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Hausdorff Dimension and Limit Sets of Quasiconformal Groups

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1. Introduction

There is an extensive theory, due initially to Patterson and Sullivan, intertwining the isometric, conformal, and ergodic properties of Kleinian groups. Our purpose here is to begin to expand this theory to the setting of quasiconformal groups. In particular, we wish to explore the connection between the Hausdorff dimension of limit sets of quasiconformal groups and the exponent of convergence of the Poincaré series. It is well known that, for a large class of finitely generated Kleinian groups, the Hausdorff dimension of the limit set is the exponent of convergence [BiJo; S2]. We are concerned with the facet of Patterson–Sullivan theory that relates the exponent of convergence to the Hausdorff dimension of the limit set, and our techniques are primarily from the analytic theory of quasiconformal mappings. We will, however, directly apply techniques and results from Patterson–Sullivan theory in the sequel to this paper [BT].

The Poincaré series of a Kleinian group has been the object of much refined study; see, for example, [BiJo; Mc; Pa; S1; S2; Tu].

Because a quasiconformal group no longer acts isometrically on hyperbolic space, it is to be expected that the whole of Patterson–Sullivan theory does not generalize directly to quasiconformal groups. Thus our purpose in this paper is twofold: we record positive results and then provide counterexamples that demonstrate ways in which the Patterson–Sullivan theory fails for discrete quasiconformal groups.

The central part of the paper consists of Sections 4 and 5, where we provide examples to demonstrate differences between the quasiconformal and the conformal case. We find that *the exponent of convergence can be strictly greater than the Hausdorff dimension of the conical limit set* (Example 4.1) and that *the Hausdorff dimension can "jump up" in the limit on convergent sequences of quasiconformal groups* (see Example 4.2). For convergent sequences of Kleinian groups, the Hausdorff dimension of the limit set is lower semicontinuous (see [BiJo] and [Mc]). We also provide an example (Example 4.3) of a *discrete quasiconformal group whose limit set consists entirely of conical limit points*; *however, the group has the property that the Hausdorff measure of the limit set at the critical dimension has zero*

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mass. This example helps to motivate our second paper on the subject [BT], in which we extend our analysis by constructing an analog of the Patterson–Sullivan measure on limit sets of quasiconformal groups and then use this measure to analyze the local properties of limit sets.

The paper proceeds as follows: Section 2 contains the basic definitions and results that we will need from both discrete group theory and the theory of quasiconformal mappings. Section 3 will show that the calculation of the exponent of convergence does not depend on the choice of extension of a quasiconformal group. Section 4 and Section 5 contain the construction of counterexamples with which we contrast and compare various properties of Hausdorff dimension on limit sets of quasiconformal groups and convergence groups to Möbius groups.

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2. Basics

We recall that a *K-quasiconformal group G* acting on $\overline{\mathbb{R}^n}$ is a group of mappings, each of which is a K-quasiconformal homeomorphism of $\overline{\mathbb{R}^n}$. Such a group G is *discrete* if there exists no sequence of mappings in the group that converges uniformly on $\overline{\mathbb{R}^n}$ to the identity mapping. We note that, from the theory of quasiconformal mappings, a discrete 1-quasiconformal group is a discrete group of Möbius transformations, that is, a *Kleinian group.*

The action of a discrete quasiconformal group G partitions $\overline{\mathbb{R}^n}$ into two disjoint sets. The *regular set* $\Omega(G) \subset \overline{\mathbb{R}^n}$ is the largest open set on which G acts discontinuously; the *limit set* $L(G)$ is the complement of $\Omega(G)$ in $\overline{\mathbb{R}^n}$. It is easy to see that $L(G)$ is a closed set; if $L(G)$ contains more than two points, then $L(G)$ is a perfect set (and thus uncountable) that is either nowhere dense or all of $\overline{\mathbb{R}^n}$. For the basics in the theory of quasiconformal groups see [GM1]; for discrete Möbius groups ($K = 1$) see [Ma].

An important tool in the theory of Kleinian groups acting on $\overline{\mathbb{R}^{n-1}}$ is the hyperbolic metric on $\mathbb{H}^n$. Note that every Kleinian group Γ acting on $\overline{\mathbb{R}^{n-1}}$ extends to a group of hyperbolic isometries acting discontinuously on hyperbolic n-space $(\mathbb{H}^n, \rho)$. If Γ has no finite-order elements, then the quotient space $\mathbb{H}^n/\Gamma$ is a complete Riemannian n-manifold of constant sectional curvature -1. One of the great difficulties in the analysis of (non-Möbius) quasiconformal groups is that, though individually every quasiconformal map of $\overline{\mathbb{R}^{n-1}}$ extends to a quasiconformal map of $\mathbb{H}^n$, it is not known whether extensions of all elements of the group exist so that the group structure is preserved. Furthermore, even in the case where such an extension exists, the extended group no longer acts isometrically on hyperbolic n-space.

Conformally equivalent to the extension problem from $\overline{\mathbb{R}^{n-1}}$ to $\mathbb{H}^n$ is the problem of extending quasiconformal groups acting on $\mathbb{S}^{n-1}$ to $\mathbb{B}^n$. Note that, by reflection, a quasiconformal group acting on $\mathbb{B}^n$ extends to a quasiconformal group acting on $\overline{\mathbb{R}^n}$ with the same dilatation.

