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C^1 Immersed Hypersurfaces Separate $\mathbb{R}^n$

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1. Introduction

Clearly an immersed 2-sphere separates $\mathbb{R}^3$. As evidenced by the papers of Vaccaro, Feighn, and M. D. Hirsch, this statement is true—but it is less than clear. We summarize the known results for proper immersions $f: M^m \to N^n$ where the codimension $k = n - m$ is ≥ 1 and M, N are boundaryless. Recall that a map is *proper* if pre-images of compact sets in the target are compact in the domain.

(a) Vaccaro [7]. If f is merely PL (piecewise linear) then everything fails; the counterexample is the house with two rooms, which is a nonseparating PL immersion of the 2-sphere in $\mathbb{R}^3$. The illustration in Figure 1 is drawn as piecewise smooth. See also Rourke and Sanderson [6] or Bing [1].

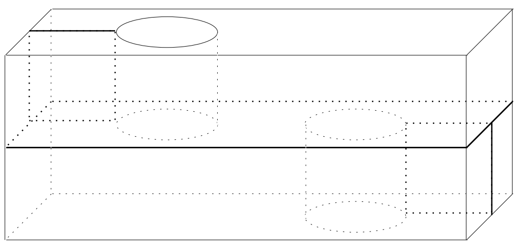

Figure 1 The house with two rooms

(b) Vaccaro [7]. If f is C^1 and if fM is a subcomplex of a C^1 triangulation of N then $H_m(fM; \mathbb{Z}_2) \neq 0$, which by Alexander duality implies that fM separates when $N = \mathbb{R}^{m+1}$.

(c) Feighn [2]. If f is C^2, $k = n - m = 1$, and $H_1(N; \mathbb{Z}_2) = 0$, then fM separates N.

(d) Hirsch [4]. If f is C^2 then fM *k-separates* N in the sense that the kth homology and the kth homotopy groups of the pair $(N, N \setminus fM)$ are nontrivial. The coefficient group for the homology can be either $\mathbb{Z}$ or $\mathbb{Z}_2$.

Note that k-separation is also referred to as "k-piercing."

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In this paper we analyze C^1, non-C^2, immersions. As the following proposition shows, (c) is a consequence of (d), and accordingly we concentrate on (d). Our main result is as follows.

THEOREM A. *A proper C^1 immersion $f\colon M^m \to N^n$ k-separates N when $k = n - m \geq 1$. In particular, a C^1 immersion of a compact M^m in $\mathbb{R}^{m+1}$ separates $\mathbb{R}^{m+1}$.*

Note, too, that (d) is a local statement whereas (c) concerns the global topology of N.

PROPOSITION. *If $H_1(N;\mathbb{Z}_2) = 0$ then 1-separation implies topological separation.*

Proof. The set fM separates N if and only if the reduced homology group $H_0^{\#}(N \setminus fM;\mathbb{Z}_2)$ is nonzero. The long exact reduced homology sequence of $(N, N \setminus fM)$ is

$$\cdots \to H_1(N;\mathbb{Z}_2) \to H_1(N, N \setminus fM;\mathbb{Z}_2) \to H_0^{\#}(N \setminus fM;\mathbb{Z}_2).$$

By assumption, the first group is zero and the second is nonzero. Exactness implies that the third is nonzero. $\square$

Vaccaro's proof relies on simplicial topology, which is why his theorem ignores immersions like that shown in Figure 2.

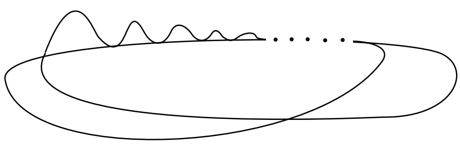

Figure 2 An immersion to which Vaccaro's method does not apply

Feighn employs standard Morse theory, which is why he assumes that f is C^2. Feighn is willing to confront infinitely complicated immersions, as in Figure 2, so his result is topologically more general than Vaccaro's. Hirsch also employs standard Morse theory and needs f to be C^2. Using smoother analysis and rougher functions, we show how to lower the differentiability hypotheses on f from C^2 to C^1, retaining the other generalities of Feighn and Hirsch.

In Section 2 we review Feighn's counting argument, in Section 3 we modify his ideas to suit the C^1 codimension-1 case, in Section 4 we generalize to higher codimensions, and in Section 5 we investigate some related but curious low-differentiability phenomena.

2. Good Points and Bad Points

To include the possibility that M, N have nonempty boundaries, we adopt Feighn's assumption that the proper immersion $f\colon M^m \to N^{m+1}$ satisfies

$$f^{-1}(\partial N) = \partial M \quad \text{and} \quad f \text{ is transverse to } \partial M.$$

Since f is a proper immersion, the f-pre-image of any $y \in fM$ is a finite set of points, $x_1, \ldots, x_s$. The number s is the *multiplicity* of y with respect to f. Feighn calls a point $y \in fM$ *good* if fM does not branch at y and *bad* if it is not good. See Figure 3. For a good point y there are neighborhoods $U_1, \ldots, U_s$ of the pre-image points $x_1, \ldots, x_s$ such that $f(U_i) = f(U_j)$ when $1 \le i, j \le s$.

