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Parabolic Manifolds for Semi-Attractive Holomorphic Germs

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1. Introduction

The purpose of this paper is to study the local behavior of *semi-attractive* holomorphic self-maps of $\mathbb{C}^m$ $(m > 2)$ in a neighborhood of a fixed point that we assume to be the origin. Such transformations are the ones whose differential at 0 has one eigenvalue equal to 1 while the remaining ones, say $\beta_1, \dots, \beta_s$ with $s \geq 1$, have modulus strictly less than 1.

Semi-attractive transformations such that 0 is not an isolated fixed point have been studied by Nishimura [N], who considered analytic automorphisms F of complex manifolds admitting a q-dimensional complex submanifold M of attracting fixed points for F. Then, for each point $p_0 \in M$, one can choose local coordinates $(w, z) \in \mathbb{C}^q \times \mathbb{C}^{m-q}$ in a neighborhood U of p_0 such that $U \cap M$ has equation $z = 0$; hence the map F can be locally written as

$$\begin{aligned} w_1 &= w + O(\|z\|), \\ z_1 &= C(w)z + O(\|z\|^2), \end{aligned}$$

where $C(w)$ is a $(m-q) \times (m-q)$ matrix whose elements are holomorphic functions on U and whose eigenvalues $\beta_1(w), \dots, \beta_{m-q}(w)$ have modulus strictly less than 1.

Let $\Omega = \{p \in U \mid F^n(p) \to p_0,\ p_0 \in M\}$, which is an open set containing M. Nishimura proved that if these eigenvalues have no relations in any point of M, that is, if for each multi-index $j = (j_1, \dots, j_{m-q})$ with $\sum_{k=1}^{m-q} j_k \geq 2$ and $1 \leq i \leq m-q$ we have $\beta_1^{j_1}(w) \cdots \beta_{m-q}^{j_{m-q}}(w) \neq \beta_i(w)$, then there exists a biholomorphism $S \colon N \to \Omega$, where $\pi \colon N \to M$ is the normal bundle of M, which conjugates F to the automorphism G of N induced by F and given in $\pi^{-1}(U \cap M)$ by

$$\begin{aligned} s_1 &= s, \\ u_1 &= C(s)u. \end{aligned}$$

When 0 is an isolated fixed point, the problem has been studied only for maps such that the eigenvalue 1 has multiplicity $q = 1$. Fatou [F] and subsequently Ueda [U1; U2] studied the dynamics of such transformations F in two complex variables. Fatou found a system of coordinates (w, z) such that F has the following expression in a neighborhood of 0:

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$$w_1 = a_1(z)w + a_2(z)w^2 + \cdots,$$
$$z_1 = bz + b_1(z)w + b_2(z)w^2 + \cdots, \tag{1.1}$$

where b is the eigenvalue such that $0 < |b| < 1$ and where $a_i(z), b_i(z)$ are holomorphic functions in a neighborhood of $0 \in \mathbb{C}$ such that $a_1(0) = 1$ and $b_1(0) = 0$; he then proved the existence of a basin of attraction to the origin when $a_2(0) \neq 0$.

The same result was obtained by Ueda after reducing F to a simpler form of type (1.1), where (for fixed $k \geq 2$) the coefficients $a_i(z)$ are constants and $b_i(z)$ are linear monomials for $i \leq k$. Ueda gave a precise description of the basin of attraction Ω and showed that if F is an automorphism then Ω is biholomorphic to $\mathbb{C}^2$; he also proved that F is conjugated to the translation $(x, y) \mapsto (x+1, y)$ on an open subset of Ω.

Finally, Hakim [H1] considered semi-attractive maps of $\mathbb{C}^m$ with $m \geq 2$; for each $k \geq 2$ she proved the existence of a local system of coordinates $(w, z) \in \mathbb{C} \times \mathbb{C}^{m-1}$ with respect to which F has the form

$$w_1 = w + a_2 w^2 + \cdots + a_k w^k + a_{k+1}(z) w^{k+1} + \cdots,$$
$$z_1 = g(z) + z h(w, z), \tag{1.2}$$

where $a_2, \ldots, a_k$ are constants; $a_j(z)$ $(j \geq k+1)$, $g(z)$, and $h(w, z)$ are germs of holomorphic maps from $\mathbb{C}^{m-1}$ to $\mathbb{C}$, from $\mathbb{C}^{m-1}$ to itself, and from $\mathbb{C}^m$ to $\mathbb{C}^{m-1}$ (respectively), with $h(0,0) = 0$, $g(0) = 0$, and $dg(0)$ as the eigenvalues of $dF(0)$ that have modulus strictly less than 1. If $a_j \equiv 0$ for each j, then there exists a curve of fixed points $(w, z(w))$ passing through the origin; otherwise (i.e., if $a_2 = \cdots = a_h = 0$ and $a_{h+1} \neq 0$ for some $h \geq 1$) she found an attracting domain. Precisely, Hakim proved the following result.

THEOREM 1.1 [H1]. *Let F be a semi-attractive holomorphic transformation of $\mathbb{C}^m$ such that the origin is an isolated fixed point and the eigenvalue 1 of $dF(0)$ has algebraic multiplicity 1. Then: either* (a) *there exists a curve of fixed points or* (b) *$F - \mathrm{id}_{\mathbb{C}^m}$ has finite multiplicity $h+1$ at 0 and there exists an attracting domain D of 0 with h petals.*

Moreover, in the latter case, if F is a global automorphism of $(\mathbb{C}^2, 0)$ then every petal of D is biholomorphic to $\mathbb{C}^2$.

We are interested in the general case in $\mathbb{C}^m$, where the algebraic multiplicity of the eigenvalue 1 is $q > 1$. The center stable manifold theorem [R, p. 32] guarantees the existence of a closed analytic stable manifold of dimension $m - q$ tangent at 0 to the generalized eigenspace E corresponding to the eigenvalues $\beta_1, \ldots, \beta_s$ with modulus strictly less than 1. In particular, in a neighborhood of 0 and choosing local coordinates $(w, z) \in \mathbb{C}^q \times \mathbb{C}^{m-q}$ such that this manifold has equation $w = 0$, we can write our map in the form

$$w_1 = f_1(w, z) = A(z)w + P_{2,z}(w) + P_{3,z}(w) + \cdots,$$
$$z_1 = f_2(w, z) = G(z) + B(w, z)w, \tag{1.3}$$

where: $A(z)$ is a $q \times q$ matrix, whose elements are holomorphic functions in z, such that $A(0) = J$ is the Jordan block of $dF(0)$ corresponding to the eigenvalue 1;

$P_{i,z}\colon \mathbb{C}^q \to \mathbb{C}^q$ are homogeneous polynomials of degree i whose coefficients are holomorphic functions in z; G is a holomorphic transformation of $\mathbb{C}^{m-q}$ such that $G(0) = 0$ and $dG(0)$ has eigenvalues $\beta_1, \ldots, \beta_s$; and $B(w, z)$ is an $(m-q) \times q$ matrix whose elements are holomorphic functions of $\mathbb{C}^m$ vanishing at $(w, z) = (0, 0)$.

Note that, if the first component f_1 is just the identity in $\mathbb{C}^q$, then by the implicit mapping theorem we obtain a q-dimensional manifold of fixed points by solving the equation $z_1 = z$ in $z = z(w)$.

We consider the case where the origin is an isolated fixed point, and we shall prove the existence of larger stable manifolds with the origin in the boundary. First, we generalize Hakim's method to separate the contracting part from the neutral part of F to sufficiently high order (Section 2). Then we distinguish the case where the algebraic and geometric multiplicities of 1 are equal (Sections 3 and 4) from the one where they are different (Sections 5 and 6).

For semi-attractive maps such that the eigenvalue 1 of $dF(0)$ has Jordan block J equal to the identity matrix, we extend Hakim's argument about germs of holomorphic transformations F_1 of $(\mathbb{C}^q, 0)$ with differential $dF_1(0) = I$ (see [H2; H3] and also [W]): by studying the blow-up of F_1, Hakim proved that, for each nondegenerate complex direction V invariant under the homogeneous polynomial of lowest degree $h+1$ in the expansion of $F_1 - \mathrm{id}_{\mathbb{C}^q}$, we can obtain h stable curves Γ^i tangent to V at 0, with $0 \in \partial\Gamma^i$, as fixed points of suitable operators on Banach spaces. She also introduced some invariants associated to V and determined conditions ensuring the existence of h attracting domains to 0 such that each orbit inside them converges tangentially to V.

To describe the result obtained for our maps, we need to recall some definitions introduced by Hakim. Given a transformation $F\colon \mathbb{C}^m \to \mathbb{C}^m$, we say that F is *tangent to the identity at order* $h+1$ if, in a neighborhood of the origin, it has homogeneous expansion $X_1 = X + P_{h+1}(X) + P_{h+2}(X) + \cdots$ with $P_{h+1} \neq 0$.

Let $P_d\colon \mathbb{C}^m \to \mathbb{C}^m$ be a homogeneous polynomial of degree $d \geq 2$; a *characteristic direction* for P_d is a direction $V \in \mathbb{C}^m \setminus \{0\}$ such that $P_d(V) = \lambda V$ for some $\lambda \in \mathbb{C}$; it is called *nondegenerate* if $P_d(V) \neq 0$. Given a nondegenerate characteristic direction V for P_d, the projection $\hat{P}_d\colon [X] \mapsto [P_d(X)]$ on the projective complex space $\mathbb{P}^{m-1}(\mathbb{C})$ is defined in a neighborhood of V, and $[V]$ is a fixed point for $\hat{P}_d$. Then the matrix associated to the linear map

$$\frac{1}{d-1}(d\hat{P}_d[V] - \mathrm{id})\colon T_{[V]}\mathbb{P}^{m-1}(\mathbb{C}) \to T_{[V]}\mathbb{P}^{m-1}(\mathbb{C})$$

is called the *matrix associated* to V and denoted by $A(V)$. A simple computation shows that this definition is equivalent to the one given by Hakim. Moreover, if all characteristic directions of P_d are nondegenerate, then $\hat{P}_d$ is defined in the whole projective space $\mathbb{P}^{m-1}(\mathbb{C})$ and hence the number of characteristic directions is equal to the number of fixed points of $\hat{P}_d$, that is, $(d^m - 1)/(d-1)$ (see [FS]).

We say that $M \subset \mathbb{C}^m$ is a *holomorphic manifold at the origin of dimension* n if there exist a domain S of $\mathbb{C}^n$, with $0 \in \partial S$, and an injective holomorphic map $\psi\colon S \to \mathbb{C}^m$ such that $\psi(S) = M$ and $\lim_{X\to 0} \psi(X) = 0$. The manifold M is said to be *tangent* (at 0) to a vector space E if, for any sequence $\{X_k\} \subset S$ such that $X_k \to 0$ and $[\psi(X_k)] \to [V]$ in $\mathbb{P}^{m-1}(\mathbb{C})$, we have $V \in E$. Finally, M is said to be *parabolic* if it is F-invariant and if, for each point $p \in M$, the forward orbit $F^n(p)$ converges to 0.

By these definitions, the result obtained by Hakim for holomorphic self-maps tangent to the identity at 0 can be stated as follows.

THEOREM 1.2 [H2; H3]. *Let F be a holomorphic germ of transformations of $\mathbb{C}^m$ tangent to the identity at order $h+1$.*

(i) *For every nondegenerate characteristic direction V of F, there exist h parabolic curves $\Gamma^1, \dots, \Gamma^h$ tangent to V at the origin.*
(ii) *If the associated matrix $A(V)$ of V has all the eigenvalues with strictly positive real part, then there exist h disjoint attracting domains D^i $(i = 1, \dots, h)$, such that $0 \in \partial D^i$ and $\Gamma^i \subset D^i$, in which every point is attracted to the origin along an orbit tangent to V at 0.*

In our case we derive the following.

THEOREM 1.3 (Parabolic Manifold Theorem). *Let $m > 2$ and let F be a holomorphic germ of a semi-attractive transformation of $\mathbb{C}^m$, fixing the origin, such that the eigenvalue 1 of $dF(0)$ has the same algebraic and geometric multiplicity $q > 1$. Assume that F is of the form* (1.3). *If $f_1(w, 0)$ is tangent to the identity at order $h+1$ and if V is a nondegenerate characteristic direction for $P_{h+1,0}$, then there exist h disjoint parabolic manifolds of dimension $m-q+1$ and tangent to $\mathbb{C}V \oplus E$, where E is generated by the generalized eigenspaces associated to the eigenvalues of $dF(0)$ with modulus strictly less than* 1*.*

In Remark 2.1 we shall see that the number of these manifolds is strictly less than the multiplicity of $F - \mathrm{id}_{\mathbb{C}^m}$ at 0.

We shall also see that the eigenvalues of the matrix associated to a nondegenerate characteristic direction are still invariants under change of coordinates and, moreover, give us a sufficient condition for the existence of attracting domains.

THEOREM 1.4. *Assume the hypotheses of Theorem 1.3. If $P_{h+1,0}$ has a nondegenerate characteristic direction V such that its associated matrix $A(V)$ has all eigenvalues with strictly positive real part, then there exist h disjoint attracting domains for F with the origin in their boundary.*

If the algebraic multiplicity q of 1 is different from the geometric multiplicity, then the idea is to reduce this situation to the previous one by generalizing to our case the following theorem of Abate.