Because we wish to use geometric arguments involving the hyperbolic metric, we will consider only those K-quasiconformal groups acting on $\overline{\mathbb{R}^n}$ that preserve the unit ball $\mathbb{B}^n$. We call such a group G a *quasiconformal Fuchsian group,* and we label such a G by the symbol "QCF". If G is a QCF group then $\partial\mathbb{B}^n$ is also invariant under G, and if G is discrete in addition then it acts discontinuously in $\mathbb{B}^n$, and $L(G) \subset \partial\mathbb{S}^{n-1}$ by [GM1, Cor. 3.8]. In this case it is easy to show that G is countable.

There is a well-developed theory relating the hyperbolic action of a Kleinian group on $\mathbb{B}^n$ to its conformal action on $\mathbb{S}^{n-1}$. One can extend parts of this theory to the class of quasiconformal groups. The *Poincaré series* with exponent s of a discrete QCF group G is defined as

$$\Sigma(s, x, y) := \sum_{g\in G} e^{-s\rho(x,g(y))} \quad \text{for } x, y \in \mathbb{B}^n \text{ and } s > 0.$$

DEFINITION 2.1 (Exponent of convergence). For a discrete QCF group G acting on $\overline{\mathbb{R}^n}$, we call

$$\delta(G) := \inf\{s \mid \Sigma(s, 0, 0) < \infty\}$$

the *exponent of convergence* of G.

The following proposition shows that the convergence or divergence of the series $\Sigma(s, x, y)$ is independent of the choice of x and y (and so $\delta(G)$ can be computed with any base points x, y in the place of 0). We omit the proof, since it is based on the well-known result in the Kleinian group setting (see e.g. [N]).

PROPOSITION 2.2. *Let G be a discrete QCF group acting on $\overline{\mathbb{R}^n}$. Then, for all $x, y \in \mathbb{B}^n$,*

$$\begin{aligned}\inf_s\{\Sigma(s, x, y) < \infty\} &= \inf_s\{\Sigma(s, 0, 0) < \infty\}\\ &= \inf_s\left\{\sum_{g\in G}(1 - |g(x)|)^s < \infty\right\}\\ &= \inf_s\left\{\sum_{g\in G}(1 - |g(0)|)^s < \infty\right\}\\ &= \limsup_{r\to\infty} \frac{1}{r}\log N(r, x, y) = \limsup_{r\to\infty} \frac{1}{r}\log N(r, 0, 0),\end{aligned}$$

where $N(x, y, r) = \#\{g \in G \mid \rho(x, g(y)) < r\}$.

It is not clear a priori that $\delta(G)$ is finite for a discrete QCF group G. Lemma 2.3 shows that in fact it is bounded above by $n-1$, that is, the dimension of the sphere at infinity of hyperbolic space $\mathbb{B}^n$. This is a well-known fact for Kleinian groups; see [N, Thm. 1.6.1]. A proof of Lemma 2.3 is in [GM2]; this has not yet appeared in print, so we provide an outline of the argument for the reader's convenience.

LEMMA 2.3 (Gehring–Martin). *Let G be a discrete QCF group acting on $\overline{\mathbb{R}^n}$. Then $\delta(G) \le n-1$.*

Proof. Because G acts discontinuously on $\mathbb{B}^n$, there exists a point $x \in \mathbb{B}^n$ that is stabilized only by the identity in G. Since Möbius transformations act as isometries on $(\mathbb{B}^n, \rho)$, it is easy to see (using Proposition 2.2) that $\delta(G)$ is invariant under conjugation of G by a Möbius transformation. Thus we can assume that $x = 0$.

The geometric idea behind the Gehring–Martin argument is as follows. First they show that there exists a fixed A (independent of r) such that

$$N(r, 0, 0) < Ae^{r(n-1)}. \tag{2.1}$$

The key geometric fact in showing this is that, because G is discrete and consists of uniformly K-quasiconformal mappings, there exists a hyperbolic distance d such that all of the orbit points of 0 are separated by a distance of at least d.

It is immediate that

$$\sum_{\substack{g\in G \\ \rho(0,g(0))<r}} e^{-\alpha\rho(0,g(0))} = N(r, 0, 0)e^{-\alpha r} + \alpha \int_0^r N(t, 0, 0)e^{-\alpha t}\, dt.$$

Substituting (2.1) into the equation just displayed and taking the limit as $r \to \infty$, we see that the Poincaré series converges for all $\alpha > n - 1$. □

REMARK 2.4. In Section 5 we will demonstrate, using a more general class of discrete groups, that Lemma 2.3 is no longer true in this enlarged class.

There is a connection between the Hausdorff dimension of the limit set and the exponent of convergence for a Kleinian group Γ acting conformally on $\mathbb{S}^{n-1}$. The relationship is especially nice in the setting of geometrically finite groups.

THEOREM 2.5 [Pa; S1; Tu]. *Let Γ be a geometrically finite Kleinian group acting on $\mathbb{S}^{n-1}$. Then*

$$\delta(\Gamma) = \dim(L(\Gamma)).$$

If Γ is geometrically finite and purely loxodromic, then $L(\Gamma)$ consists entirely of a certain type of limit points. More generally, let G be a discrete QCF group. Then a point $x \in L(G)$ is a *conical limit* point if there is a sequence of orbit points in $\mathbb{B}^n$ that converges to x inside a Euclidean nontangential cone with vertex at x. We denote by $L_c(G)$ the full collection of conical limit points in $L(G)$.

With respect to the conical limit set, Bishop and Jones [BiJo] were able to remove the assumption of geometric finiteness.