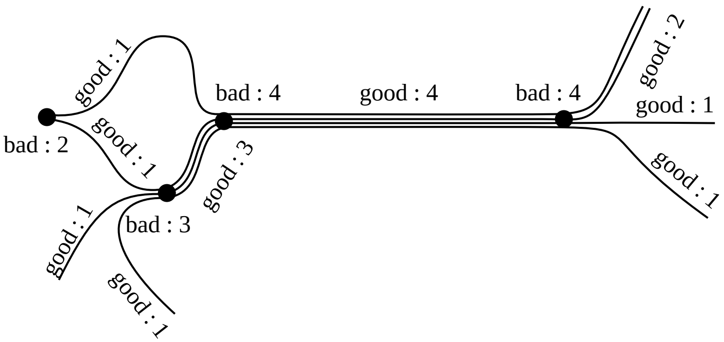

Figure 3 Good and bad points of fM labeled with their multiplicities

It is standard to see that the set of good points in fM is open-dense, and the pre-image is also open-dense in M.

Feighn's strategy is to analyze the multiplicity assuming that fM does not 1-separate N homotopically. There are four steps.

Step 1. Every multiplicity is even. Feighn then chooses a good y_0 whose multiplicity s_0 has the smallest even factor 2^{α_0}. All other multiplicities are divisible by 2^{α_0}.

Step 2. From the assumption that fM fails to 1-separate N, Feighn constructs a smooth map of the 2-disc into N, $g\colon D \to N$, such that g and $g|_{\partial D}$ are transverse to fM and $y_0 \in g(\partial D)$. Also there is a unique $z_0 \in D$ with $g(z_0) = y_0$, and this z_0 lies in ∂D.

Step 3. Feighn next examines the joint pullback

$$P = \{(x, z) \in M \times D : fx = gz\}.$$

It is a compact 1-manifold whose boundary lies in $M \times z_0$. It consists of $s_0/2$ arcs (and some Jordan curves that are irrelevant). Thus $\chi(P) = s_0/2$, which implies that 2^{α_0} does not divide $\chi(P)$.

Step 4. Feighn then constructs a function $\tau\colon D \to \mathbb{R}$ such that the composite

$$\mu\colon (x, z) \mapsto z \mapsto \tau(z)$$

is a Morse function on P that has noncritical maxima at the boundary points of P. The Euler characteristic of P is the sum of the Morse indices at the critical points of μ. Because μ is a composite, its critical points occur not singly and independently but rather in whole fibers,

$$P(y) = \{(x, z) \in P : f(x) = y\},$$

where all points in the fiber share the same Morse index. The multiplicity of y is the cardinality of the fiber, and all multiplicities are divisible by 2^{α_0}; hence the Morse-theoretic Euler characteristic is divisible by 2^{α_0}, which contradicts Step 3.

> *Feighn's construction of τ uses the distance function in the ambient space N, and this is where he uses the C^2 hypothesis. In Section 3 we produce such a Morse function by different means.*

Here are some details about Steps 1–3. The standing assumption is that fM fails to 1-separate N homotopically.

Step 1. For any $y^* \in fM$, draw a short arc γ_0 that crosses fM transversely at y^*. Since fM does not 1-separate N homotopically, there is a second arc γ_1 homotopic to γ_0 (the homotopy keeps the endpoints in $N \setminus fM$ fixed) that is disjoint from fM. Rounding off corners and smoothing the homotopy leads to a smooth map of the 2-disc into N, $h\colon D \to N$, such that:

(a) h and $h|_{\partial D}$ are transverse to f;
(b) h embeds ∂D and sends a unique point $z^* \in \partial D$ to y^*;
(c) $h(\partial D) \cap fM = \{y^*\}$; and
(d) $h(\partial D) \cap \partial N = \emptyset$.

This construction is standard (see Figure 4). Transversality implies that $f \times h\colon M \times D \to N \times N$ is transverse to the diagonal Δ_N. The pre-image is the joint pullback

$$P = P_h = (f \times h)^{-1}(\Delta_N) = \{(x, z) : fx = hz\}.$$

The diagram

$$\begin{tikzcd} & P \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_D"] & \\ M \arrow[dr, "f"'] & & D \arrow[dl, "h"] \\ & N & \end{tikzcd}$$

commutes, and P is a compact 1-dimensional submanifold of $M \times D$ whose boundary necessarily lies in $M \times \partial D$. Commutativity implies that the only points of P that