THEOREM 1.5 [A] (Diagonalization Theorem). *Let F be a holomorphic germ of a transformation of $\mathbb{C}^m$ such that $F(0) = 0$ and $dF(0)$ is invertible and nondiagonalizable. Then there exist a complex m-dimensional manifold M, a holomorphic projection $\pi: M \to \mathbb{C}^m$, a canonical point $\tilde{e} \in M$, and a holomorphic self-map $\tilde{F}: M \to M$ such that:*

(i) *π restricted to $M\backslash\pi^{-1}(0)$ is a biholomorphism between $M\backslash\pi^{-1}(0)$ and $\mathbb{C}^m\backslash\{0\}$;*
(ii) *$\pi \circ \tilde{F} = F \circ \pi$; and*
(iii) *$\tilde{F}(\tilde{e}) = \tilde{e}$ and $d\tilde{F}(\tilde{e})$ is diagonalizable.*

This theorem is proved by making iterated blow-ups of $\mathbb{C}^m$ along submanifolds starting from the blow-up at the origin. We cannot use Abate's result directly because it requires $dF(0)$ to be invertible whereas for our map this matrix may be singular. But, in order to apply subsequently Theorem 1.3, we need only to diagonalize the Jordan block J, which is always invertible. Hence, following the same line of proof as for the diagonalization theorem, but starting now from the blow-up along the center stable manifold, we shall prove our next theorem.

THEOREM 1.6 (Partial Diagonalization Theorem). *Let F be a holomorphic germ of a semi-attractive transformation of $\mathbb{C}^m$ fixing the origin, and let X be the center stable manifold for F. Then there exist a complex m-dimensional manifold M, a holomorphic projection $\pi\colon M \to \mathbb{C}^m$, a canonical point $\tilde{e} \in M$, and a holomorphic self-map $\tilde{F}\colon M \to M$ such that*:

(i) *π restricted to $M\backslash\pi^{-1}(X)$ is a biholomorphism between $M\backslash\pi^{-1}(X)$ and $\mathbb{C}^m\backslash X$;*
(ii) *$\pi \circ \tilde{F} = F \circ \pi$; and*
(iii) *$\tilde{e}$ is a fixed point of $\tilde{F}$ and $d\tilde{F}(\tilde{e}) = \mathrm{diag}\{I, J_0\}$, where I is the identity matrix whose order is equal to the algebraic multiplicity of the eigenvalue* 1 *and where J_0 is the Jordan block of $dF(0)$ corresponding to the eigenvalues with modulus strictly less than* 1.

Moreover, in a neighborhood of $\tilde{e}$, there is a system of coordinates such that $\tilde{F}$ takes the form (1.3) *with $A(z) \equiv I$ and $P_{2,0} \neq 0$.*

Finally, Theorem 1.6 allows us to find parabolic manifolds also for semi-attractive germs with geometric multiplicity of 1 strictly less than q. In fact, under generic conditions, we shall prove the existence of an "allowable" nondegenerate characteristic direction for $P_{2,0}$ and then of a $\tilde{F}$-parabolic manifold at the origin that projects to a F-parabolic one (see Corollary 6.1 and Corollary 6.2).

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2. Change of Coordinates

Let $m \geq 2$ and let F be a germ of holomorphic self-maps of $\mathbb{C}^m$ with $F(0) = 0$ and let $1, \beta_1, \ldots, \beta_s$ be the eigenvalues of $dF(0)$, where $|\beta_i| < 1$ for each $i = 1, \ldots, s$. Let $q \geq 1$ be the algebraic multiplicity of 1 and let J be its corresponding Jordan block. In this section, we reduce the expression (1.3) of F to a form where the terms of the first component do not depend on the variable z up to an order k. If $A(0) \neq I$ then this order will be $k = 2$; otherwise, it will be arbitrary.

PROPOSITION 2.1. *Assume F in the form* (1.3). *Then there exists a local system of coordinates $(w,z)\in\mathbb{C}^q\times\mathbb{C}^{m-q}$ such that, in a neighborhood of the origin, F takes the expression*

$$\begin{aligned} w_1 &= \tilde{f}_1(w,z) = Jw + \tilde{P}_{2,z}(w) + \tilde{P}_{3,z}(w) + \cdots, \\ z_1 &= \tilde{f}_2(w,z) = G(z) + \tilde{B}(w,z)w, \end{aligned} \tag{2.1}$$

where the $\tilde{P}_{i,z}$ are homogeneous polynomials of degree i in $\mathbb{C}^q$ whose coefficients are holomorphic functions in z and where $\tilde{B}(w,z)$ is an $(m-q)\times q$ matrix whose elements are holomorphic functions of $\mathbb{C}^m$ with $\tilde{B}(0,0)=0$.

Proof. Let us consider, in a neighborhood $N_1\times N_2\subset\mathbb{C}^q\times\mathbb{C}^{m-q}$ of the origin, the change of coordinates

$$\chi_1 : \begin{cases} W = U(z)w, \\ Z = z; \end{cases} \qquad \chi_1^{-1} : \begin{cases} w = U(Z)^{-1}W, \\ z = Z. \end{cases}$$

Here $U\colon N_2\to \mathrm{Gl}(\mathbb{C},q)$ is holomorphic. Then we have

$$\begin{aligned} W_1 = U(z_1)w_1 &= U(G(z)+B(w,z)w)[A(z)w + P_{2,z}(w)+\cdots] \\ &= U(G(Z)+\tilde{B}(W,Z)U(Z)^{-1}W)[A(Z)U(Z)^{-1}W \\ &\qquad\qquad + P_{2,Z}(U(Z)^{-1}W)+\cdots] \\ &= U(G(Z))A(Z)U(Z)^{-1}W + \tilde{P}_{2,Z}(W)+\cdots . \end{aligned}$$

We want

$$U(G(Z))A(Z)U(Z)^{-1} = J$$

or, equivalently,

$$JU(Z) = U(G(Z))A(Z).$$

Then it easy to see that

$$U(z) = \lim_n J^{-n-1}\prod_{i=0}^{n} A(G^{n-i}(z))$$

is the solution. This limit converges in a neighborhood of 0 by the following facts.

(a) For $\|z\|$ small enough, there exists $0<\gamma<1$ such that $A(G^i(z)) = J + B_i$ with $\|B_i\| = O(\gamma^i\|z\|)$.

(b) Since

$$\begin{aligned} J^{-n}(J+B_k) &= J^{-1}(J+B_k)J^{-n+1} + (J^{-n}B_k - J^{-1}B_kJ^{-n+1}) \\ &= (I + J^{-1}B_k)J^{-n+1} + (J^{-n}B_kJ^{n-1}J^{-n+1} - J^{-1}B_kJ^{-n+1}) \\ &= (I + J^{-1}J^{-n+1}B_kJ^{n-1})J^{-n+1}, \end{aligned}$$

by iteration we obtain

$$\begin{aligned} J^{-n-1}\prod_{i=0}^{n}(J+B_{n-i}) &= J^{-n-1}(J+B_n)\prod_{i=1}^{n}(J+B_{n-i}) \\ &= (I+J^{-1}J^{-n}B_nJ^n)J^{-n}\prod_{i=1}^{n}(J+B_{n-i}) \\ &= \prod_{i=0}^{n}(I+J^{-1}J^{-n+i}B_{n-i}J^{n-i}). \end{aligned}$$

(c) Finally,

$$\begin{aligned} \lim_n \sum_{i=0}^{n}\|J^{-1}J^{-n+i}B_{n-i}J^{n-i}\| &\le \|J^{-1}\|\sum_{k=0}^{\infty}\|J^{-k}B_kJ^k\| \\ &\le \|J^{-1}\|\sum_{k=0}^{\infty}O(k^2\gamma^k\|z\|) \end{aligned}$$

is convergent.

Moreover, in a small enough neighborhood of $0\in\mathbb{C}^{m-q}$, $U(z)$ is an invertibile matrix because $U\colon N_2\to\mathbb{C}^{q^2}$ is a holomorphic transformation and $U(0)=I$. $\square$

Proposition 2.2. *Assume F in the form* (1.3) *with* $A(0)=I$. *Then, for each $k\ge 2$, there exists a local system of coordinates $(w,z)\in\mathbb{C}^q\times\mathbb{C}^{m-q}$ such that F takes the following expression in a neighborhood of the origin*:

$$\begin{aligned} w_1 &= \tilde f_{1,k}(w,z) = w + P_2(w) + \cdots + P_{k-1}(w) + \tilde P_{k,z}(w) + \cdots, \\ z_1 &= \tilde f_2(w,z) = G(z) + \tilde B(w,z)w, \end{aligned} \tag{2.2$_k$}$$

where P_i (resp., $\tilde P_{i,z}$) are homogeneous polynomials of degree i in $\mathbb{C}^q$ with constant (resp., holomorphic functions in z) coefficients and where $\tilde B(w,z)$ is an $(m-q)\times q$ matrix whose elements are holomorphic functions of $\mathbb{C}^m$ with $\tilde B(0,0)=0$.

Moreover, if $f_1(w,0)$ is tangent to the identity at order $h+1$, then for each $k>h+1$ we have $P_2=\cdots=P_h\equiv 0$ and $P_{h+1}=P_{h+1,0}\ne 0$.

Proof. By Proposition 2.1, the assertion is true for $k=2$, so we reason by induction. Suppose that there are coordinates (w,z) such that F takes the form

$$\begin{aligned} w_1 &= w + P_2(w) + \cdots + P_{k-1}(w) + \tilde P_{k,z}(w) + \cdots, \\ z_1 &= G(z) + \tilde B(w,z)w, \end{aligned}$$

and consider the coordinate transformation

$$\chi_k : \begin{cases} W = w + Q_{k,z}(w), \\ Z = z; \end{cases} \qquad \chi_k^{-1} : \begin{cases} w = W - Q_{k,Z}(W) + \cdots, \\ z = Z. \end{cases}$$

Here $Q_{k,z}\colon\mathbb{C}^q\to\mathbb{C}^q$ is a homogeneous polynomial of degree k whose coefficients are holomorphic functions in the variable z in a neighborhood of 0. Then we have

$$\begin{aligned} W_1 &= w_1 + Q_{k,z_1}(w_1) \\ &= w + P_2(w) + \cdots + P_{k-1}(w) + \tilde{P}_{k,z}(w) + Q_{k,G(z)}(w) + P^*_{k+1,z}(w) + \cdots \\ &= W - Q_{k,Z}(W) + P_2(W) + \cdots \\ &\quad + P_{k-1}(W) + \tilde{P}_{k,Z}(W) + Q_{k,G(Z)}(W) + \bar{P}_{k+1,Z}(W) + \cdots . \end{aligned}$$

We want

$$Q_{k,Z}(W) - Q_{k,G(Z)}(W) = \tilde{P}_{k,Z}(W) - \tilde{P}_{k,0}(W); \tag{2.3}$$

this implies equalities between the coefficients $\tilde{p}_i, q_i\colon \mathbb{C}^{m-q} \to \mathbb{C}$ of the polynomials $\tilde{P}_{k,z}$ and $Q_{k,z}$. Then, for each $i = 1, \ldots, q\binom{q+k-1}{k}$, we need that

$$\begin{aligned} q_i(z) - q_i(G(z)) &= \tilde{p}_i(z) - \tilde{p}_i(0), \\ q_i(G(z)) - q_i(G^2(z)) &= \tilde{p}_i(G(z)) - \tilde{p}_i(0), \\ &\vdots \\ q_i(G^n(z)) - q_i(G^{n+1}(z)) &= \tilde{p}_i(G^n(z)) - \tilde{p}_i(0); \end{aligned}$$

hence, the solutions are

$$q_i(z) = \sum_{n=0}^{\infty} [\tilde{p}_i(G^n(z)) - \tilde{p}_i(0)].$$

For each i, the series converges in a neighborhood of 0 because G is a contraction and $\tilde{p}_i(z) - \tilde{p}_i(0) = 0$ for $z = 0$. In particular we have $Q_{k,0} \equiv 0$ for each $k \geq 2$.

Finally, assume $f_1(w, 0)$ tangent to the identity at order $h+1$, that is, $h+1 = \min\{d \mid P_{d,0} \neq 0\}$. Then the second part of the assertion is proved because $\chi_k(w, 0) = \mathrm{id}_{\mathbb{C}^q}$ for each k and thus $\tilde{f}_{1,k}(w, 0) = f_1(w, 0)$ for each $k \in \mathbb{N}$. □

REMARK 2.1. Suppose $f_1(w, 0)$ is tangent to the identity at order $h+1$; then assume F in the form $(2.2)_k$ with $k > h+1$. Let $\mathrm{mult}_0(F)$ and $\mathrm{ord}_0(F)$ be (respectively) the multiplicity and the order of F at 0. Since $G'(0) - I_{m-q}$ is invertibile, we can solve locally $z_1 - z = 0$ in $z = z(w)$. Then $\mathrm{mult}_0(F - \mathrm{id}_{\mathbb{C}^m}) = \mathrm{mult}_0(\tilde{f}_{1,k}(w, z(w)) - \mathrm{id}_{\mathbb{C}_q})$ (see [C, p. 108, Lemma 2]).

Moreover, for a holomorphic self-map of a neighborhood of 0 in $\mathbb{C}^m$ such that 0 is an isolated fixed point, the multiplicity at 0 is greater than or equal to $\prod_{j=1}^m \mathrm{ord}_0(p^j_*)$, where p^j_* is the initial homogeneous polynomial at 0 of the jth component of the map (see [C, p. 112, Thm. 2]).

Hence $\mathrm{mult}_0(F - \mathrm{id}_{\mathbb{C}^m}) > \mathrm{ord}_0(\tilde{f}_{1,k}(w, z(w)) - \mathrm{id}_{\mathbb{C}_q}) = h+1$.

3. Parabolic Manifold Theorem

Given $f, g_1, \ldots, g_s\colon \mathbb{C}^m \to \mathbb{C}^n$, from now on we denote

$$\begin{aligned} f = O(g_1, \ldots, g_s) \iff \exists C_1, \ldots, C_s > 0 \mid \|f(w)\| &\leq C_1\|g_1(w)\| \\ &\quad + \cdots + C_s\|g_s(w)\|; \end{aligned}$$

$$f = o(g) \iff \frac{\|f(w)\|}{\|g(w)\|} \to 0 \text{ as } w \to 0.$$

Also, for $z \in \mathbb{C}^p$ we set $O_z(g_1, \ldots, g_s) := O(\|z\|g_1, \ldots, \|z\|g_s)$. Under the hypotheses of Theorem 1.3, we can assume F in the form $(2.2)_{3h+1}$ with $P_2 = \cdots = P_h \equiv 0$ and $P_{h+1} = P_{h+1,0} \neq 0$, by Proposition 2.2. We split $w = (x, y) \in \mathbb{C} \times \mathbb{C}^{q-1}$ and set $P_i(x, y) = (p_i(x, y), q_i(x, y))$ and $\tilde{P}_{i,z}(x, y) = (\tilde{p}_{i,z}(x, y), \tilde{q}_{i,z}(x, y))$.