THEOREM 2.6 (Bishop–Jones). *Let Γ be a Kleinian group whose limit set contains more than two points. Then $\delta(\Gamma) = \dim(L_c(\Gamma))$.*

In Example 4.1 we see that Theorem 2.6 does not generalize to the full class of QCF groups. Though Theorem 2.6 is not true in this more general setting, an argument in [BiJo] (Theorem 2.1), showing that the Hausdorff dimension of the conical limit set is bounded above by the exponent of convergence, generalizes to the case of QCF groups.

THEOREM 2.7. *Let G be a discrete QCF group acting on $\overline{\mathbb{R}^n}$. Then $\delta(G) \geq \dim(L_c(G))$.*

3. Independence of Extension

Our next goal is to show that the exponent of convergence of a discrete QCF group depends only upon the action of the group on $\mathbb{S}^{n-1}$.

We first note that a quasiconformal mapping $f\colon \mathbb{B}^n \to \mathbb{B}^n$ is a quasi-Möbius mapping; hence it is immediate that, if f is the identity on the boundary, then the hyperbolic distance from $x \in \mathbb{B}^n$ to $f(x)$ is bounded above by a constant depending only on K and n.

LEMMA 3.1. *For each $n \in \mathbb{N}$, $n \geq 2$, and $K \geq 1$, there exists a constant $c(K,n) > 0$ such that the following holds: If $f\colon \mathbb{B}^n \to \mathbb{B}^n$ is K-quasiconformal and extends to the identity on $\mathbb{S}^{n-1}$, then*

$$\rho(x, f(x)) \leq c(K,n) \quad \textit{for all } x \in \mathbb{B}^n.$$

REMARK 3.2. Every K-quasiconformal mapping $f\colon \mathbb{B}^n \to \mathbb{B}^n$ extends uniquely to a K-quasiconformal mapping of $\mathbb{S}^{n-1}$; see [V].

Using Lemma 3.1, we now show that the exponent of convergence of a discrete QCF group on $\overline{\mathbb{R}^n}$ depends only on the action of the group on $\mathbb{S}^{n-1}$.

THEOREM 3.3. *Let G and $\tilde{G}$ be two discrete QCF groups acting on $\overline{\mathbb{R}^n}$ that agree on $\mathbb{S}^{n-1}$; that is, there exists an isomorphism $\varphi\colon G \to \tilde{G}$ such that $g|_{\mathbb{S}^{n-1}} = \varphi(g)|_{\mathbb{S}^{n-1}}$ for all $g \in G$. Then $\delta(G) = \delta(\tilde{G})$.*

Proof. Choose $K \geq 1$ large enough so that both groups G and $\tilde{G}$ are K-quasiconformal. Choose $x \in \mathbb{B}^n$. For $g \in G$ let $\tilde{g} := \varphi(g) \in \tilde{G}$ be the corresponding element so that g and $\tilde{g}$ agree on $\mathbb{S}^{n-1}$. By the triangle inequality we have that

$$\rho(x, \tilde{g}(x)) \leq \rho(x, g(x)) + \rho(g(x), \tilde{g}(x)).$$

Since g^{-1} is K-quasiconformal, we obtain furthermore that

$$\rho(g(x), \tilde{g}(x)) \leq \Phi_K(\rho(x, g^{-1}\tilde{g}(x))),$$

where $\Phi_K(t) = 4d \max\{t, t^\alpha\}$ with $d = d(K,n)$ and $\alpha = K^{1/(1-n)}$ (see [GM2, Cor. 2.10]). Observe now that $g^{-1} \circ \tilde{g}$ is a K^2-quasiconformal map on $\mathbb{B}^n$ that extends to the identity on $\mathbb{S}^{n-1}$. By Lemma 3.1 we obtain that $\rho(x, g^{-1}\tilde{g}(x)) \leq c(K^2, n)$. Hence

$$\rho(x, \tilde{g}(x)) \leq \rho(x, g(x)) + C,$$

where $C = \Phi_K(c(K^2,n))$ depends only on K and n. In the same way we obtain

$$\rho(x, g(x)) \leq \rho(x, \tilde{g}(x)) + C.$$

Thus, for any $s > 0$ we have that

$$e^{-s\rho(x,g(x))}e^{-sC} \leq e^{-s\rho(x,\tilde{g}(x))} \leq e^{-s\rho(x,g(x))}e^{sC}.$$

This implies that

$$\sum_{\tilde{g} \in \tilde{G}} e^{-s\rho(x,\tilde{g}(x))} < \infty \quad \text{if and only if} \quad \sum_{g \in G} e^{-s\rho(x,g(x))} < \infty,$$

and this completes the proof. □

4. Counterexamples

This section contains a collection of basic counterexamples highlighting the different behavior of the exponent of convergence with respect to Hausdorff dimension that discrete quasiconformal groups exhibit. This list is not complete in detailing the pathological behavior (relative to Kleinian groups) of the action of quasiconformal groups; we have limited ourselves here to examples using results in Sections 2 and 3 and to the use of elementary analytical techniques. In [BT] we will explain why such phenomena occur by "localizing" Patterson–Sullivan theory.

Recall that a Kleinian group Γ acting on $\overline{\mathbb{R}^n}$ is a *Fuchsian group* if Γ keeps the unit ball $\mathbb{B}^n$ (or the upper half-space $\mathbb{H}^n$) invariant. A Fuchsian group is a QCF group with $K = 1$. Our first counterexample shows that, unlike in the Kleinian case (cf. Theorem 2.6), the Hausdorff dimension of the conical limit set of a QCF group can be strictly smaller than the exponent of convergence of the group (note, however, that it is always true that $\dim L_c(G) \leq \delta(G)$; see Theorem 2.7).

