}}{|z \bullet z|}\right) \quad \text{for } (z, z_{n+1}) \in \mathbb{M}. \tag{5.2}$$

Proposition 5.1. *Suppose that f is a $\bar\partial$-closed $(0,1)$-form of class $\mathcal{C}^1$ defined in a neighborhood of $\mathbb{B}_*$. Then the solution $T(\pi^* f)$ given by Theorem 4.2 satisfies*

$$(T(\pi^* f))(z, z_{n+1}) = (T(\pi^* f))(z, -z_{n+1}) \quad \forall (z, z_{n+1}) \in \mathbb{M}.$$

Proof. Suppose that $f \in \mathcal{C}^1_{0,1}(r\mathbb{B}_*)$ for some $r > 1$. Since $r\mathbb{B}_*$ is pseudoconvex, there exists a $u \in \mathcal{C}^1(\overline{\mathbb{B}_*})$ such that $\bar\partial u = f$ in $\mathbb{B}_*$. Then it follows from (4.3) that

$$T(\pi^* f)(z) = u_{\pi^* f} = (\pi^* u)(z) - \mathcal{S}_{\mathbb{M}}[\pi^* u](z) \quad \forall z \in \mathbb{M}. \tag{5.3}$$

In view of formulas (2.11) and (5.1), it can be checked that

$$\mathcal{S}_{\mathbb{M}}[\pi^* u](z, z_{n+1}) = \mathcal{S}_{\mathbb{M}}[\pi^* u](z, -z_{n+1}) \quad \forall z \in \mathbb{M}.$$

This, combined with equality (5.3), completes the proof. □

We are now in position to prove the main theorem.

First we assume that f is a $\bar\partial$-closed $(0,1)$-form of class $\mathcal{C}^1$ defined in a neighborhood of $\mathbb{B}_*$. The general case will be treated later.

In view of Proposition 5.1, we can define

$$(Tf)(z) := T(\pi^* f)(z, z_{n+1}) \quad \forall (z, z_{n+1}) \in \mathbb{M}. \tag{5.4}$$

Combining Theorem 4.2 and Proposition 5.1, we see that the solution operator Tf satisfies

$$\bar\partial Tf = f \ \text{ in } \mathbb{B}_*;$$

$$\|Tf\|_\infty \le C\|f\|_\infty;$$

$$|(Tf)(z) - (Tf)(w)| \le C\|f\|_\infty\Big(|z - w| + \min_{\varepsilon \in \{-1,1\}} |\sqrt{z \bullet z} + \varepsilon\sqrt{w \bullet w}|\Big)^{\frac{1}{2}}$$

$$\forall z, w \in \mathbb{B}_*.$$

Let $z, w \in \mathbb{B}_*$. Then there exist $z_{n+1} \in \mathbb{C}$ and $w_{n+1} \in \mathbb{C}$ such that $(z, z_{n+1}) \in \mathbb{M}$, $(w, w_{n+1}) \in \mathbb{M}$, and $|z_{n+1} - w_{n+1}| \le |z_{n+1} + w_{n+1}|$.

Using the latter estimate, we obtain

$$|z_{n+1} - w_{n+1}| \le |z_{n+1}^2 - w_{n+1}^2|^{\frac{1}{2}} = |z \bullet z - w \bullet w|^{\frac{1}{2}} \le C|z - w|^{\frac{1}{2}}.$$

Hence

$$|z - w| + |z_{n+1} - w_{n+1}| \le C|z - w|^{\frac{1}{2}}. \tag{5.5}$$

On the other hand, we have

$$|w_{n+1}| = \frac{|w_{n+1} - z_{n+1} + w_{n+1} + z_{n+1}|}{2} \le |z_{n+1} + w_{n+1}|.$$

This implies that

$$|w_{n+1}||z_{n+1} - w_{n+1}| \le |z_{n+1}^2 - w_{n+1}^2| \le C|z - w|.$$

Therefore,

$$\begin{aligned} &|\bar{w}_1(z_1 - w_1) + \cdots + \bar{w}_{n+1}(z_{n+1} - w_{n+1})| \le C|z - w|, \\ &|w_1(z_1 - w_1) + \cdots + w_{n+1}(z_{n+1} - w_{n+1})| \le |z - w|. \end{aligned} \tag{5.6}$$

We consider two cases.

Case 1: $N_*(z) \le 1 - r_0/4$, $N_*(w) \le 1 - r_0/4$. Here r_0 is the number appearing in the proof of Theorem 4.7.

In view of estimate (4.7) and Lemma 4.1, it is clear that

$$|J_1(a) - J_1(b)| \le C\|f\|_\infty |a - b| \quad \forall a, b \in \mathbb{M}_{1-r_0/4}.$$

Combining the latter estimate with (4.8) and formula (4.1), we see that

$$\begin{aligned} &|(Tf)(z) - (Tf)(w)| \\ &\qquad \le C\|f\|_\infty (|z - w| + |z_{n+1} - w_{n+1}|)|\log(|z - w| + |z_{n+1} - w_{n+1}|)|. \end{aligned}$$

Hence, by virtue of (5.5),

$$|(Tf)(z) - (Tf)(w)| \le C\|f\|_\infty |z - w|^{\frac{1}{2}} |\log|z - w||.$$

This completes case 1.