Up to a linear change of coordinates, we can assume that P_{h+1} has a nondegenerate characteristic direction V equal to $(1, 0)$. Then, making the blow-up $y = ux$ with $u \in \mathbb{C}^{q-1}$, we obtain

$$\begin{aligned}
x_1 &= x + p_{h+1}(1, u)x^{h+1} + \cdots + p_{3h}(1, u)x^{3h} + \tilde{p}_{3h+1,z}(1, u)x^{3h+1} + \cdots,\\
u_1 &= u + r(u)x^h + O(x^{h+1}) + O_z(x^{3h}),\\
z_1 &= G(z) + O(x^2, \|z\|x),
\end{aligned}$$

where $r(u) = q_{h+1}(1, u) - p_{h+1}(1, u)u$.

Since V is a nondegenerate characteristic direction, $r(0) = 0$ and $p_{h+1}(1, 0) \neq 0$. If we then replace $r(u)$ by its power series expansion at 0 and change the coordinate x into λx with $\lambda^h = -p_{h+1}(1, 0)h$, it follows that

$$\begin{aligned}
x_1 &= x - \tfrac{1}{h}x^{h+1} + O(x^{h+2}, \|u\|x^{h+1}) + O_z(x^{3h+1}),\\
u_1 &= (I - Ax^h)u + O(x^{h+1}, \|u\|^2x^h) + O_z(x^{3h}),\\
z_1 &= G(z) + O(x^2, \|z\|x),
\end{aligned}$$

where $A = r'(0)/(hp_{h+1}(1, 0))$ is just the matrix $A(V)$ associated to V.

Moreover, we can find a polynomial function $g(x)$ such that, changing x with $g(x)$, the map F has the following form (see [B, p. 122, Thm. 6.5.7]):

$$\begin{aligned}
x_1 &= x - \tfrac{1}{h}x^{h+1} + O(x^{2h+1}, \|u\|x^{h+1}) + O_z(x^{3h+1}),\\
u_1 &= (I - Ax^h)u + O(x^{h+1}, \|u\|^2x^h) + O_z(x^{3h}),\\
z_1 &= G(z) + O(x^2, \|z\|x).
\end{aligned} \tag{3.1}$$

PROPOSITION 3.1. *Let $V = (1, u_0)$ be a nondegenerate characteristic direction for P_{h+1}. Then the class of similarity of the associated matrix $A(V)$ is invariant under changes of coordinates; in particular, its eigenvalues are invariants.*

Proof. Let $(X, Y, Z) = \chi(x, y, z)$ be a local diffeomorphism that fixes the origin and preserves the characteristic direction of P_{h+1}, that is, maps $(1, u_0, z)$ into $(1, U_0, Z)$ for each $z \in \mathbb{C}^{m-q}$, where $(1, U_0)$ is another characteristic direction of P_{h+1}. Then the first q components of χ do not depend on z. Since P_{h+1} is a homogeneous polynomial in the first q variables $x, y^1, \ldots, y^{q-1}$, we can simply consider the restriction of χ and F to $\mathbb{C}^q$. In this case the linear part of F is the identity and $P_i \equiv 0$ for $i = 2, \ldots, h$, so $\tilde{F} = \chi^{-1} \circ F \circ \chi$ is still tangent to the identity at order $h + 1$ and $\tilde{P}_{h+1}$ is conjugated to P_{h+1} by the linear part of χ. Then, by our definition of associated matrix, it follows that $A(1, u_0)$ and $A(1, U_0)$ belong to the same class of similarity. $\square$

Later we will find domains $D^1, \dots, D^h$ such that, for each point $(x, u, z) \in D^i$, all iterates x_n are contained in disjoint simply connected domains that omit the origin. We can then apply the following argument to the restriction of F to D^i.

PROPOSITION 3.2. *Let F be a holomorphic transformation of type* (3.1) *and let $l_1, \dots, l_c$ be the eigenvalues of $A(V)$ such that $hl_j \in \mathbb{N}$ for each $j = 1, \dots, c$. Then, for each $k \in \mathbb{N}$, there exist an integer d_k and an analytic function $\tilde{u}_k(x, z)$ in $x, x^{hl_1}, \dots, x^{hl_c}$ with values in $\mathbb{C}^{q-1}$ such that, after the change of coordinates $u - \tilde{u}_k$, the transformation F takes the form*

$$\begin{aligned} x_1 &= f(x,u,z) = x - \tfrac{1}{h}x^{h+1} + O(x^{2h+1}\log x, \|u\|x^{h+1}) + O_z(x^{3h+1}),\\ u_1 &= \phi(x,u,z) = (I - Ax^h)u + O(\|u\|^2x^h, \|u\|x^{h+1}\log x)\\ &\qquad\qquad + O_z(\|u\|x^{3h}) + x^{h(k+1)}\varphi_k(x,z),\\ z_1 &= \nu(x,u,z) = G(z) + O(x^2, \|z\|x), \end{aligned} \tag{3.2}$$

where $\varphi_k(x, z) = O(x(\log x)^{d_k}, \|z\|)$ and the maps $f, \phi, \nu, x^{h(k+1)}\varphi_k(x, z)$ are analytic in the variables $x, x^{hl_1}\log x, \dots, x^{hl_c}\log x, u, z$.

Let $\alpha_1, \dots, \alpha_{q-1}$ be the eigenvalues of $A(V)$ and set $\lambda := \max_j\{\operatorname{Re}\alpha_j\}$. Assuming $k > 1 + \lambda$ and $k > 3$, we can take F in the form $(3.2)_k$. Let $D_r^+ = \{x \in \mathbb{C} \mid |x - r| < r\}$ and let $\Pi_r^i, \dots, \Pi_r^h$ be the h branches of $z^{1/h}$ in D_r^+. Then, for each $i = 1, \dots, h$ we shall prove the existence of a holomorphic local manifold M^i of dimension $m - q + 1$, with the origin on its boundary, by finding $r, \delta \in \mathbb{R}^+$ and a holomorphic function

$$\begin{aligned} u^i\colon \Pi_r^i \times \Delta_\delta^{m-q} &\to \mathbb{C}^{q-1}\\ (x,z) &\mapsto u^i(x,z), \end{aligned}$$

where $\Delta_\delta^{m-q} := \{z \in \mathbb{C}^{m-q} \mid \|z\| < \delta\}$, such that $u^i(x, z) \to 0$ as $(x, z) \to 0$.

Observe that $M^i = \{(x, u, z) \in \mathbb{C}^m \mid u = u^i(x, z)\}$ is invariant under F if and only if its points satisfy

$$u^i\big(f(x,u^i(x,z),z), \nu(x,u^i(x,z),z)\big) = \phi(x,u^i(x,z),z). \tag{3.3}$$

Let $v \in \mathbb{C}^{q-1}$ defined by

$$u = x^{hA}v = \exp(hA\log x)v,$$

and let $H(x, u, z)$ be the map deduced from ϕ by the equality

$$H(x,u,z) := x^{hA}(v - v_1) = u - x^{hA}x_1^{-hA}u_1.$$

By $(3.2)_k$ we have

$$\begin{aligned} u_1 - (I - Ax^h)u = O(&x^{h(k+1)+1}(\log x)^{d_k}, \|u\|^2x^h, \|u\|x^{h+1}\log x)\\ &+ O_z(x^{h(k+1)}, \|u\|x^{3h}); \end{aligned}$$

moreover,

$$x_1^{hA} = [I - Ax^h + O(x^{2h}\log x, \|u\|x^h) + O_z(x^{3h})]x^{hA}$$

also implies

$$
\begin{aligned}
u_1 - (I - Ax^h)u &\\
&= (I - Ax^h)x^{hA}(v_1 - v) + [O(x^{2h}\log x, \|u\|x^h) + O_z(x^{3h})]x^{hA}x_1^{-hA}u_1\\
&= (I - Ax^h)x^{hA}(v_1 - v)\\
&\quad + O(x^{h(k+3)+1}(\log x)^{d_k+1}, \|u\|^2x^h, \|u\|x^{2h}\log x)\\
&\quad + O_z(x^{h(k+3)}\log x, \|u\|x^{3h}).
\end{aligned}
$$

Then H is analytic in $x, x^{hl_1}\log x, \ldots, x^{hl_c}\log x, u, z$ and

$$
\begin{aligned}
H(x,u,z) = O(x^{h(k+1)+1}(\log x)^{d_k}, \|u\|^2x^h, \|u\|x^{h+1}\log x)&\\
+ O_z(x^{h(k+1)}, \|u\|x^{3h}).&
\end{aligned}
\tag{3.4}
$$

Moreover, the component ϕ of F can now be written as $v_1 = v - x^{-hA}H(x,u,z)$. Since the functional equation (3.3) means $u^i(x_1, z_1) = u_1^i(x, z)$, it is equivalent to $v^i(x_1, z_1) = v_1^i(x, z)$, that is,

$$
x^{-hA}u^i(x,z) - x_1^{-hA}u^i(x_1, z_1) = x^{-hA}H(x, u^i(x,z), z). \tag{3.5}
$$

LEMMA 3.1. *Let $u(x,z) = x^{2h}t(x,z)$ with $t\colon \Pi_r^i \times \Delta_\delta^{m-q} \to \mathbb{C}^{q-1}$ holomorphic and bounded. Let $\{(x_n, z_n)\}$ be the orbit of a point $(x,z) \in \Pi_r^i \times \Delta_\delta^{m-q}$ under the transformation F_u given by*

$$
\begin{aligned}
x_1 &= f_u(x,z) = f(x, u(x,z), z),\\
z_1 &= \nu_u(x,z) = \nu(x, u(x,z), z).
\end{aligned}
$$

If r, δ are small enough, then for each $(x,z) \in \Pi_r^i \times \Delta_\delta^{m-q}$ it follows that $(x_n, z_n) \in \Pi_r^i \times \Delta_\delta^{m-q}$ and that $x_n, \|z_n\|$ are $O(1/n^{1/h})$ for each $n \in \mathbb{N}$.

Proof. By hypothesis, F_u has the form

$$
\begin{aligned}
x_1 &= f(x, u(x,z), z) = x\big[1 - \tfrac{1}{h}x^h + ax^{2h}\log x + bx^{2h} + o(x^{2h})\big],\\
z_1 &= \nu(x, u(x,z), z) = G(z) + O(x^2, \|z\|x),
\end{aligned}
$$

where a and b are two constants. By elevating the first component of F_u to the power h, we obtain

$$
\frac{1}{x_1^h} = \frac{1}{x^h} + 1 + x^h(1 - b - a\log x) + o(x^h);
$$

so, for r and δ sufficiently small, there exists some $K > 0$ such that

$$
\left|\frac{1}{x_1^h} - \frac{1}{x^h} - 1\right| < K|x|^{h/2} < \frac{K}{R^{1/2}},
$$

where $R = 1/(2r)$.

Then, proceeding as in [H1, Prop. 3.1], we derive the assertion for semi-attractive transformations with the eigenvalue 1 having multiplicity 1. □

Now let $B^i_{k,d,r,\delta}$ be the Banach space formed by functions of type $u(x,z) = x^{hk-1}(\log x)^d t(x,z)$, with t holomorphic bounded from $\Pi^i_r \times \Delta^{m-q}_\delta$ to $\mathbb{C}^{q-1}$, endowed with the norm $\|u\|_B = \|t\|_\infty$. Given $\varepsilon > 0$, for $|x|$ small enough it is not difficult to show that

$$\|x^{-A}\| \leq |x|^{-(\lambda+\varepsilon)}; \tag{3.6}$$

then, for each element $u(x,z)$ in this space,

$$x^{-hA}H(x,u(x,z),z) = O(x^{h(k+1-\lambda-\varepsilon)}(\log x)^{d+1}).$$

Since for each $n \in \mathbb{N}$ we have that $z_n \in \Delta^{m-q}_\delta$ and that $x^h_n = O(1/n)$ uniformly on $\Pi^i_r \times \Delta^{m-q}_\delta$, and since $k > 1+\lambda$, the series

$$\sum_{n=0}^{\infty} x_n^{-hA} H(x_n, u(x_n,z_n), z_n)$$

is normally convergent. We can thus define

$$Tu(x,z) = x^{hA}\sum_{n=0}^{\infty} x_n^{-hA} H(x_n, u(x_n,z_n), z_n).$$

PROPOSITION 3.3. *T is an operator on $B^i_{k,d,r,\delta}$. Moreover, there exists a closed subset S^i_T of $B^i_{k,d,r,\delta}$ such that T restricted to S^i_T is a contraction.*

Therefore, T has a fixed point $\tilde{u}^i \in S^i_T$ and so $\tilde{u}^i$ satisfies equation (3.5). Hence $u = \tilde{u}^i(x,z)$ is the equation of M^i defined on $\Pi^i_r \times \Delta^{m-q}_\delta$. By (3.3), if we make the change $u - \tilde{u}^i(x,z)$ in the form $(3.2)_k$ of F, we find that

$$\begin{aligned}
x_1 &= x - \tfrac{1}{h}x^{h+1} + O(x^{2h+1}\log x, \|u\|x^{h+1}),\\
u_1 &= (I - Ax^h)u + O(\|u\|^2 x^h, \|u\|x^{h+1}\log x),\\
z_1 &= G(z) + O(x^2, \|z\|x),
\end{aligned}$$

and the equation of M^i becomes $u = 0$. Then, for each $(x,0,z) \in M^i$, by the same argument as used in Lemma 3.1 we see that x_n and $\|z_n\|$ are $O(1/n^{1/h})$. Therefore M^i is stable and tangent to $\mathbb{C}V \oplus E$ at 0.