EXAMPLE 4.1. There exists a QCF group G acting on $\overline{\mathbb{R}^2}$ such that $\delta(G) > \dim L_c(G)$.

Construction. It will be more convenient to work in the upper half-space model $\mathbb{H}^2$. Let Γ be a finitely generated Fuchsian group of the second kind acting on $\mathbb{H}^2$ that contains a parabolic element with fixed point at ∞. Necessarily $\delta(\Gamma) < 1$. One can, for example, choose Γ to be the Hecke group generated by $z \mapsto z + 1$ and $z \mapsto -\lambda^{-2}/z$ for some $\lambda > 2$. We can assume that $\gamma(z) = z + 1$ is a generator of Γ. Choose $K > 1$ so large that $K/(K + 1) > \delta(\Gamma)$, and define

$$\varphi(z) := \begin{cases} z & \text{if } |z| \leq 1, \\ |z|^{1/K-1} z & \text{if } |z| > 1. \end{cases}$$

One easily verifies that, for large $n \in \mathbb{Z}$, we have

$$\rho(i, \varphi\gamma^n\varphi^{-1}(i)) = \rho(i, \varphi(i + n)) \sim \log(|n|^{1/K+1}),$$

where "$\sim$" means that the two quantities differ only by an additive constant that is bounded independently of n. Hence, using Proposition 2.2, we observe that

$$\begin{aligned} \delta(G) &\geq \delta(\langle \varphi\gamma\varphi^{-1} \rangle) \\ &= \inf\left\{ s > 0 \,\middle|\, \sum_{n \in \mathbb{Z}} e^{-s\rho(i, \varphi\gamma^n\varphi^{-1}(i))} < \infty \right\} \\ &= \inf\left\{ s > 0 \,\middle|\, 2\sum_{n \in \mathbb{N}} \frac{1}{n^{s(1/K+1)}} < \infty \right\} \\ &= \frac{K}{K+1} > \delta(\Gamma). \end{aligned}$$

On the other hand, φ is locally bi-Lipschitz at every $x \in \mathbb{R}$ (but not at ∞), and this implies that $\dim(L_c(G)) = \dim(L_c(\Gamma))$. Since $\dim(L_c(\Gamma)) = \delta(\Gamma)$ by [Pa] and [S1], we conclude that $\dim(L_c(G)) < \delta(G)$.

We will return to this example in more detail in [BT]. In particular, we will localize the definitions for the exponent of convergence and the Hausdorff dimension of the limit set and thus show that the pathology behind this example can be described in terms of these localized quantities.

We conjecture that Example 4.1 is sharp in the sense that the exponent of convergence of a K-quasiconformal conjugate of a parabolic cyclic Fuchsian group acting on $\mathbb{H}^2$ cannot exceed $K/(K+1)$. Note that if we replace φ in the example by $\varphi(z) = |z|^{1/K-1}z$ then we have $\delta(\langle\varphi\gamma\varphi^{-1}\rangle) = 1/(K+1)$. Recall also that the exponent of convergence of any parabolic cyclic Fuchsian group acting on $\mathbb{H}^2$ is $\frac{1}{2}$.

Conjecture. *Let Γ be a parabolic cyclic Fuchsian group acting on $\mathbb{H}^2$, and let $\varphi\colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ be a K-quasiconformal mapping that keeps $\mathbb{H}^2$ invariant. Then*

$$\frac{1}{K+1} \leq \delta(\varphi\Gamma\varphi^{-1}) \leq \frac{K}{K+1}.$$

Our second counterexample contrasts the behavior of the exponent of convergence on sequences of QCF groups with its behavior on sequences of Kleinian groups. We start by defining in what sense these sequences are converging.

Fix an abstract finitely generated group H, and let

$$\rho_i\colon H \to \mathcal{QC}(K)$$

be a discrete faithful representation of H into the space $\mathcal{QC}(K)$ of K-quasiconformal mappings on $\overline{\mathbb{R}^n}$ endowed with the compact-open topology. We say that a sequence

$$\{\rho_i\colon H \to \mathcal{QC}(K)\}$$

converges algebraically to a discrete faithful representation $\rho_\infty\colon H \to \mathcal{QC}(K)$ if $\rho_i(h) \to \rho_\infty(h)$ for each generator $h \in H$.

In dimension 2, if $\{\rho_i\}$ is a sequence of discrete faithful representations into the space of Möbius transformations and H is not cyclic, then it is a fundamental result of Jørgensen [J] that the limit representation is automatically discrete and faithful. In any dimension $n \geq 2$, if H does not contain elements of finite order, if $\rho_i(H)$ is a non-elementary quasiconformal Fuchsian group for each i, and if K is sufficiently close to 1, then we have shown [BM] that the limit representation is again discrete and faithful.

It is clear that, by definition, algebraic convergence preserves the group structure. Our next type of convergence respects, in the Kleinian case, the geometric structure on compact subsets of the quotient manifolds. A sequence

$$\{G_i \in \mathcal{QC}\}$$

converges geometrically to G_∞ if:

(1) for each $g \in G_\infty$, there exists a sequence $\{g_i \in G_i\}$ such that $g_i \to g$; and
(2) if $\{g_{i_k} \in G_{i_k}\}$ so that $\{g_{i_k}\}$ converges to g, then $g \in G_\infty$.