Case 2: $N_*(w) > 1 - r_0/4$. We may assume without loss of generality that $|z - w| < r_0/8$. Thus $N_*(z) > 1 - r_0/2$.

Let $T'_{(w, w_{n+1})}$ denote the complex tangent space to $\partial\mathbb{B}_{|(w, w_{n+1})|}$ at (w, w_{n+1}). Then, by virtue of (5.6), we obtain

$$\begin{aligned} \operatorname{dist}((z, z_{n+1}), T'_{(w, w_{n+1})}) &= \frac{|\bar{w}_1(z_1 - w_1) + \cdots + \bar{w}_{n+1}(z_{n+1} - w_{n+1})|}{|(w, w_{n+1})|} \\ &\le C|z - w|, \\ \operatorname{dist}((z, z_{n+1}), T_{(w, w_{n+1})}) &= \frac{|w_1(z_1 - w_1) + \cdots + w_{n+1}(z_{n+1} - w_{n+1})|}{|(w, w_{n+1})|} \\ &\le C|z - w|. \end{aligned} \tag{5.7}$$

Since $(\bar{w}, \bar{w}_{n+1})$ is normal to $T_{(w,w_{n+1})}$, (w, w_{n+1}) is normal to $T'_{(w,w_{n+1})}$, and (w, w_{n+1}) is orthogonal to $(\bar{w}, \bar{w}_{n+1})$, a geometric argument shows that

$$[\operatorname{dist}((z, z_{n+1}), T_{(w,w_{n+1})} \cap T'_{(w,w_{n+1})})]^2 = [\operatorname{dist}((z, z_{n+1}), T_{(w,w_{n+1})})]^2 + [\operatorname{dist}((z, z_{n+1}), T'_{(w,w_{n+1})})]^2.$$

This, combined with (5.7), yields that

$$\operatorname{dist}((z, z_{n+1}), T_{(w,w_{n+1})} \cap T'_{(w,w_{n+1})}) \le C|z-w|. \tag{5.8}$$

Let $v \in T_{(w,w_{n+1})} \cap T'_{(w,w_{n+1})}$ such that

$$|(z, z_{n+1}) - v| = \operatorname{dist}((z, z_{n+1}), T_{(w,w_{n+1})} \cap T'_{(w,w_{n+1})}). \tag{5.9}$$

Then there exists a curve $\gamma \in \mathcal{C}_1^2(\mathbb{M})$ such that

$$\begin{gathered}\gamma(0) = (w, w_{n+1}),\\ |\gamma(2|v-(w, w_{n+1})|) - v| \le C|v-(w, w_{n+1})|^2.\end{gathered} \tag{5.10}$$

We estimate

$$\begin{aligned}|(Tf)(z) - (Tf)(w)| &\le |T(\pi^* f)(z, z_{n+1}) - T(\pi^* f)(v)|\\ &\quad + |T(\pi^* f)(v) - T(\pi^* f)(\gamma(2|v-(w, w_{n+1})|)|\\ &\quad + |T(\pi^* f)(\gamma(2|v-(w, w_{n+1})|) - T(\pi^* f)(\gamma(0))|\\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

Using the estimates (5.5) and (5.8)–(5.10) and then applying Theorem 4.2, we immediately majorize I and II by $C\|f\|_\infty |z-w|^{\frac{1}{2}}$.

Applying Theorem 4.7 yields that

$$\begin{aligned}\mathrm{III} &\le C\|f\|_\infty |v-(w, w_{n+1})||\log|v-(w, w_{n+1})||\\ &\le C\|f\|_\infty |(z, z_{n+1}) - (w, w_{n+1})||\log|(z, z_{n+1}) - (w, w_{n+1})||\\ &\le C\|f\|_\infty |z-w|^{\frac{1}{2}} |\log|z-w||.\end{aligned}$$

Hence,

$$|(Tf)(z) - (Tf)(w)| \le C\|f\|_\infty |z-w|^{\frac{1}{2}} |\log|z-w||.$$

This completes case 2.

It remains now to treat the general case. If merely $f \in \mathcal{A}^\infty_{0,1}(\mathbb{B}_*)$ then we can regularize f by convolution with a $\mathcal{C}_0^\infty$ function of sufficiently small support. Then the same limiting argument as in [22, pp. 361–362] shows that the conclusion of the theorem also holds for such f. The proof of the theorem is complete. □

We conclude this article with some remarks.

REMARK 5.2. Making use of formulas (5.1), (5.2), (5.4), and (4.1), we can write down explicitly the solution operator Tf. In this case, Tf has the form stated in Section 1. We can also obtain another expression for Tf by applying (5.2) to $u_{\pi^* f}$.

REMARK 5.3. In view of Remark 4.6 and formulas (5.1) and (5.4), we have the following characterization: Tf is the solution of $\bar{\partial}u = f$ that has smallest $L^2\big(\frac{d\theta(\zeta)}{|\zeta\bullet\zeta|}\big)$-norm.