Proof of Proposition 3.2

The following result is a generalization of Proposition 3.5 in [H2], and it can be proved with the same argument used by Hakim.

PROPOSITION 3.4. *Let (f^*, ϕ^*) be a holomorphic transformation of $\mathbb{C} \times \mathbb{C}^{q-1}$ of type*

$$\begin{aligned}
x_1 &= f^*(x,u) = x\big(1 - \tfrac{1}{h}x^h\big) + O(x^{2h+1}, \|u\|x^{h+1}),\\
u_1 &= \phi^*(x,u) = (I - Ax^h)u + O(\|u\|^2x^h, \|u\|x^{h+1}) + x^{h+1}\psi(x).
\end{aligned} \tag{3.8}$$

Let $\{l_1, \ldots, l_c\}$ be the eigenvalues of A such that $hl_i \in \mathbb{N}$ for $i = 1, \ldots, c$. Then, for each $n \in \mathbb{N}$, there are an integer b_n and a function $u_n(x)$ with values in $\mathbb{C}^{q-1}$

such that (a) $u_n(0) = 0$, (b) *its components are polynomials in* $x, x^{hl_1} \log x, \dots,$ $x^{hl_c} \log x$ *of total degree* n *in* x, *and* (c) *after the change of* u *in* $s = u - u_n$, *the transformation takes the expression*

$$x_1 = \hat{f}^*(x, s) = x\big(1 - \tfrac{1}{h}x^h\big) + O(x^{2h+1}\log x, \|s\|x^{h+1}),$$
$$s_1 = \hat{\phi}^*(x, s) = (I - Ax^h)s + O(\|s\|^2 x^h, \|s\|x^{h+1}\log x) + x^{h+1+n}\psi_n(x).$$

Here $\psi_n(x) = R_n^*(\log x) + o(x\log x)$, *with* $R_n^*(t)$ *a polynomial of degree* b_n, *and* $\hat{f}^*, \hat{\phi}^*, x^{h+n}\psi_n(x)$ *are analytic in* $x, x^{hl_1}\log x, \dots, x^{hl_c}\log x, s$.

The expression (3.1) of our map can be rewritten as

$$\begin{aligned} x_1 &= f^*(x, u) + O_z(x^{3h+1}),\\ u_1 &= \phi^*(x, u) + O_z(\|u\|x^{3h}) + x^{3h}\varphi(x, z),\\ z_1 &= G(z) + O(x^2, \|z\|x), \end{aligned}$$

where (f^*, ϕ^*) is of type (3.8) and $\varphi(x, 0) \equiv 0$. For each $n \in \mathbb{N}$, let $u_n(x)$ be the function given by Proposition 3.4 for the map (f^*, ϕ^*); then, if we change u with $s = u - u_n(x)$, by the same proposition we have

$$\begin{aligned} s_1 = (I - Ax^h)s &+ O(\|s\|^2 x^h, \|s\|x^{h+1}\log x) + x^{h+1+n}\psi_n(x)\\ &+ O_z(\|s\|x^{3h}) + x^{3h}\varphi(x, z). \end{aligned}$$

Hence, by choosing $n = 2h$ we can assume F in the form $(3.2)_2$, where $\varphi_2(x, z) = xR_{2h}^*(\log x) + o(x) + O(\|z\|)$.

Now let us consider the simpler case $q = 2$, where $A(V)$ is a complex number α.

LEMMA 3.2. *Let* F *be a self-map of* $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{m-2}$ *in the form*

$$\begin{aligned} x_1 &= f(x, u, z) = x\big(1 - \tfrac{1}{h}x^h\big) + O(x^{2h+1}\log x, ux^{h+1})O_z(x^{3h+1}),\\ u_1 &= \phi(x, u, z) = (1 - \alpha x^h)u + O(u^2x^h, ux^{h+1}\log x)\\ &\qquad\qquad + O_z(ux^{3h}) + x^{3h}\varphi_2(x, z),\\ z_1 &= \nu(x, u, z) = G(z) + O(x^2, \|z\|x), \end{aligned}$$

with $\varphi_2(x, z) = xR_2(\log x) + o(x) + O(\|z\|)$. *Then there exist sequences* $\{d_n\}$ *of integers and* $\{\hat{u}_n\}$ *of polynomials* $\hat{u}_n(x, z)$ *in* $x, x^{h\alpha}\log x$ *of degree* $3h + n$ *in* x *whose coefficients are holomorphic functions in* z *such that, for each* $n \in \mathbb{N}^*$, *we have* $\hat{u}_n(0, 0) = 0$ *and*

$$\begin{aligned} \phi(x, \hat{u}_{n-1}(x, z), z) = \hat{u}_{n-1}\big(&f(x, \hat{u}_{n-1}(x, z), z), \nu(x, \hat{u}_{n-1}(x, z), z)\big)\\ &+ x^{3h+n}\varphi_n(x, z), \end{aligned} \tag{3.9}$$

with $\varphi_n(x, z) = xR_n(\log x) + o(x) + O(\|z\|)$ *and where* $R_n(t)$ *is a polynomial of degree* d_n. *Moreover,* $\hat{u}_{-1} \equiv 0$ *and*

$$\hat{u}_n(x, z) = \hat{u}_{n-1}(x, z) + [x^{1-h}Q_n(\log x) + c_n(z)]x^{3h+n} \quad \textit{for } n \geq 0,$$

where $Q_n(t)$ is a polynomial and

$$c_n(z) = \sum_{i=0}^{\infty} \varphi_n(0, G^i(z)).$$

Proof. For $n = 0$ the claim is obvious, by hypothesis. So by induction we assume there exists a function $\hat{u}_{n-1}(x, z)$ such that the assertion is satisfied for $n \geq 1$. Define

$$\hat{u}_n(x, z) = \hat{u}_{n-1}(x, z) + [x^{1-h} Q_n(\log x) + c_n(z)] x^{3h+n}$$

with $c_n(0) = 0$ and let $p_n \geq d_n$ be the degree of Q_n. Since

$$\hat{u}_{n-1} = O(x^{2h+1} (\log x)^{p_1}, \|z\| x^{3h}),$$

we have

$$\begin{aligned}
&\hat{u}_n\big(f(x, \hat{u}_n(x, z), z), \nu(x, \hat{u}_n(x, z), z)\big) \\
&\quad = \hat{u}_{n-1}\big(f(x, \hat{u}_{n-1}(x, z), z), \nu(x, \hat{u}_{n-1}(x, z), z)\big) \\
&\quad\quad + Q_n(\log x) x^{2h+1+n} + c_n(G(z)) x^{3h+n} \\
&\quad\quad - \left[\left(2 + \frac{n+1}{h}\right) Q_n(\log x) + \frac{1}{h} Q_n'(\log x) + O(\|z\|)\right] x^{3h+n+1} \\
&\quad\quad + O(x^{3h+n+2} (\log x)^{p_n+1}).
\end{aligned}$$

Since $\hat{u}_{n-1}(x, z)$ satisfies equality (3.9) and $\varphi_n(x, z) = x R_n(\log x) + o(x) + O(\|z\|)$, it follows that

$$\begin{aligned}
&\phi(x, \hat{u}_n(x, z), z) - \hat{u}_n\big(f(x, \hat{u}_n(x, z), z), \nu(x, \hat{u}_n(x, z), z)\big) \\
&\qquad = [c_n(z) - c_n(G(z)) + \varphi_n(0, z)] x^{3h+n} + \varphi_{n+1}(x, z) x^{3h+n+1},
\end{aligned}$$

where

$$\begin{aligned}
\varphi_{n+1}(x, z) = {} & \left(2 - \alpha + \frac{n+1}{h}\right) Q_n(\log x) + \frac{1}{h} Q_n'(\log x) \\
& + R_n(\log x) + O(x (\log x)^{p_n+1}, \|z\|).
\end{aligned}$$

Then we want

$$\begin{gathered}
c_n(G(z)) - c_n(z) = \varphi_n(0, z) \\
\vdots \\
c_n(G^{i+1}(z)) - c_n(G^i(z)) = \varphi_n(0, G^i(z)).
\end{gathered}$$

Since G is a contraction and $\varphi_n(0, 0) = 0$ by induction, the solution is

$$c_n(z) = \sum_{i=0}^{\infty} \varphi_n(0, G^i(z))$$

because the series converges. Since $G(0) = 0$, this implies also that $c_n(0) = 0$.

Moreover, Q_n is the unique solution of the differential equation

$$\left(2 - \alpha + \frac{n+1}{h}\right) Q_n(t) + \frac{1}{h} Q_n'(t) = -R_n(t),$$

so $p_n = d_n$ if $2 - \alpha + \frac{n+1}{h} \neq 0$; otherwise, $p_n = d_n + 1$. Hence, by the inductive construction and equality (3.9), $\varphi_{n+1}(x, z) = xR_{n+1}(\log x) + o(x) + O(\|z\|)$. In particular, $\varphi_{n+1}(0, 0) = 0$ and $\varphi_{n+1}(0, z)$ is analytic.

By induction, $x^{3h+n}R_n(\log x)$ is a polynomial in x and $x^{h\alpha}\log x$; then also $\hat{u}_n$ is a polynomial in x and $x^{h\alpha}\log x$ and $x^{3h+n}\varphi_{n+1}(x, z)$ is analytic in x, $x^{h\alpha}\log x$, z. □

PROPOSITION 3.5. *Let F be in the form* $(3.2)_2$ *and let $l_1, \dots, l_c$ be the eigenvalues of A such that $hl_i \in \mathbb{N}$ for each $i = 1, \dots, c$. Then for each $n \in \mathbb{N}$ there exist an integer d_n and a function $\hat{u}_n(x, z)$ with values in $\mathbb{C}^{q-1}$ such that, after the change of coordinates $u - \hat{u}_n$, the transformation F takes the form*

$$\begin{aligned} x_1 &= f(x, u, z) = x - \tfrac{1}{h}x^{h+1} + O(x^{2h+1}\log x, \|u\|x^{h+1}) + O_z(x^{3h+1}), \\ u_1 &= \phi(x, u, z) = (I - Ax^h)u + O(\|u\|^2 x^h, \|u\|x^{h+1}\log x) \\ &\qquad\qquad + O_z(\|u\|x^{3h}) + x^{3h+n+1}\varphi_{n+1}(x, z), \\ z_1 &= \nu(x, u, z) = G(z) + O(x^2, \|z\|x), \end{aligned} \tag{3.10}$$

where $\varphi_n(x, z) = O(x(\log x)^{d_k}, \|z\|)$ and the maps $f, \phi, \nu, x^{3h+n}\varphi_n(x, z)$ are analytic in the variables $x, x^{hl_1}\log x, \dots, x^{hl_c}\log x, u, z$.

Proof. By Lemma 3.2, the assertion is true when $q = 2$. In the general case we again reason by induction on n. If the matrix $A = (a^i_j)$ is in triangular Jordan form, then the previous argument shows that the components $\hat{u}^j_n$ of $\hat{u}_n$ are determined for the j in decreasing order from $q - 1$ to 1. □

Finally, for each $k \in \mathbb{N}$, if we take $\tilde{u}_k(x, z) = \hat{u}_{h(k-2)-1}$ (where $\hat{u}_n(x, z)$ are the functions given in Proposition 3.5), then Proposition 3.2 is proved.

Proof of Proposition 3.3

To simplify the notation, we shall prove Proposition 3.3 for $h = 1$. The same argument can be used also when $h > 1$.

Let $u(x, z) = x^2 t(x, z)$ with $t\colon D^+_r \times \Delta^{m-q}_\delta \to \mathbb{C}^{q-1}$ holomorphic and bounded, and let $\{(x_n, z_n)\}$ be the orbit of (x, z) under the transformation

$$\begin{aligned} x_1 &= f_u(x, z) = f(x, u(x, z), z), \\ z_1 &= \nu_u(x, z) = \nu(x, u(x, z), z). \end{aligned}$$

By Lemma 3.1, there are positive constants r and δ such that, for each $(x, z) \in \bar{D}^+_r \times \Delta^{m-q}_\delta$, one has $(x_n, z_n) \in \bar{D}^+_r \times \Delta^{m-q}_\delta$ and $x_n = O(1/n)$ for each $n \in \mathbb{N}$. Moreover, there is an integer l such that

$$x_1 = x - x^2 + ax^3 + bx^3\log x + O(x^4(\log x)^l),$$

where the constants a and b do not depend on z. Hence

$$\frac{1}{x_1} = \frac{1}{x} + 1 + x(1 - a - b\log x) + O(x^2(\log x)^l).$$

Then, arguing as in [H2, Lemma 4.2 and Cor. 4.3], we find

$$|x_n| \leq 2\frac{|x|}{|1+nx|} \quad \forall n \in \mathbb{N} \tag{3.11}$$

and, for each real number $\mu > 1$ and for each integer l, there exists a constant $C_{\mu,l}$ such that

$$\sum_{n=0}^{\infty} |x_n|^{\mu} |\log x_n|^l \leq C_{\mu,q} |x|^{\mu-1} |\log|x||^l. \tag{3.12}$$

LEMMA 3.3. *There are positive constants r and δ such that, for every t satisfying $\|t\|_\infty \leq 1$, $\left\|x\frac{\partial t}{\partial x}\right\|_\infty \leq 1$, and $\left\|\frac{\partial t}{\partial z}\right\|_\infty \leq 1$ and for each $(x, z) \in \bar{D}_r^+ \times \Delta_\delta^{m-q}$,*

$$\left|\frac{\partial x_n}{\partial x}\right| \leq 2\frac{|x_n|^2}{|x|^2}, \qquad \left\|\frac{\partial x_n}{\partial z}\right\| \leq |x_n|^2,$$
$$\left\|\frac{\partial z_n}{\partial x}\right\| \leq 1, \qquad \left\|\frac{\partial z_n}{\partial z}\right\| \leq 1,$$

for all $n \in \mathbb{N}$.