For Kleinian groups, the topology induced by geometric convergence is equivalent to the Gromov topology, that is, quasi-isometric convergence on compact sets of the quotient manifolds; see [Th] or [BePe]. Observe that, if a sequence $\{\rho_i\}$ is converging algebraically and geometrically, then it is clear from the definition of geometric convergence that the algebraic limit is contained in the geometric limit. Should a sequence $\{\rho_i\}$ converge algebraically and geometrically to the same discrete group G, we say that the sequence *converges strongly*.

We will show that the following two facts, which are true in the Möbius category, are not true in general for QCF groups. The first theorem is true in more generality than we give here; however, the statement given makes the contrast between Kleinian groups and non-Kleinian QCF groups explicit.

THEOREM 4.1 [CT2; Mc]. *Suppose that H is not a free group, and let $\{\rho_i : H \to$ $\text{Möb}(n)\}$ be a sequence of Kleinian groups converging strongly to a geometrically finite group $\rho_\infty(H)$. Then*

$$\lim_{i\to\infty} \delta(\rho_i(H)) = \delta(\rho_\infty(H)).$$

Via Patterson–Sullivan theory, we see that Theorem 4.1 can be restated to conclude that the Hausdorff dimension function is continuous on strongly convergent sequences (with geometrically finite limits) of Kleinian groups provided the groups are not free.

If one assumes only algebraic convergence, then the best one can do is lower semicontinuity.

THEOREM 4.2 [BiJo]. *If H is a finitely generated abstract group and $\{\rho_i : H \to$ $\text{Möb}(n)\}$ is a sequence of discrete faithful representations converging algebraically to ρ_∞, then*

$$\dim(L(\rho_\infty(H)) \le \liminf_{i\to\infty} \dim(L(\rho_i(H)).$$

REMARK 4.3. There are well-known examples where the dimension of the limit set of the limit group is *strictly less* than the limit inferior of the dimensions of the limit sets along the sequence. An example of such phenomena is a sequence of degenerate groups on the boundary of a Bers' slice converging to a maximal cusp [C].

PROPOSITION 4.4. *Let Γ be a Kleinian group, and let $\{\varphi_n \in \mathcal{QC}(K)\}$ so that $\{\varphi_n\}$ converges to a mapping $\varphi \in \mathcal{QC}(K)$ in the compact-open topology.*

Then the sequence $\{\rho_n\}$ of representations of Γ to $\mathcal{QC}$ groups defined by

$$\{\rho_n : \Gamma \mapsto \varphi_n \Gamma \varphi_n^{-1}\}$$

is strongly convergent to the representation

$$\rho : \Gamma \mapsto \varphi \Gamma \varphi^{-1}.$$

The proof of this proposition is exactly the same as the proof for Kleinian groups, so we omit it.

The following example shows that Theorems 4.1 and 4.2 do not generalize to the setting of QCF groups. Example 4.2(i) contrasts Theorem 4.1, and Example 4.2(ii) contrasts Theorem 4.2.

EXAMPLE 4.2. (i) There exist a Kleinian group Γ acting on $\overline{\mathbb{R}^2}$ and a sequence $\{\varphi_n\}$ of uniformly quasiconformal mappings on $\overline{\mathbb{R}^2}$ that satisfy the following: $\varphi_n \to \mathrm{id}$ as $n \to \infty$, but

$$\delta(\varphi_n \Gamma \varphi_n^{-1}) \not\to \delta(\Gamma) \ \text{ as } n \to \infty.$$

(ii) There exist a finitely generated Kleinian group Γ acting on $\overline{\mathbb{R}^2}$ and a sequence $\{\psi_n\}$ of uniformly quasiconformal mappings on $\overline{\mathbb{R}^2}$ that satisfy the following: $\psi_n \to \psi$ quasiconformal as $n \to \infty$, but

$$\dim(L(G_n)) = 1 \ \text{ for all } n \quad \text{and} \quad \dim(L(G)) > 1,$$

where $G_n = \psi_n \Gamma \psi_n^{-1}$ and $G = \psi \Gamma \psi^{-1}$; that is,

$$\dim(L(G)) \not\leq \liminf_{n\to\infty} \dim(L(G_n)).$$

Construction. (i) Let Γ be a finitely generated and purely loxodromic Fuchsian group, so that $L_c(\Gamma) = L(\Gamma) = \partial\mathbb{D}$. For each $n \in \mathbb{N}$ we construct φ_n as follows. Let P_n be a regular polygon with 2^n sides, inscribed in the unit disk and placed in such a way that two of its corners are on the x-axis. Then the interior of P_n is contained in the interior of P_{n+1} for all n. Replace now each of the sides of P_n by a regular snowflake curve and call the resulting curve C_n (see Figure 1).

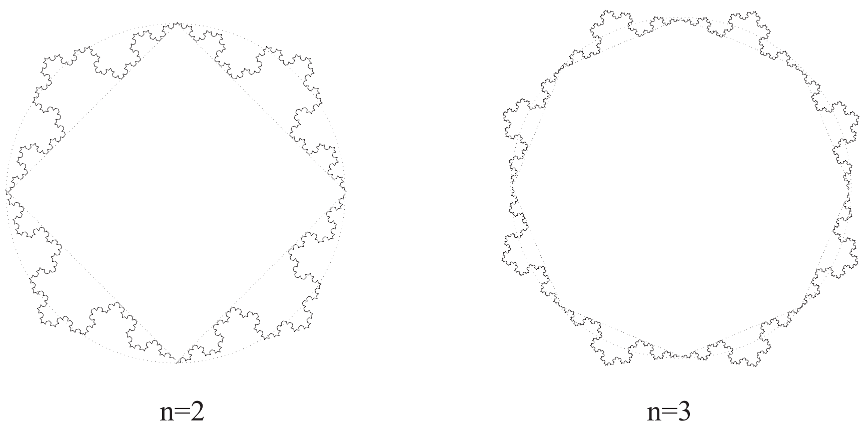

Figure 1 The curves C_n

Then C_n is a K-quasicircle, where K can be chosen independently of n. Furthermore, $\dim(C_n) = \log 4/\log 3$ for all n, and $C_n \to \partial\mathbb{D}$ in the Hausdorff set topology as $n \to \infty$.