REMARK 5.4. We do not know whether the Lipschitz $\widetilde{\frac{1}{2}}$-estimate can be improved to Lipschitz $\frac{1}{2}$.

References

[1] B. Berndtsson, *A formula for interpolation and division in* $\mathbb{C}^n$, Math. Ann. 263 (1983), 399–418.

[2] A. Bonami and P. Charpentier, *Solutions de l'equation* $\bar{\partial}$ *et zéros da la classe de Nevanlinna dans certains domaines faiblement pseudo-convexes,* Ann. Inst. Fourier (Grenoble) 32 (1982), 53–89.

[3] J. Bruna and J. del Castillo, *Hölder and* L^p*-estimates for the* $\bar{\partial}$*-equation in some convex domains with real-analytic boundary,* Math. Ann. 269 (1984), 527–539.

[4] P. Charpentier, *Formules explicites pour les solutions minimales de l'équation* $\bar{\partial}u = f$ *dans la boule et dans le polydisque de* $\mathbb{C}^n$, Ann. Inst. Fourier (Grenoble) 30 (1980), 121–154.

[5] A. Cumenge, *Estimées Lipschitz optimales dans les convexes de type fini,* C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), 1077–1080.

[6] K. Diederich, B. Fischer, and J. E. Fornæss, *Hölder estimates on convex domains of finite type,* Math. Z. 232 (1999), 43–61.

[7] K. Diederich, J. E. Fornæss, and J. Wiegerinck, *Sharp Hölder estimates for* $\bar{\partial}$ *on ellipsoids,* Manuscripta Math. 56 (1986), 399–417.

[8] J. E. Fornæss and N. Sibony, *On* L^p*-estimates for* $\bar{\partial}$, Proc. Sympos. Pure Math., 52, pp. 129–163, Amer. Math. Soc., Providence, RI, 1991.

[9] L. Gruman and P. Lelong, *Entire functions of several complex variables,* Grundlehren Math. Wiss., 282, Springer-Verlag, Berlin, 1986.

[10] K. T. Hahn and P. Pflug, *On a minimal complex norm that extends the real Euclidian norm,* Monatsh. Math. 105 (1988), 107–112.

[11] G. Henkin and A. Iordan, *Compactness of the Neumann operator for hyperconvex domains with non-smooth* B*-regular boundary,* Math. Ann. 307 (1997), 151–168.

[12] G. Henkin and J. Leiterer, *Andreotti–Grauert theory by integral formula*, Progr. Math., 74, Birkhäuser, Boston, 1988.

[13] K. T. Kim, *Automorphism groups of certain domains in* $\mathbb{C}^n$ *with singular boundary,* Pacific J. Math. 151 (1991), 54–64.

[14] S. G. Krantz, *Estimates for integral kernels of mixed type, fractional integration operators, and optimal estimates for the* $\bar{\partial}$ *operator,* Manuscripta Math. 30 (1979), 21–52.

[15] G. Mengotti, *Duality theorems for certain analytic spaces on the minimal ball,* Arch. Math. (Basel) 75 (2000), 389–394.

[16] G. Mengotti and E. H. Youssfi, *The weighted Bergman projection and related theory on the minimal ball,* Bull. Sci. Math. 123 (1999), 501–525.

[17] K. Oeljeklaus, P. Pflug, and E. H. Youssfi, *The Bergman kernel of the minimal ball and applications,* Ann. Inst. Fourier (Grenoble) 47 (1997), 915–928.

[18] K. Oeljeklaus and E. H. Youssfi, *Proper holomorphic mappings and related automorphism groups,* J. Geom. Anal. 7 (1997), 623–636.

[19] N. Ourimi, *Applications holomorphes propres entre certains domaines bornés de* $\mathbb{C}^n$, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), 1063–1068.
[20] P. Pflug and E. H. Youssfi, *The Lu Qi-Keng conjecture fails for strongly convex algebraic domains,* Arch. Math. 71 (1998), 240–245.
[21] R. M. Range, *On Hölder estimates for* $\bar\partial u = f$ *on weakly pseudoconvex domains,* Several complex variables (Cortona, Italy, 1976/77), pp. 247–267, Scuola Norm. Sup. Pisa, Pisa, 1978.
[22] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, Berlin, 1980.
[23] N. Sibony, *Une classe de domaines pseudoconvexes,* Duke Math. J. 55 (1987), 299–319.
[24] N. Viêt Anh, *Fatou and Korànyi–Vàgi type theorems on the minimal balls,* preprint, 1999.
[25] E. H. Youssfi, *Proper holomorphic lifting of the Bergman and Szëgo kernels for some complex domains,* preprint, 1998.

N. Viêt Anh
L.A.T.P. U.M.R. C.N.R.S 6632
Centre de Mathématiques
et d'Informatique
Université de Provence, F-13453
Marseille Cedex 13
France

vietanh@gyptis.univ-mrs.fr

E. H. Youssfi
L.A.T.P. U.M.R. C.N.R.S 6632
Centre de Mathématiques
et d'Informatique
Université de Provence, F-13453
Marseille Cedex 13
France

youssfi@gyptis.univ-mrs.fr