Proof. Let $\mu = G'(0)$; since $G(0) = 0$, we get $z_1 = \mu z + O(x^2, \|t\|_\infty x^4, \|z\|x, \|z\|^2)$. By the hypotheses on t and $\|\mu\| < 1$, for r and δ sufficiently small there exist positive constants K_1, K_2, K_3, K_4 such that

$$\left\|\frac{\partial z_1}{\partial x}\right\| \leq K_1|x| + K_2\|z\| \leq 1,$$
$$\left\|\frac{\partial z_1}{\partial z}\right\| \leq \|\mu\| + K_3|x| + K_4\|z\| \leq 1.$$

Since

$$x_1 = f(x, u, z) = x - x^2 + ax^3 + bx^3 \log x + O(x^4(\log x)^l, \|u\|x^2) + O_z(x^4)$$

for a suitable l, it follows that

$$\frac{1}{x_1} + (1-a)\log x_1 - \frac{b}{2}(\log x_1)^2$$
$$= \frac{1}{x} + 1 + (1-a)\log x - \frac{b}{2}(\log x)^2 + \psi(x, u, z),$$

where

$$\psi(x, u, z) = O(x^2(\log x)^l, \|u\|, \|z\|x^2).$$

Setting $u = u(x, z)$ yields

$$\psi(x, u(x,z), z) = O(x^2(\log x)^l, x^2\|t\|_\infty, \|z\|x^2),$$

$$\left|\frac{\partial}{\partial x}\psi(x, u(x,z), z)\right| \leq C_1|x||\log|x||^l + C_2\left(\|t\|_\infty + \left\|x\frac{\partial t}{\partial x}\right\|_\infty\right)|x|, \tag{3.13}$$

$$\left\|\frac{\partial}{\partial z}\psi(x, u(x,z), z)\right\| \leq C_3\left(\left\|\frac{\partial t}{\partial z}\right\|_\infty + 1\right)|x|^2, \tag{3.14}$$

where C_1, C_2, C_3 are positive constants.

Differentiating with respect to x (resp., z) the relation

$$\frac{1}{x_n} + (1-a)\log x_n - \frac{b}{2}(\log x_n)^2 = \frac{1}{x} + n + (1-a)\log x - \frac{b}{2}(\log x)^2 + \sum_{j=0}^{n-1} \psi(x_j, u(x_j, z_j), z_j),$$

we have

$$\frac{1-(1-a)x_n + bx_n \log x_n}{x_n^2}\frac{\partial x_n}{\partial x} = \frac{1-(1-a)x + bx\log x}{x^2} - \sum_{j=0}^{n-1} \frac{\partial}{\partial x_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial x_j}{\partial x} - \sum_{j=0}^{n-1} \frac{\partial}{\partial z_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial z_j}{\partial x}$$

and (resp.)

$$\frac{1-(1-a)x_n + bx_n \log x_n}{x_n^2}\frac{\partial x_n}{\partial z} = -\sum_{j=0}^{n-1} \frac{\partial}{\partial x_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial x_j}{\partial z} - \sum_{j=0}^{n-1} \frac{\partial}{\partial z_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial z_j}{\partial z}.$$

By (3.11), $|x_n| \le 2|x|$. Moreover, $\arg x_n \to 0$ and then, for $|x|$ small, it follows that

$$\left|\frac{\partial x_1}{\partial x}\right| \le \frac{|1-(1-a)x + bx\log x| + C_1|x|^3|\log|x||^l + 2C_2|x|^3}{|1-(1-a)x_1 + bx_1 \log x_1|}\frac{|x_1|^2}{|x|^2} \le 2\frac{|x_1|^2}{|x|^2},$$

$$\left\|\frac{\partial x_1}{\partial z}\right\| \le \frac{2C_3|x|^2}{|1-(1-a)x_1 + bx_1\log x_1|}|x_1|^2 \le |x_1|^2.$$

We argue by induction on n. Suppose that, for each $j < n$, all inequalities of the assertion are satisfied; by (3.13) and (3.14) and then applying (3.12), we obtain

$$\sum_{j=0}^{n-1}\left|\frac{\partial}{\partial x_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial x_j}{\partial x}\right| \le \tilde{C}_1|\log|x||^l + \tilde{C}_2,$$

$$\sum_{j=0}^{n-1}\left\|\frac{\partial}{\partial z_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial z_j}{\partial x}\right\| \le \tilde{C}_3|x|,$$

$$\sum_{j=0}^{n-1}\left\|\frac{\partial}{\partial x_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial x_j}{\partial z}\right\| \le C_1'|x|^2|\log|x||^l + C_2'|x|^2,$$

$$\sum_{j=0}^{n-1}\left\|\frac{\partial}{\partial z_j}[\psi(x_j, u(x_j, z_j), z_j)]\frac{\partial z_j}{\partial z}\right\| \le C_3'|x|.$$

Hence, for $|x|$ small we have

$$\left|\frac{\partial x_n}{\partial x}\right| \le \frac{|1-(1-a)x+bx\log x| + \tilde{C}_1|x|^2|\log|x||^l + \tilde{C}_2|x|^2 + \tilde{C}_3|x|^3}{|1-(1-a)x_n+bx_n\log x_n|}\frac{|x_n|^2}{|x|^2}$$
$$\le 2\frac{|x_n|^2}{|x|^2},$$

$$\left\|\frac{\partial x_n}{\partial z}\right\| \le \frac{C_1'|x|^2|\log|x||^l + C_2'|x|^2 + C_3'|x|}{|1-(1-a)x_n+bx_n\log x_n|}|x_n|^2 \le |x_n|^2.$$

Finally, by the inductive hypothesis and (3.11) (which implies $|x_j|/|x| \le 2$),

$$\left\|\frac{\partial z_j}{\partial x}\right\| \le \|\mu\| + \tilde{K}_1|x_{j-1}| + \tilde{K}_2\|z_{j-1}\| \le 1$$

and

$$\left\|\frac{\partial z_j}{\partial z}\right\| \le \|\mu\| + \tilde{K}_3|x_{j-1}| + \tilde{K}_4\|z_{j-1}\| \le 1.$$ □

LEMMA 3.4. *Let T be the operator on the Banach space $B_{k,d,r,\delta}$ defined previously. Given a constant $R_0 \ge 0$, we can find r, δ small enough such that, if*

$$\|u(x,z)\| \le R_0|x|^{k-1}|\log|x||^d \quad \forall (x,z) \in D_r^+ \times \Delta_\delta^{m-q}, \tag{3.15}$$

then $\|Tu(x,z)\|$ satisfies the same inequality in $D_r^+ \times \Delta_\delta^{m-q}$.

Proof. Recall that

$$Tu(x,z) = \sum_{n=0}^{\infty}\left(\frac{x_n}{x}\right)^{-A} H(x_n, u(x_n,z_n), z_n).$$

By (3.4), for each R_0 the hypothesis implies

$$H(x,u(x,n),z) = O(|x|^{k+1}|\log|x||^{d+1}, \|z\||x|^{k+1}).$$

Since $z_n \in \Delta_\delta^{m-q}$ for each n, for r, δ small enough it follows that there exists a positive constant C_1 such that, in $D^+ \times \Delta_\delta^{m-q}$,

$$\|H(x_n, u(x_n,z_n), z_n)\| \le C_1|x_n|^{k+1}|\log|x_n||^{d+1}.$$

Since $k > \max\{3, 1+\lambda\}$, by (3.6) and (3.12) we obtain

$$\|Tu(x,z)\| \le C_1\sum_{n=0}^{\infty}\left|\frac{x_n}{x}\right|^{-(\lambda+\varepsilon)}|x_n|^{k+1}|\log|x_n||^{d+1} \le C_1'|x|^k|\log|x||^{d+1}.$$

Hence, when $|x|$ is small enough, $\|Tu(x,z)\|$ satisfies the inequality. □

Lemma 3.4 tells us that, if $u(x,z) \in B_{k,d,r,\delta}$ with $\|u\|_B \le R_0$, then also $Tu(x,z)$ belongs to the same Banach space and its norm is bounded by R_0. Hence T is an operator on the Banach space $B_{k,d,r,\delta}$ and

$$\|T\| = \sup_{\|u\|_B \le 1}\|Tu\|_B \le 1.$$

LEMMA 3.5. *Let T be the operator previously defined, and assume the hypotheses of Lemma 3.3 are satisfied. Given two positive constants R_1 and R_2, we can find r and δ such that, if $u(x,z)$ satisfies inequality* (3.15) *and if*

$$\left\| \frac{\partial u}{\partial x}(x,z) \right\| \leq R_1 |x|^{k-2} |\log|x||^d, \tag{3.16}$$

$$\left\| \frac{\partial u}{\partial z}(x,z) \right\| \leq R_2 |x|^{k-1} |\log|x||^d \tag{3.17}$$

for every $(x,z) \in D_r^+ \times \Delta_\delta^{m-q}$, then also the partial derivatives of $Tu(x,z)$ satisfy (3.16) *and* (3.17).

Proof. Inequalities (3.15)–(3.17) imply that

$$\|H(x_n, u(x_n,z_n), z_n)\| \leq C_1 |x_n|^{k+1} |\log|x_n||^{d+1},$$

$$\left\| \frac{\partial}{\partial x} H(x, u(x,z), z) \right\| \leq C_2 |x|^k |\log|x||^{d+1},$$

$$\left\| \frac{\partial}{\partial u} H(x, u(x,z), z) \right\| \leq C_3 |x|^2 |\log|x||,$$

$$\left\| \frac{\partial}{\partial z} H(x, u(x,z), z) \right\| \leq C_4 |x|^{k+1} |\log|x||^{d+1}$$

for some positive constants.

Differentiating with respect to x the formula

$$Tu(x,z) = x^A \sum_{n=0}^{\infty} x_n^{-A} H(x_n, u(x_n,z_n), z_n),$$

we have

$$\frac{\partial}{\partial x} Tu(x,z) = s_1 + s_2 + s_3 + s_4,$$

where

$$s_1 = \frac{dx^A}{dx} \sum_{n=0}^{\infty} x_n^{-A} H(x_n, u(x_n,z_n), z_n),$$

$$s_2 = x^A \sum_{n=0}^{\infty} x_n^{-A} \left[-A x_n^{-1} H(x_n, u(x_n,z_n), z_n) + \frac{\partial}{\partial x_n} H(x_n, u(x_n,z_n), z_n) \right] \frac{\partial x_n}{\partial x},$$

$$s_3 = x^A \sum_{n=0}^{\infty} x_n^{-A} \frac{\partial}{\partial u} H(x_n, u(x_n,z_n), z_n) \left[\frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial x} + \frac{\partial u}{\partial z_n} \frac{\partial z_n}{\partial x} \right],$$

$$s_4 = x^A \sum_{n=0}^{\infty} x_n^{-A} \frac{\partial}{\partial z_n} H(x_n, u(x_n,z_n), z_n) \frac{\partial z_n}{\partial x}.$$

Since $\frac{dx^A}{dx} = Ax^{-1}x^A$, the proof of the previous lemma shows that there exists a constant K_1 such that

$$\|s_1\| \leq K_1|x|^{k-1}|\log|x||^{d+1}.$$

Lemma 3.3 and (3.12) imply that

$$\begin{aligned}\|s_2\| &\leq 2\sum_{n=0}^{\infty}(\|A\|C_1 + C_2)\left|\frac{x_n}{x}\right|^{-(\lambda+\varepsilon)}\frac{|x_n|^{k+2}}{|x|^2}|\log|x_n||^{d+1}\\ &\leq K_2|x|^{k-1}|\log|x||^{d+1},\end{aligned}$$

where C_1 is the constant given in Lemma 3.4.

Similarly, by inequalities (3.16) and (3.17), we find

$$\|s_3\| \leq K_3|x|^{k-1}|\log|x||^{d+1},$$

$$\|s_4\| \leq K_4|x|^{k}|\log|x||^{d+1}.$$

Because s_1, s_2, s_3, and s_4 are of order higher than $|x|^{k-2}|\log|x||^d$, it follows that $\frac{\partial}{\partial x}Tu(x,z)$ satisfies inequality (3.16) when r, δ are small enough.

Similarly, differentiating with respect to z the formula of $Tu(x,z)$, we have

$$\frac{\partial}{\partial z}Tu(x,z) = \tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3,$$

where

$$\tilde{s}_1 = x^A\sum_{n=0}^{\infty}x_n^{-A}\left[-Ax_n^{-1}H(x_n, u(x_n, z_n), z_n) + \frac{\partial}{\partial x_n}H(x_n, u(x_n, z_n), z_n)\right]\frac{\partial x_n}{\partial z},$$

$$\tilde{s}_2 = x^A\sum_{n=0}^{\infty}x_n^{-A}\frac{\partial}{\partial u}H(x_n, u(x_n, z_n), z_n)\left[\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial z} + \frac{\partial u}{\partial z_n}\frac{\partial z_n}{\partial z}\right],$$

$$\tilde{s}_3 = x^A\sum_{n=0}^{\infty}x_n^{-A}\frac{\partial}{\partial z_n}H(x_n, u(x_n, z_n), z_n)\frac{\partial z_n}{\partial z}.$$

Then

$$\|\tilde{s}_1\| \leq \tilde{K}_1|x|^{k+1}|\log|x||^{d+1},$$

$$\|\tilde{s}_2\| \leq \tilde{K}_2|x|^{k}|\log|x||^{d+1},$$

$$\|\tilde{s}_3\| \leq \tilde{K}_3|x|^{k}|\log|x||^{d+1}.$$

Hence $\frac{\partial}{\partial z}Tu(x,z)$ satisfies inequality (3.17) when r, δ are small enough. □

Let R_0, R_1, R_2 be positive constants. Let $S_T(r, \delta, R_0, R_1, R_2)$ be the closed subset of $B_{k,d,r,\delta}$ formed by functions $u(x,z)$ such that, for each $(x,z) \in D_r^+ \times \Delta_\delta^{m-q}$, the inequalities (3.15)–(3.17) are satisfied. Then, by the preceding lemmas, the operator T sends $S_T(r, \delta, R_0, R_1, R_2)$ into itself.