Let φ_n map $\mathbb{D}$ conformally onto the interior of C_n, where $\varphi_n(0) = 0$ and $\varphi_n'(0) > 0$. Then $\varphi_n \to \mathrm{id}$ by Carathéodory's theorem on kernel convergence (see e.g. [Po]). Extend each φ_n to a quasiconformal map of $\overline{\mathbb{R}^2}$, and denote the extension again by φ_n. Then all φ_n are uniformly quasiconformal, and $\varphi_n \to \mathrm{id}$ as $n \to \infty$.

Define $G_n := \varphi_n \circ \Gamma \circ \varphi_n^{-1}$. Since the quasiconformal conjugacy maps the limit set of Γ onto the limit set of G_n and in fact preserves the property of being conical, we have that $L_c(G_n) = C_n$; hence

$$\dim(L_c(G_n)) = \frac{\log 4}{\log 3} \text{ for all } n, \quad \text{but}$$
$$\dim(L_c(\Gamma)) = 1.$$

Now extend each φ_n to a quasiconformal map of $\mathbb{H}^3$ and choose a subsequence (again denoted by $\{\varphi_n\}$) so that $\varphi_n \to \varphi$ as $n \to \infty$, where φ is quasiconformal and $\varphi|_{\overline{\mathbb{R}^2}} = \text{id}$. Using Theorem 2.7 and Theorem 3.3, we obtain that

$$\delta(\varphi \circ \Gamma \circ \varphi^{-1}) = \delta(\Gamma) = \dim(L_c(\Gamma)) = 1,$$

but

$$\delta(G_n) \geq \dim(L_c(G_n)) = \frac{\log 4}{\log 3} > 1 \quad \text{for all } n \in \mathbb{N}.$$

(ii) Let Γ be a finitely generated Fuchsian group with $L(\Gamma) = \partial\mathbb{D}$. Let ψ_n map $\mathbb{D}$ conformally onto the interior of the nth approximation of the regular snowflake curve (see Figure 2), where $\psi_n(0) = 0$ and $\psi_n'(0) > 0$.

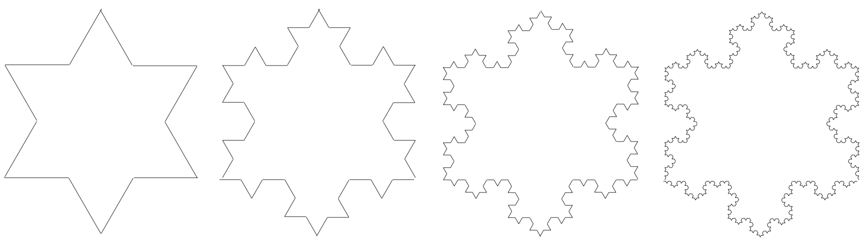

Figure 2 Approximations of the snowflake curve

Then $\{\psi_n\}$ converges to the the map ψ, which maps $\mathbb{D}$ conformally onto the interior of the snowflake curve. Each ψ_n can be extended to a quasiconformal mapping of $\overline{\mathbb{R}^2}$, where the quasiconformal dilatation is independent of n. Choose a subsequence such that $\psi_n \to \tilde{\psi}$ for some quasiconformal mapping $\tilde{\psi} : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$. Then $\tilde{\psi}|_{\mathbb{D}} = \psi$, so that $\tilde{\psi}$ is an extension of ψ to $\overline{\mathbb{R}^2}$.

Defining $G_n := \psi_n \circ \Gamma \circ \psi_n^{-1}$ and $G := \tilde{\psi} \circ \Gamma \circ \tilde{\psi}^{-1}$, we obtain

$$L(G_n) = \psi_n(L(\Gamma)) = \psi_n(\partial\mathbb{D})$$

and hence

$$\dim(L(G_n)) = 1 \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, we have

$$L(G) = \psi(\partial\mathbb{D}) = \text{standard snowflake curve}$$

and so

$$\dim(L(G)) = \frac{\log 4}{\log 3} > 1.$$

Using a similar construction to that of Example 4.2, we can produce an example that illustrates how different the Hausdorff measure behaves on limit sets of K-quasiconformal groups as compared to how it behaves on limit sets of Kleinian groups. To draw the comparison out, note that we can take the quasiconformal group G constructed in Example 4.3 to be finitely generated so that its limit set consists entirely of conical limit points. For Kleinian groups Γ satisfying $L_c(\Gamma) = L(\Gamma)$, the Hausdorff measure H_α at the critical dimension α (restricted to the limit set) is finite and positive [S1]. We explore this dichotomy between quasiconformal groups and Kleinian groups more fully in [BT].

Example 4.3. There exists a discrete quasiconformal group G acting on $\overline{\mathbb{R}^2}$ that satisfies $H_\alpha(L(G)) = 0$, where $\alpha = \dim(L(G))$.