LEMMA 3.6. *Let*

$$u_1(x,z) = x^{k-1}(\log x)^d t_1(x,z) \quad \textit{and} \quad u_2(x,z) = x^{k-1}(\log x)^d t_2(x,z)$$

be two functions in $S_T(r, \delta, R_0, R_1, R_2)$. *Let* $\{(x_n, z_n)\}$ *and* $\{(x'_n, z'_n)\}$ *be the iterates of* (x, z) *by* (f_{u_1}, ν_{u_1}) *and* (f_{u_2}, ν_{u_2}), *respectively. Then there exist positive constants* K_1, K_2 *such that, for each* $n \in \mathbb{N}$,

$$|x'_n - x_n| \le K_1 |x_n||x|^{k-1} |\log|x||^d \|t_2 - t_1\|_\infty,$$
$$\|z'_n - z_n\| \le K_2 |x|^{k-1} |\log|x||^d \|t_2 - t_1\|_\infty.$$

Proof. If r, δ are small enough, then for each $(x, z) \in D_r^+ \times \Delta_\delta^{m-q}$ there exists a $C_1 > 0$ such that

$$|x'_1 - x_1| = |f(x, u_2(x, z), z) - f(x, u_1(x, z), z)| \le C_1 |x_1||x|^k |\log|x||^d \|t_2 - t_1\|_\infty.$$

Suppose, by induction, that there exists some $K > C_1$ such that

$$|x'_{n-1} - x_{n-1}| \le K |x_{n-1}| \sum_{i=0}^{n-2} |x_i|^k |\log|x_i||^d \|t_2 - t_1\|_\infty.$$

Since $x'_n = x_n + o(x_n)$ and $z_n, z'_n \in \Delta_\delta^{m-q}$, we obtain

$$\begin{aligned}
|x'_n - x_n| &\le |x'_{n-1} - x_{n-1}||1 - x'_{n-1} - x_{n-1} + o(x_{n-1})| \\
&\quad + K|x_n||x_{n-1}|^k |\log|x_{n-1}||^d \|t_2 - t_1\|_\infty \\
&\le K\biggl(|x_{n-1}||1 - x'_{n-1} - x_{n-1} + o(x_{n-1})| \sum_{i=0}^{n-2} |x_i|^k |\log|x_i||^d \\
&\quad + |x_n||x_{n-1}|^k |\log|x_{n-1}||^d\biggr) \|t_2 - t_1\|_\infty \\
&\le K|x_n| \sum_{i=0}^{n-1} |x_i|^k |\log|x_i||^d \|t_2 - t_1\|_\infty.
\end{aligned}$$

Then, by (3.12),

$$|x'_n - x_n| \le K_1 |x_n||x|^{k-1} |\log|x||^d \|t_2 - t_1\|_\infty.$$

A similar argument can be used for ν:

$$\|z'_1 - z_1\| = \|\nu(x, u_2(x, z), z) - \nu(x, u_1(x, z), z)\| \le C_2 |x|^k |\log|x||^d \|t_2 - t_1\|_\infty;$$

since $\|G(z') - G(z)\| \le \gamma \|z' - z\|$ with $0 < \gamma < 1$ and since x_n, x'_n are $O(1/n)$ by the estimate on $|x'_n - x_n|$, we have

$$\begin{aligned}
&\|z'_n - z_n\| \\
&\quad \le \|z'_{n-1} - z_{n-1}\|(\gamma + O(|x_{n-1}|)) + C_3 |x'_{n-1} - x_{n-1}||x_{n-1}| \\
&\qquad + C_4 |x_{n-1}|^{k+1} |\log|x_{n-1}||^d \|t_2 - t_1\|_\infty \\
&\quad \le \|z'_{n-1} - z_{n-1}\| + C_3 K_1 |x_{n-1}|^2 |x|^{k-1} |\log|x||^d \|t_2 - t_1\|_\infty \\
&\qquad + C_4 |x_{n-1}|^{k+1} |\log|x_{n-1}||^d \|t_2 - t_1\|_\infty \\
&\quad \le \biggl(C_3 K_1 |x|^{k-1} |\log|x||^d \sum_{i=0}^{n-1} |x_i|^2 + C_4 \sum_{j=0}^{n-1} |x_j|^k |\log|x_j||^d\biggr) \|t_2 - t_1\|_\infty \\
&\quad \le K_2 |x|^{k-1} |\log|x||^d \|t_2 - t_1\|_\infty.
\end{aligned}$$

□

Finally, the previous inequalities allow us to argue as Hakim (see [H2, Prop. 4.8]) to prove that, for r, δ small enough, the restriction to the metric space $S_T(r, \delta, R_0, R_1, R_2)$ of the operator T is a contraction. This establishes Proposition 3.3.

4. Existence of Attracting Domains

Given a semi-attractive self-map F of $\mathbb{C}^m$ such that the eigenvalue 1 of $dF(0)$ has the same algebraic and geometric multiplicities, we have seen that F can assume the form $(2.2)_k$ for $k \geq 2$. Then Theorem 1.4 is an immediate consequence of Proposition 3.1 and of the following proposition.

PROPOSITION 4.1. *Let F be a holomorphic germ of semi-attractive self-maps of $\mathbb{C}^m$ such that $F(0) = 0$, and assume the eigenvalue 1 of $dF(0)$ has algebraic and geometric multiplicity $q > 1$. Choose a local coordinate system $(w, z) \in \mathbb{C}^q \times \mathbb{C}^{m-q}$ such that F takes the form $(2.2)_{3h+1}$ in a neighborhood of 0, with $P_2 = \cdots = P_h \equiv 0$ and $P_{h+1} \neq 0$. Let $V \in \mathbb{C}^q$ be a nondegenerate characteristic direction for P_{h+1}. If the matrix $A = A(V)$ associated to V has all eigenvalues with strictly positive real part, then there exist h attracting domains $D^1, \dots, D^h$ for F such that $0 \in \partial D^i$ and each point in D^i has the first q components of its orbit converging tangentially to V.*

Proof. Under a linear change of coordinates, we can assume that $V = (1, 0) \in \mathbb{C} \times \mathbb{C}^{q-1}$. We have seen that for each $i = 1, \dots, h$ there is a local system of coordinates, analytic in a sector

$$\{(x, y, z) \in \mathbb{C} \times \mathbb{C}^{q-1} \times \mathbb{C}^{m-q} \mid x \in \Pi_r^i,\ \|y\| \leq c|x|,\ \|z\| < \delta\},$$

such that, after the blow-up $y = ux$, the transformation takes the form (3.7). Let $\{\alpha_1, \dots, \alpha_{q-1}\}$ be the eigenvalues of A, and let λ be a positive constant such that $\operatorname{Re} \alpha_j > \lambda$ for each $j = 1, \dots, q-1$.

For $j = 1, \dots, q-1$, let $D^+_{1/|\alpha_j|} = \{x \in \mathbb{C} \mid |1 - \alpha_j x| < 1\}$. Since the real part of every eigenvalue α_j is positive, there exist two constants η, ρ such that the sector

$$S_{\eta,\rho} := \{x \in \mathbb{C} \mid |\operatorname{Im} x| \leq \eta \operatorname{Re} x,\ |x| \leq \rho\}$$

is contained in the intersection of $\bar{D}^+_r$ and the $\bar{D}^+_{1/|\alpha_j|}$. Then, for each $i = 1, \dots, h$, the sets $\Pi^i_{\eta,\rho} = \{x \in \Pi^i_r \mid x^h \in S_{\eta,\rho}\}$ are disjoint nonempty domains with the origin in their boundary.

Choose a system of coordinate such that A is in an almost diagonal Jordan form, that is, the elements of A that are above the diagonal are equal to 0 or ε_1, with $\varepsilon_1 > 0$ small compared to λ. Then there exists some $\varepsilon > 0$ such that

$$\|I - Ax^h\| \leq 1 - (\lambda + \varepsilon)|x|^h.$$

Then, if η, ρ, δ, c are small enough, for $(x, u, z) \in \Pi^i_{\eta,\rho} \times \Delta_c^{q-1} \times \Delta_\delta^{m-q}$ we see that

$$\|u_1\| \leq \|u\|(1 - \lambda|x|^h). \tag{4.1}$$

Hence $\|u\|$ is uniformly bounded. We have

$$\frac{1}{x_1^h} = \frac{1}{x^h} + 1 + O(x^h \log x, \|u\|)$$

and so, for ρ, c, δ sufficiently small, there exists a $K > 0$ such that

$$\left| \frac{1}{x_1^h} - \frac{1}{x^h} - 1 \right| < \tilde{c} + K|x|^h |\log x| \le \frac{1}{2}.$$

We then proceed as in Lemma 3.1 to prove that x_n and $\|z_n\|$ are $O(1/n^{1/h})$. Since $\|u\|$ is uniformly bounded, we also get $y_n = u_n x_n \to 0$.

Let μ be a positive number such that $\mu < \lambda$; then

$$x_1^{-h\mu} = x^{-h\mu}(1 + \mu x^h + O(x^{2h} \log x, \|u\| x^h))$$

and, for ρ, c, δ small enough,

$$|x_1|^{-h\mu} \le |x|^{-h\mu}(1 + \lambda |x|^h).$$

Thus, by inequality (4.1),

$$\|u_1\| |x_1|^{-h\mu} \le \|u\| |x|^{-h\mu}(1 - \lambda^2 |x|^{2h}) < \|u\| |x|^{-h\mu}.$$

Hence there exists a constant C such that $\|u_n\| \le C|x_n|^{h\mu}$. In particular, $u_n \to 0$ faster than $1/n^{\lambda}$ for each positive number λ such that $\operatorname{Re} \alpha_j > \lambda$ ($j = 1, \dots, q-1$), and (x_n, y_n) converges to 0 tangentially to V with $y_n = o(1/n^{\lambda+1})$. $\square$

5. Iterated Blow-up of $\mathbb{C}^m$ along Submanifolds

We shall refer to the notion given in [A] of *blow-up* of a complex m-manifold M along a closed complex submanifold $X \subset M$ and, for increased clarity, we shall use the same notation.

Let $N_{X/M}$ be the normal bundle of X in M, and let $E_X = \mathbb{P}(N_{X/M})$ be the projective normal bundle whose fiber over $p \in X$ is $E_p = \mathbb{P}(T_p M / T_p X)$. Then the blow-up of M along X is the set $\tilde{M}_X = (M \setminus X) \cup E_X$ endowed with the complex structure that we shall describe, together with the projection $\sigma \colon \tilde{M}_X \to M$ defined by $\sigma|_{M \setminus X} = \mathrm{id}_{M \setminus X}$ and $\sigma|_{E_p} \equiv \{p\}$ for each $p \in X$.

Given $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ and a splitting $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$ of $\{1, \dots, m\}$ of weight $0 \le r < m$ (i.e., $\mathcal{P}' = \{i_1, \dots, i_r\}$ and $\mathcal{P}'' = \{i_{r+1}, \dots, i_m\}$, where $i_1 < \cdots < i_r$ and $i_{r+1} < \cdots < i_m$), we shall write $z' = (z_{i_1}, \dots, z_{i_r})$ and $z'' = (z_{i_{r+1}}, \dots, z_{i_m})$.

A chart $\phi = (z_1, \dots, z_m) \colon V \to \mathbb{C}^m$ is said to be *adapted* to X if there is a splitting $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$ of $\{1, \dots, m\}$ of weight $r = \dim X$ such that $V \cap X = \{z'' = 0\}$. Choose a chart (V, ϕ) adapted to X and, for $j \in \mathcal{P}''$ and $p \in V \cap X$, set $X_j = \{z_j = 0\} \subset V$, $L_{j,p} = \mathbb{P}(\operatorname{Ker}(dz_j(p))/T_p X) \subset E_p$, $L_j = \bigcup_{p \in V \cap X} L_{j,p}$, $E_{V \cap X} = \sigma^{-1}(V \cap X)$, and $V_j = (V \setminus X_j) \cup (E_{V \cap X} \setminus L_j)$.

Define $\chi_j\colon V_j \to \mathbb{C}^m$ by

$$\chi_j(p)_h = \left\{\begin{array}{ll} z_h(p) & \text{if } h \in \mathcal{P}' \cup \{j\} \\ z_h(p)/z_j(p) & \text{if } h \in \mathcal{P}''\backslash\{j\} \end{array}\right\} \quad \text{if } p \in V\backslash X_j$$

and by

$$\chi_j([v])_h = \left\{\begin{array}{ll} z_h(\sigma([v])) & \text{if } h \in \mathcal{P}' \\ dz_h(\sigma([v]))(v)/dz_j(\sigma([v]))(v) & \text{if } h \in \mathcal{P}''\backslash\{j\} \\ 0 & \text{if } h = j \end{array}\right\}$$
$$\text{if } [v] \in E_{V\cap X}\backslash L_j.$$

The family (V_j, χ_j) together with an atlas of $M\backslash X$ determines an m-dimensional complex structure on $\tilde{M}_X$ such that the projection σ is holomorphic.

Moreover,

$$\phi \circ \sigma \circ \chi_j^{-1}(w)_h = \begin{cases} w_h & \text{if } h \in \mathcal{P}' \cup \{j\}, \\ w_j w_h & \text{if } h \in \mathcal{P}''\backslash\{j\}. \end{cases} \tag{5.1}$$

The set $E_X = \sigma^{-1}(X)$ is called the *exceptional divisor* of the blow-up. If $Y \subseteq M$ is a submanifold of M, then $\tilde{Y} = \overline{\sigma^{-1}(Y\backslash X)} \subset \tilde{M}$ is called the *proper transform* of Y.