Construction. Let Γ be a Fuchsian group acting on $\overline{\mathbb{R}^2}$ with $L(\Gamma) = \partial\mathbb{D}$. Choose a sequence $\{\alpha_i\}$ with $1 < \alpha_1 < \alpha_2 < \cdots < \frac{3}{2}$ and $\lim_{i\to\infty} \alpha_i = \frac{3}{2}$. Let C_i be a snowflake curve of Hausdorff dimension α_i whose base length is $1/i^2$. Put all C_i together "head to tail", and close this homeomorphic image of [0, 1] up with a rectifiable arc so that the resulting curve forms a closed Jordan curve C; see Figure 3.

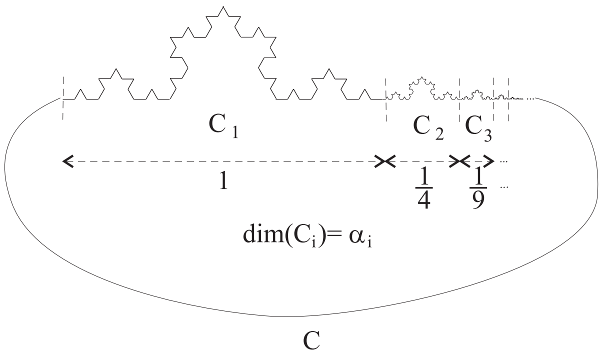

Figure 3 The curve C of dimension $\frac{3}{2}$

Let φ be a quasiconformal map of $\overline{\mathbb{R}^2}$ that maps $\mathbb{D}$ onto the interior of C, and define $G := \varphi\Gamma\varphi^{-1}$. Then we have $L(G) = C$ and moreover that $\dim C = \frac{3}{2}$, but $H_{3/2}(C) = 0$.

5. Convergence Groups

We conclude this paper with an open question. In [CT1] it was shown that an infinite-index geometrically finite subgroup $\tilde{\Gamma}$ of a Kleinian group Γ has the property that $\dim(L(\tilde{\Gamma})) < \dim(L(\Gamma))$. We show in Example 5.3 that the analogous statement is not true in the very general setting of convergence groups. The question of whether this theorem remains true in the quasiconformal case (see Question 5.1) remains open in its full generality [ABT].

Discrete convergence groups are generalizations of Kleinian groups and were invented by Gehring and Martin [GM1]. The definition is as follows: A *discrete convergence group* G on $\overline{\mathbb{R}^n}$ consists of homeomorphisms acting on $\overline{\mathbb{R}^n}$ with the property that, for every sequence $\{f_k\}$ of distinct elements in G, there is a subsequence $\{f_{k_j}\}$ and two points $a, b \in \overline{\mathbb{R}^n}$ such that the sequence $\{f_{k_j}\}$ converges to the point a locally uniformly in $\overline{\mathbb{R}^n} \setminus \{b\}$ and the sequence $\{f_{k_j}^{-1}\}$ converges to the point b locally uniformly in $\overline{\mathbb{R}^n} \setminus \{a\}$. It is clear that each element is isolated in a discrete convergence group; no sequence of elements converges to the identity.

Möbius groups and quasiconformal groups are examples of convergence groups (see [GM1]). Homeomorphic conjugates of quasiconformal groups are also convergence groups, so that the class of convergence groups of $\overline{\mathbb{R}^n}$ is strictly larger than the class of quasiconformal groups. Convergence groups in many essential ways resemble their conformal counterparts. For instance, as with Möbius groups one defines the *limit set* $L(G)$ and the *regular set* $\Omega(G)$ of the convergence group G in exactly the same way.

First we remark that Proposition 2.2 is no longer true if we require the group G only to be a discrete convergence group. However, we can still define the exponent of convergence for a discrete convergence group G acting on $\overline{\mathbb{R}^n}$ (and keeping $\mathbb{B}^n$ invariant) to be $\delta(G) := \inf\{s > 0 \mid \sum_{g\in G} e^{-s\rho(0,g(0))<\infty}\}$, with the convention that $\inf(\emptyset) = \infty$.

EXAMPLE 5.1. There exists a discrete convergence group G on $\overline{\mathbb{R}^n}$ that keeps $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ invariant but satisfies $\delta(G) = \infty$.

Construction. Let Γ be a Fuchsian Möbius group on $\overline{\mathbb{R}^n}$ that satisfies $\delta(\Gamma) \geq 1$, and let $\varphi\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be the homeomorphism defined by $\varphi(0) = 0$, $\varphi = \text{id}$ on $\mathbb{S}^{n-1}$, $\varphi(x)/|\varphi(x)| = x/|x|$, and $\rho(0, \varphi(x)) = (\rho(0,x))^{1/2}$ for $x \in \mathbb{B}^n \setminus \{0\}$ (and extend φ to all of $\overline{\mathbb{R}^n}$ by reflection). Then $G := \varphi\Gamma\varphi^{-1}$ satisfies $\delta(G) = \infty$.

This example actually shows more. Since the groups G and Γ agree on $\mathbb{S}^{n-1}$ (note that $\varphi = \text{id}$ on $\mathbb{S}^{n-1}$), we immediately have the following.

EXAMPLE 5.2. There exist two discrete convergence groups G and $\tilde{G}$ on $\overline{\mathbb{R}^n}$ that agree on $\mathbb{S}^{n-1}$ but have different exponents of convergence.

Hence, Theorem 3.3 cannot be extended beyond the quasiconformal class.

We now end with the example announced previously.

EXAMPLE 5.3. There exists a discrete convergence group G acting on $\overline{\mathbb{R}^2}$ with an infinite index subgroup $\tilde{G}$ such that

$$\dim(L(\tilde{G})) = \dim(L(G)).$$

Moreover, G and $\tilde{G}$ are finitely generated groups that are topologically conjugate to Schottky groups.