Let $\mathrm{End}(M, X)$ be the set of germs at X of holomorphic self-maps of M such that $F(X) \subseteq X$, and let $F \in \mathrm{End}(M, X)$. Take $p \in X$ and choose charts (V, ϕ) and $(\tilde{V}, \tilde{\phi})$ adapted to X so that $p \in V$ and $F(p) \in \tilde{V}$. Then, setting $H = \tilde{\phi} \circ F \circ \phi$, in a neighborhood of p we can write the homogeneous expansion of H'' as

$$H''(z) = \sum_{l \geq 1} P_{l,z'}(z'').$$

Let

$$\nu_X(F, p) = \min\{l \mid P_{l,\phi(p)'} \neq 0\} \geq 1$$

be the *order of F at p*, and let

$$\nu_X(F) = \min\{\nu_X(F, p) \mid p \in X\}$$

be the *order of F along X*. Then F is said *nondegenerate along X* if $F^{-1}(X) \subseteq X$ and if, for each $p \in X$, we have: $\nu_X(F, p) = \nu_X(F)$ and $P_{\nu_X(F),\phi(p)'}(v) = 0$ iff $v = 0 \in \mathbb{C}^{m-r}$.

PROPOSITION 5.1 [A]. *Let M be a complex manifold of dimension m, and let $X \subset M$ be a closed submanifold of dimension $r \geq 0$. Let $F \in \mathrm{End}(M, X)$ be nondegenerate along X. Then there exists a unique $\tilde{F} \in \mathrm{End}(\tilde{M}_X, E_X)$ such that $F \circ \sigma = \sigma \circ \tilde{F}$. Furthermore, if $p \in X$ and (V, ϕ), $(\tilde{V}, \tilde{\phi})$ are charts adapted to X with $p \in V$ and $F(p) \in \tilde{V}$, then for all $[v] \in E_p$ we have*

$$\tilde{F}([v]) = (i_{F(p),\tilde{\phi}})^{-1}\big(\big[P_{\nu_X(F),\phi(p)'}(i_{p,\phi}([v]))\big]\big),$$

where $i_{p,\phi}\colon E_p \to \mathbb{P}^{m-r-1}(\mathbb{C})$ is the canonical isomorphism defined by the chart ϕ.

Arguing as in [A], it is possible to prove the following results.

PROPOSITION 5.2. *Let M be a complex manifold of dimension m, and let $X \subset M$ be a closed submanifold of dimension $r \geq 0$. Let $F \in \mathrm{End}(M, X)$ be nondegenerate along X with $\tilde{F} \in \mathrm{End}(\tilde{M}_X, E_X)$ its lifting. Let $Y \subseteq M$ be a submanifold of M of dimension $r + s$ (with $s \geq 1$) and let $\tilde{Y} \subseteq \tilde{M}$ be its proper transform. Assume that*:

(i) *Y properly contains X*;
(ii) *$F(Y) \subseteq Y$ and $F^{-1}(Y) \subseteq Y$*; *and*
(iii) *there exists a local system of coordinates such that*

$$dF(p) = \mathrm{diag}\{J_1(p), J_2(p)\},$$

where $J_1(p) = d(F|_X)(p)$ and $J_2(p) \in M_{m-r,m-r}(\mathbb{C})$ is invertible for all $p \in Y$.

Then $\tilde{F}$ is nondegenerate along $\tilde{Y}$ and $d\tilde{F}(\tilde{p}) = \mathrm{diag}\{\tilde{J}_1(\tilde{p}), \tilde{J}_2(\tilde{p})\}$ with $\tilde{J}_2(\tilde{p})$ invertible for all $\tilde{p} \in \tilde{Y}$.

PROPOSITION 5.3. *Let M be a complex manifold of dimension m, and let $X \subset M$ be a closed submanifold of dimension $r \geq 0$. Let $F \in \mathrm{End}(M, X)$ be nondegenerate along X, with $\tilde{F} \in \mathrm{End}(\tilde{M}_X, E_X)$ its lifting. Take $p \in X$ and a linear subspace $L \subseteq E_p$ of dimension $s - 1$ (with $s \geq 1$). Assume that*:

(i) *$\tilde{F}(L) \subseteq L$*; *and*
(ii) *there exists a local system of coordinates such that*

$$dF(p) = \mathrm{diag}\{J_1(p), J_2(p)\},$$

where $J_1(p) = d(F|_X)(p)$ and $J_2(p) \in M_{m-r,m-r}(\mathbb{C})$ is invertible.

Then $\tilde{F}$ is nondegenerate along L and $d\tilde{F}([v]) = \mathrm{diag}\{\tilde{J}_1([v]), \tilde{J}_2([v])\}$ with $\tilde{J}_2([v])$ invertible for each $[v] \in L$.

Now we describe a precise sequence of blow-ups of $\mathbb{C}^m$ starting from the blow-up along a complex submanifold X containing the origin. Given $\rho \geq 1$, a *ρ-partition* of n is a set $\mathcal{M} = \{\mu_1, \dots, \mu_\rho\} \subset \mathbb{N}$ with $\mu_1 \geq \cdots \geq \mu_\rho \geq 1$ and $\mu_1 + \cdots + \mu_\rho = n$. The *length* of $\mathcal{M}$ is $l(\mathcal{M}) = \mu_1$ if $\mu_1 > \mu_2$ or $l(\mathcal{M}) = \mu_1 + 1$ if $\mu_1 = \mu_2$.

Let $r = \dim X$. Given a ρ-partition of $m - r$, set $\nu_1 = r$ and $\nu_j = \nu_{j-1} + \mu_{j-1}$ for $j = 2, \dots, \rho$. For each $1 \leq l \leq \rho$ and each $0 \leq k \leq \mu_1 - 1$, define also the sets

$$\mathcal{P}'_{kl} = \begin{cases} \emptyset & \text{if } k = 0, \\ \{\nu_l + 1, \dots, \nu_l + \min\{k, \mu_l\}\} & \text{if } 1 \leq k \leq \mu_1 - 1. \end{cases}$$

In addition, for $1 \leq l \leq \rho$ define

$$\mathcal{P}'_{\mu_1 l} = \begin{cases} \{\nu_l + 1, \dots, \nu_l + \mu_l\} & \text{if } l \neq 2 \text{ or } \mu_1 \neq \mu_2, \\ \{\nu_2 + 1, \dots, \nu_2 + \mu_2 - 1\} & \text{if } l = 2 \text{ and } \mu_1 = \mu_2 \end{cases}$$

and define $\mathcal{P}'_{\mu_1+1,1} = \{\nu_1 + 1, \dots, \nu_1 + \mu_1, \nu_2 + \mu_2\}$. Finally, let $\mathcal{P}'_k = \bigcup_{l=1}^{\rho} \mathcal{P}'_{kl}$ and $\mathcal{P}''_k = \{r + 1, \dots, m\} \backslash \mathcal{P}'_k$.

We now set $M^0 = \mathbb{C}^m$, $X^0 = X$, and $\mathbf{e}_0 = 0 \in X$ and let φ_0 be a local chart centered at 0 and adapted to X with respect to the standard splitting $\mathcal{P}' = \{1, \ldots, r\}$, $\mathcal{P}'' = \{r+1, \ldots, m\}$. Starting from the blow-up of M^0 along X^0, we obtain $M^1 = \tilde{M}^0_{X^0}$ and $\pi_1 = \sigma_1 \colon M^1 \to M^0$. Let $\{\partial/\partial w_1, \ldots, \partial/\partial w_m\}$ be the canonical basis of $T_0\mathbb{C}^m$; then $T_0 X = \operatorname{span}\{\partial/\partial w_1, \ldots, \partial/\partial w_r\}$. Set

$$\mathbf{p}_h = \left[\frac{\partial}{\partial w_{r+h}} + T_0 X\right] \in E^1 = \pi_1^{-1}(X^0), \quad h = 1, \ldots, m-r;$$

$$Y^k = \operatorname{span}\{\mathbf{p}_h \mid h \in \mathcal{P}'_k\} \subset E^1_0, \quad k = 1, \ldots, l(\mathcal{M}) - 1.$$

Now put $X^1 = Y^1$ and set $M^2 = \tilde{M}^1_{X^1}$. Let $X^2 \subset M^2$ be the proper transform of Y^2, and set $M^3 = \tilde{M}^2_{X^2}$. Next, let $X^3 \subset M^3$ be the proper transform (with respect to $\sigma_3 \colon M^3 \to M^2$) of the proper transform (with respect to $\sigma_2 \colon M^2 \to M^1$) of Y^3, and put $M^4 = \tilde{M}^3_{X^3}$. Proceeding in this way, for $k = 2, \ldots, l(\mathcal{M}) - 1$ we define the manifold M^{k+1} as the blow-up of M^k along the iterated proper transform X^k of Y^k; we denote by $\sigma_{k+1} \colon M^{k+1} \to M^k$ the associated projection and by $E^{k+1} = \sigma_{k+1}^{-1}(X^k)$ the exceptional divisor.

We also put $\pi_k = \sigma_1 \circ \cdots \circ \sigma_k \colon M^k \to M^0$ for $k = 1, \ldots, l(\mathcal{M})$. The set $\pi_k^{-1}(X^0)$ is called the *singular divisor* of M^k.

LEMMA 5.1. *For $1 \leq k \leq l(\mathcal{M})$, there exist $\mathbf{e}_k \in M^k$ and a canonical chart (U_k, φ_k) centered in $\mathbf{e}_k$ such that*

$$U_k \cap X^k = \varphi_k^{-1}\Bigl(\{w_{r+1} = 0\} \cap \bigcap_{h \in \mathcal{P}''_k} \{w_h = 0\}\Bigr),$$

$$U_k \cap \pi_k^{-1}(X^0) = \varphi_k^{-1}\Bigl(\bigcup_{h \in \mathcal{P}'_{k1}} \{w_h = 0\}\Bigr),$$

and, for $j = k+1, \ldots, l(\mathcal{M}) - 1$,

$$U_k \cap X^j = \varphi_k^{-1}\Bigl(\{w_{r+1} = 0\} \cap \bigcap_{h \in \mathcal{P}''_j} \{w_h = 0\}\Bigr).$$

Furthermore,

$$\varphi_0 \circ \sigma_1 \circ \varphi_1^{-1}(w) = (w_1, \ldots, w_{r+1}, w_{r+1}w_{r+2}, \ldots, w_{r+1}w_m),$$

$$\varphi_{\mu_1} \circ \sigma_{\mu_1+1} \circ \varphi_{\mu_1+1}^{-1}(w) = (w_1, \ldots, w_r, w_{r+1}w_{\mu_2+\nu_2}, w_{r+2}, \ldots, w_m),$$

and, for $2 \leq k \leq \mu_1$,

$$\varphi_{k-1} \circ \sigma_k \circ \varphi_k^{-1}(w)_h = \begin{cases} w_h & \text{if } h \in (\mathcal{P}'_{k-1} \backslash \{r+1\}) \cup \{1, \ldots, r, r+k\}, \\ w_k w_h & \text{if } h \in \{r+1\} \cup (\mathcal{P}''_{k-1} \backslash \{r+k\}). \end{cases}$$

Proof. Fix $\mathbf{e}_1 = \mathbf{p}_1$. Let (U_1, φ_1) be the canonical chart, centered at $\mathbf{e}_1$ and adapted to X^1, obtained from φ_0 via the previous construction. Then, for $k = 1$, the assertion is an immediate consequence of (5.1) and the definition of blow-up along a manifold. For $k > 1$ one argues by induction, taking

$$\mathbf{e}_k = \left[\frac{\partial}{\partial w_{r+k}} + T_{\mathbf{e}_{k-1}} X^{k-1}\right] \in \sigma_k^{-1}(\mathbf{e}_{k-1})$$

or, if $k = \mu_1 + 1$,

$$\mathbf{e}_k = \left[\frac{\partial}{\partial w_{\nu_2+\mu_2}} + T_{\mathbf{e}_{\mu_1}} X^{\mu_1}\right] \in \sigma_{\mu_1}^{-1}(\mathbf{e}_{\mu_1})$$

and defining (U_k, φ_k) as (V_{r+k}, χ_{r+k}) via (U_{k-1}, φ_{k-1}). □

Writing now $z = \varphi_0 \circ \pi_k \circ \varphi_k^{-1}(w)$, by induction we have, for $1 \leq k \leq \mu_1$,

$$z_j = \begin{cases} w_j & \text{if } j \in \{1, \dots, r\}, \\ w_{r+1} \prod_{h=r+2}^{j} (w_h)^2 \prod_{h=j+1}^{r+k} w_h & \text{if } j \in \mathcal{P}'_{k1}, \\ w_{r+1} \prod_{h=r+2}^{r+j-\nu_l} (w_h)^2 \big(\prod_{h=r+j-\nu_l+1}^{r+k} w_h\big) w_j & \text{if } j \in \mathcal{P}'_{kl},\ 2 \leq l \leq \rho, \\ w_{r+1} \prod_{h=r+2}^{r+k} (w_h)^2 w_j & \text{if } j \in \mathcal{P}''_k; \end{cases}$$

for $k = \mu_1 + 1$,

$$z_j = \begin{cases} w_j & \text{if } j \in \{1, \dots, r\}, \\ w_{r+1} \prod_{h=r+2}^{j} (w_h)^2 \big(\prod_{h=j+1}^{r+\mu_1} w_h\big) w_{\nu_2+\mu_2} & \text{if } j \in \mathcal{P}'_{\mu_1 1}, \\ w_{r+1} \prod_{h=r+2}^{r+j-\nu_l} (w_h)^2 \big(\prod_{h=r+j-\nu_l+1}^{r+\mu_1} w_h\big) w_j w_{\nu_2+\mu_2} & \text{if } j \in \mathcal{P}'_{\mu_1 l}, \\ & \quad 2 \leq l \leq \rho, \\ w_{r+1} \prod_{h=r+2}^{r+\mu_1} (w_h)^2 (w_{\nu_2+\mu_2})^2 & \text{if } j \in \mathcal{P}''_{\mu_1}. \end{cases}$$

Furthermore, if $z_{r+1}, \dots, z_{r+k} \neq 0$ then, for $1 \leq k \leq \mu_1$, we have

$$w_j = \begin{cases} z_j & \text{if } j \in \{1, \dots, r\}, \\ (z_{r+1})^2 / z_{r+k} & \text{if } j = r+1, \\ z_j / z_{j-1} & \text{if } j \in \mathcal{P}'_{k1} \backslash \{r+1\}, \\ z_j / z_{r+j-\nu_l} & \text{if } j \in \mathcal{P}'_{kl},\ 2 \leq l \leq \rho, \\ z_j / z_{r+k} & \text{if } j \in \mathcal{P}''_k; \end{cases}$$

for $k = \mu_1 + 1$,

$$w_j = \begin{cases} z_j & \text{if } j \in \{1, \dots, r\}, \\ (z_{r+1})^2 / z_{\nu_2+\mu_2} & \text{if } j = r+1, \\ z_j / z_{j-1} & \text{if } j \in \mathcal{P}'_{\mu_1 1} \backslash \{r+1\}, \\ z_j / z_{r+j-\nu_l} & \text{if } j \in \mathcal{P}'_{\mu_1 l},\ 2 \leq l \leq \rho, \\ z_j / z_{r+\mu_1} & \text{if } j \in \mathcal{P}''_{\mu_1}. \end{cases}$$

6. Partial Diagonalization Theorem

Let $F \in \mathrm{End}(\mathbb{C}^m, 0)$ be semi-attractive, let X be the center stable manifold of F, and let q be the algebraic multiplicity of the eigenvalue 1. Then X is F-invariant, $0 \in X$, and $\dim X = m - q$. Moreover, F is nondegenerate along X with order $\nu_X(F) = 1$.