Construction. Let Γ be a Schottky group on $\overline{\mathbb{R}^2}$; that is, Γ is a finitely generated, free, purely loxodromic Kleinian group with non-empty regular set [Ma]. We further assume that Γ is generated by $\gamma_1, \gamma_2, \gamma_3$ such that, for $i = 1, 2, 3$, γ_i maps the exterior of the (round) disk $D_a^{(i)} \subset \mathbb{R}^2$ onto the interior of the disk $D_b^{(i)} \subset \mathbb{R}^2$ and also the interior of $D_a^{(i)}$ onto the exterior of $D_b^{(i)}$. Let all six disks be mutually disjoint and have radius 1. Let Ω be the common exterior of all six disks (with $\infty \in \Omega$).

Now define a homeomorphism φ as follows. Set $\varphi = \mathrm{id}$ on Ω. Define φ on $D_a^{(1)} \setminus \gamma_1^{-1}(D_a^{(1)} \cup D_a^{(2)} \cup D_b^{(2)} \cup D_a^{(3)} \cup D_b^{(3)})$ so that the images of $\gamma_1^{-1}(\partial D_a^{(1)})$, $\gamma_1^{-1}(\partial D_a^{(2)})$, and $\gamma_1^{-1}(\partial D_b^{(2)})$ are circles of radius $1/5$ and, on the other hand, the images of $\gamma_1^{-1}(\partial D_a^3)$ and $\gamma_1^{-1}(\partial D_b^3)$ are circles of radius $(1/5)^2$.

In the same way, define φ on $D_b^{(1)} \setminus \gamma_1(D_b^{(1)} \cup D_a^{(2)} \cup D_b^{(2)} \cup D_a^{(3)} \cup D_b^{(3)})$: Let the images of $\gamma_1(\partial D_b^{(1)})$, $\gamma_1(\partial D_a^{(2)})$, and $\gamma_1(\partial D_b^{(2)})$ be circles of radius $1/5$ and, on the other hand, let the images of $\gamma_1(\partial D_a^3)$ and $\gamma_1(\partial D_b^3)$ be circles of radius $(1/5)^2$.

Proceed inductively in the same manner. The images under φ of circles that are images of $\partial D_{a,b}^{(1),(2)}$ under an element of $\langle \gamma_1, \gamma_2 \rangle$ are circles of radius $R/5$, where R is the radius of the "motherdisk". All other "children" in a disk of radius R have size $(R/5)^2$ (see Figure 4).

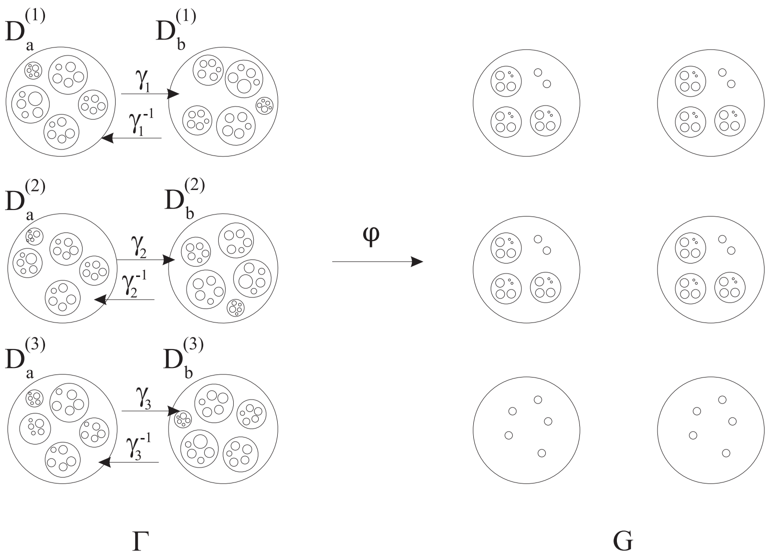

Figure 4 Construction of the group G

Now let $\tilde{\Gamma} = \langle \gamma_1, \gamma_2 \rangle$ and $\tilde{G} = \varphi(\Gamma)$. It is easy to see that $\dim L(\tilde{G}) = \log 3/\log 5 =: d$. On the other hand, for any $n \in \mathbb{N}$, the limit set of G can be covered by $4 \cdot 3^n$ disks of radius $1/5^n$ and $4 \cdot (5^n - 3^n) + 2 \cdot 5^n$ disks of radius $(1/5^n)^2$. For this cover $\{U_\alpha\}$, we have

$$\sum (\operatorname{diam} U_\alpha)^d = 4 \cdot 3^n \left(\frac{2}{5^n}\right)^d + [4 \cdot (5^n - 3^n) + 2 \cdot 5^n] \left(\frac{2}{5^{2n}}\right)^d$$
$$= 4 \cdot 2^d + 2^d [4 \cdot (5^n - 3^n) + 2 \cdot 5^n] 3^{-2n} \to 4 \cdot 2^d \text{ as } n \to \infty.$$

Hence $\dim L(G) \le d$, but since $L(\tilde{G}) \subset L(G)$ we conclude that $L(G)$ and $L(\tilde{G})$ have the same Hausdorff dimension.

However, our question does remain open in the quasiconformal setting.

QUESTION 5.1. Let G be a discrete quasiconformal group acting on $\overline{\mathbb{R}^n}$, so that $\dim(L(G)) < n$. Let $\tilde{G}$ be a finitely generated infinite index subgroup of G.
Is $\dim L(\tilde{G}) < \dim L(G)$?

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