In order to prove Theorem 1.6, we assume F in the form (2.1) and invert the variables (we set $(x, y) = (z, w) \in \mathbb{C}^{m-q} \times \mathbb{C}^q$):

$$\begin{aligned} x^1 &= G(x) + \tilde{B}(x, y)y, \\ y^1 &= Jy + \tilde{P}_{2,x}(y) + \tilde{P}_{3,x}(y) + \cdots . \end{aligned} \tag{6.1}$$

Then the second component of its linear part does not depend on the x-variables and is in Jordan form. Hence the linear spaces Y^k, defined in Section 5, will be invariant under the lifting of F to the blow-up of $\mathbb{C}^m$ along the center stable manifold. This fact is fundamental for the iteration of the liftings of F.

Consider the sequence of blow-ups of $\mathbb{C}^m$ defined before. By Proposition 5.1, its lifting $\tilde{F}_1 \in \mathrm{End}(M^1, E^1)$ exists and $\tilde{F}_1|_{E^1}$ is induced by J. Moreover $\mathbf{e}_1 \in E^1$ is a fixed point of $\tilde{F}_1$ and $\tilde{F}_1(Y^k) = Y^k$ for $k = 1, \ldots, \mu_1$.

By Proposition 5.3, $\tilde{F}_1$ is nondegenerate along X^1 and so Proposition 5.1 yields $\tilde{F}_2$. By Proposition 5.2, $\tilde{F}_2$ is nondegenerate along X^2 and thus we have $\tilde{F}_3$. By Proposition 5.2, $\tilde{F}_3$ is nondegenerate along X^3 because, outside of $E^2 \subset X^2$, $d\tilde{F}_2 = d\tilde{F}_1$ and then Proposition 5.1 yields $\tilde{F}_4$. Hence we can repeat this procedure for all $k \leq l(\mathcal{M})$ to obtain $\tilde{F}_k$. By the Jordan structure of $dF(0)$ and the definition of $\tilde{F}_k$, the point $\mathbf{e}_k$ defined in Lemma 5.1 is a fixed point for $\tilde{F}_k$.

Finally, since $F \circ \pi_{l(\mathcal{M})} = \pi_{l(\mathcal{M})} \circ \tilde{F}_{l(\mathcal{M})}$, we also have

$$F \circ (\varphi_0 \circ \pi_{l(\mathcal{M})} \circ \varphi_{l(\mathcal{M})}^{-1}) = (\varphi_0 \circ \pi_{l(\mathcal{M})} \circ \varphi_{l(\mathcal{M})}^{-1}) \circ (\varphi_{l(\mathcal{M})} \circ \tilde{F}_{l(\mathcal{M})} \circ \varphi_{l(\mathcal{M})}^{-1}).$$

Then, applying the formulas at the end of Section 5, and inverting the coordinates once again, we find $\tilde{F} := \tilde{F}_{l(\mathcal{M})}$ in the form (1.3) with $A(z) \equiv I$, and $P_{2,z} = (p^1_{2,z}, \ldots, p^q_{2,z})$ has the following expression:

(a) if $\mu_1 > \mu_2$,

$$p^j_{2,z}(w) = \begin{cases} -a^{\mu_1}_{11}(z)w_1^2 + 2w_1w_2 & \text{if } j = 1, \\ -w_j^2 + w_{j+1}w_j & \text{if } 2 \leq j \leq \mu_1 - 1, \\ a^{\mu_1}_{11}(z)w_1w_{\mu_1} - w^2_{\mu_1} & \text{if } j = \mu_1, \\ w_{j-\nu_l+m-q+1}(-w_j + w_{j+1}) & \text{if } m - q + j \in \mathcal{P}'_{\mu_1,l} \setminus \{\nu_l + \mu_l\}, \\ & \quad 2 \leq l \leq h, \\ -w_{\mu_l+1}w_j & \text{if } m - q + j = \nu_l + \mu_l, \\ & \quad \mu_l < \mu_1 - 1, \\ a^j_{11}(z)w_1w_{\mu_1} - w_{\mu_1}w_j & \text{if } m - q + j = \nu_l + \mu_l, \\ & \quad \mu_l = \mu_1 - 1; \end{cases}$$

(b) if $\mu_1 = \mu_2$,

$$p^j_{2,z}(w) = \begin{cases} -a_{11}^{\nu_2+\mu_2-m+q}(z)w_1^2 + 2w_1w_2 & \text{if } j = 1, \\ -w_j^2 + w_{j+1}w_j & \text{if } 2 \le j \le \mu_1 - 1, \\ -w_{\mu_1}^2 & \text{if } j = \mu_1, \\ w_{j-\nu_l+m-q+1}(-w_j + w_{j+1}) & \text{if } m-q+j \in \mathcal{P}'_{\mu_1,l} \setminus \{\nu_l + \mu_l\}, \\ & \quad 2 \le l \le h, \\ a_{11}^{\nu_2+\mu_2-m+q}(z)w_1w_{\nu_2+\mu_2-m+q} & \text{if } m-q+j = \nu_2+\mu_2, \\ 0 & \text{if } m-q+j = \nu_l + \mu_l, \\ & \quad \mu_l < \mu_1, \\ a_{11}^j(z)w_1w_{\nu_2+\mu_2-m+q} & \text{if } m-q+j = \nu_l+\mu_l, \\ & \quad \mu_l = \mu_1,\ 3 \le l \le \rho. \end{cases}$$

Here $z = x$ and $a_{11}^j(x)$ is the coefficient of y_1^2 in the jth component of $\tilde{P}_{2,x}(y)$.

COROLLARY 6.1. *Let $F \in \mathrm{End}(\mathbb{C}^m, 0)$ be semi-attractive. Let q be the algebraic multiplicity of the eigenvalue 1 of $dF(0)$, and suppose that the geometric multiplicity of 1 is strictly less than q. Let $\mathcal{M}$ be the ρ-partition of q induced by the structure of the Jordan block associated to the eigenvalue 1. Assume that $l(\mathcal{M}) = \mu_1$ and $a_{11}^{\mu_1}(0) \neq 0$.*

Then F admits a parabolic manifold of dimension $m - q + 1$ tangent to $\mathbb{C} \oplus E$ at 0, where E is generated by the generalized eigenspaces associated to the eigenvalues of $dF(0)$ with modulus strictly less than 1.

Proof. Consider the lifting $\tilde{F}_{\mu_1}$ of F given by Theorem 1.6. By Proposition 2.2, we can assume $\tilde{F}_{\mu_1}$ in the form $(2.2)_k$ with $k > 3$ and $P_2 = P_{2,0}$. Then we can apply Theorem 1.3 to obtain a $\tilde{F}_{\mu_1}$-parabolic manifold and use π_{μ_1} to project it down to F.

Not all nondegenerate characteristic directions of P_2 are acceptable; we must exclude the ones tangent to $\pi_{\mu_1}^{-1}(X)$ because they are killed when we project down by π_{μ_1}. Hence $V = (v_1, \dots, v_q)$ is an *allowable* characteristic direction of P_2 if it is not tangent to $\pi_{\mu_1}^{-1}(X)$ (i.e., iff $v_1, \dots, v_{\mu_1} \neq 0$, because of Lemma 5.1).

Then, using the expression of $P_{2,z}$ just given and imposing $P_{2,0}(V) = \lambda V$ with $\lambda \neq 0$, we see that the unique solution is

$$v_j = \begin{cases} \frac{1}{a_{11}^{\mu_1}(0)}(2\mu_1 - 1)\lambda & \text{for } j = 1, \\ (\mu_1 + j - 2)\lambda & \text{for } 2 \le j \le \mu_1, \\ 0 & \text{for } m-q+j = \nu_l + h, \\ & \quad 1 \le h \le \mu_l,\ \mu_l < \mu_1 - 1, \\ \frac{a_{11}^{\nu_l+\mu_l-m+q}(0)}{a_{11}^{\mu_1}(0)}(\mu_l + h)\lambda & \text{for } m-q+j = \nu_l + h, \\ & \quad 1 \le h \le \mu_l,\ \mu_l = \mu_1 - 1. \end{cases}$$

Thus there exists a $\tilde{F}_{\mu_1}$-parabolic manifold at the origin, tangent to $\mathbb{C}V \oplus E$ and of dimension $m-q+1$, that is contained in $M^{\mu_1}\backslash\pi_{\mu_1}^{-1}(X)$. Because this manifold is given as the image of an injective holomorphic map $\tilde{\psi}$, it follows that $\psi = \pi_{\mu_1}\circ\tilde{\psi}$ defines the parabolic manifold for F since π_{μ_1} restricted to $M^{\mu_1}\backslash\pi_{\mu_1}^{-1}(X)$ is a biholomorphism. $\square$

REMARK 6.1. With computations similar to those made in the preceding proof, we also see that if $\rho \geq 2$ and $\mu_1 = \mu_2$ then there are no allowable characteristic directions for $P_{2,0}$ of $\tilde{F}_{\mu_1+1}$.

COROLLARY 6.2. *Let $F \in \mathrm{End}(\mathbb{C}^m, 0)$ be semi-attractive such that the eigenvalue 1 of $dF(0)$ has algebraic multiplicity $q = 2$ and the corresponding Jordan block is nondiagonalizable. Assume F in the form* (6.1) *with $a^2_{11}(0) = 0$, that is,*

$$\begin{aligned}
x^1 &= G(x) + B(x,y)y,\\
y_1^1 &= y_1 + y_2 + a^1_{11}(x)y_1^2 + 2a^1_{12}(x)y_1y_2 + a^1_{22}(x)y_2^2 + \cdots,\\
y_2^1 &= y_2 + 2a^2_{12}(x)y_1y_2 + a^2_{22}(x)y_2^2 + a^2_{111}(x)y_1^3 + \cdots,
\end{aligned}$$

and set

$$\varepsilon = a^1_{11}(0) + a^2_{12}(0), \qquad \eta = (a^1_{11}(0) - a^2_{12}(0))^2 + 2a^2_{111}(0).$$

Then ε and η are projective invariants and, when $(\varepsilon,\eta) \neq (0,0)$:

(i) *if $\eta \neq 0, \varepsilon^2$, then F has two distinct parabolic manifolds of dimension $m-1$;*
(ii) *if $\eta = \varepsilon^2 \neq 0$ or $\eta = 0 \neq \varepsilon^2$, then F has one parabolic manifold of dimension $m-1$.*

Proof. For such maps, the blow-up along the center stable manifold immediately diagonalizes the Jordan block of $dF(0)$ corresponding to the eigenvalue 1. In fact we have

$$\begin{aligned}
x^1 &= G(x) + \tilde{B}(x,w)w,\\
w_1^1 &= w_1 + a^1_{11}(x)w_1^2 + w_1w_2 + O(\|w\|^3),\\
w_2^1 &= w_2 + a^2_{111}(x)w_1^2 + (2a^2_{12}(x) - a^1_{11}(x))w_1w_2 - w_2^2 + O(\|w\|^3).
\end{aligned}$$

Then a characteristic direction $V = (v_1, v_2)$ for $P_{2,0}(w)$ is allowable iff $v_1 \neq 0$. Therefore we obtain two allowable characteristic directions (up to multiplication by a constant),

$$V_\pm = \left(1, \frac{a^2_{12}(0) - a^1_{11}(0) \pm \sqrt{\eta}}{2}\right),$$

which are degenerate iff $\varepsilon \pm \sqrt{\eta} = 0$. Hence the assertion is obtained by applying Theorem 1.3. $\square$

REMARK 6.2. Note that $A(V_\pm) = \mp 2\sqrt{\eta}/(\varepsilon \pm \sqrt{\eta})$ and so $\mathrm{Re}\,A(V_\pm) > 0$ iff $\mathrm{Re}(\varepsilon/\pm\sqrt{\eta}) < -1$ when $\eta \neq 0, \varepsilon^2$. Then Theorem 1.4 also implies that F has an attracting domain when $|\mathrm{Re}(\varepsilon/\sqrt{\eta})| > 1$.

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