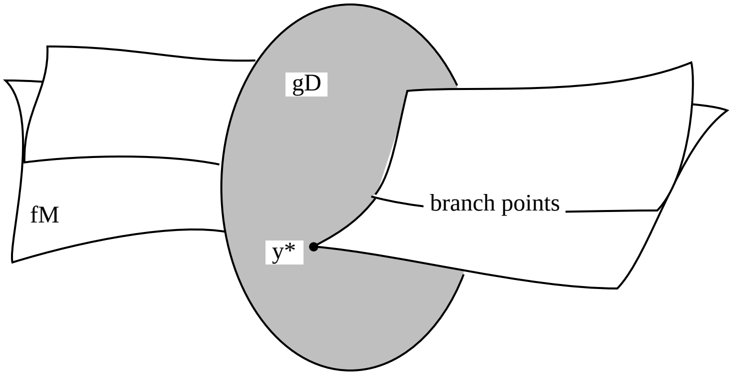

Figure 4 A smooth disc transverse to fM at a good point y^*

map to y^* are in $M \times z^*$. The fiber of P over $y \in fM$, $P(y) = \{(x, z) : fx = hz = y\}$, either is empty or consists precisely of the set of points $\{(x_1, z), \ldots, (x_s, z)\}$, where $\{x_1, \ldots, x_s\}$ is the f-pre-image of y. Thus, $P(y)$ has cardinality 0 or cardinality s.

Observe that $P(y^*)$ is simultaneously (a) the boundary points of P and (b) the product $f^{-1}(y^*) \times z^*$. The boundary points of a compact 1-manifold have even cardinality. Therefore, the multiplicity of every y^* with respect to f is even. See Figure 5.

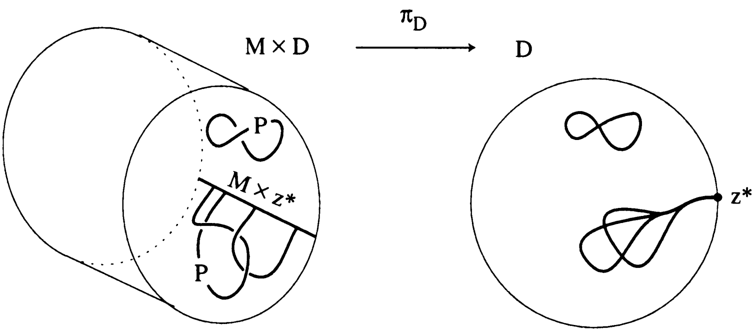

Figure 5 The joint pullback P and its boundary points (the manifold M appears to be 1-dimensional in the figure)

Step 1$^{\text{bis}}$. Choose a good point $y_0 \in fM$ with least even multiplicity s_0 among good points. Then $s_0 = 2^{\alpha_0}\ell$, ℓ is odd, and all multiplicities of good points are divisible by 2^{α_0}.

Steps 2 and 3. The preceding construction was made for a general point $y^* \in fM$. Repeating it for the special point y_0 produces $g\colon D \to N$, and Figure 5 is valid for $P = P_g$. Thus $\chi(P) = s_0/2$.

REMARK. The even multiplicity condition is often fulfilled by an immersion. For example, Boy's surface is an immersed 2-sphere in $\mathbb{R}^3$ all of whose good points have multiplicity 2. It is also easy to immerse the torus in $\mathbb{R}^3$ so that all its points have multiplicity a power of 2.

3. C^1 Immersions in Codimension 1

Consider a space W and a continuous function defined on W, say $w_0 \colon W \to \mathbb{R}$. As explained in [5], a *strong* (or "Whitney") C^0 neighborhood of w_0 consists of all functions $w \colon W \to \mathbb{R}$ for which $|w(x) - w_0(x)| < \varepsilon(x)$, where ε is a given positive continuous function defined on W. If W is compact, then ε is bounded away from 0 and the strong ε-neighborhood of w_0 is just an ordinary uniform neighborhood. If, however, W is noncompact, then the function ε can have arbitrarily fast decay toward the frontier of W; consequently, the behavior of w near the frontier is extremely like that of w_0. For example, if $W = (0,1)$ and $w_0(x) = e^{-1/x}$ then, for the correct choice of $\varepsilon \colon (0,1) \to (0,\infty)$, any strong C^0 ε-approximation to w_0 satisfies $\lim_{x\to 0} w(x)x^{-n} = 0$ for all $n \in \mathbb{N}$. If W is a manifold and w_0 is C^r $(r \geq 1)$, then it is natural to define corresponding strong C^r neighborhoods of w_0. This leads to the *strong C^r topology* on the set of C^r maps from one manifold to another. See [5].

LEMMA 1. *Any proper C^1 immersion is C^1 diffeomorphic to a proper C^1 immersion that is C^∞ on its good set.*

Proof. Let $f_0 \colon M^m \to N^n$ be a proper C^1 immersion. Its good set $G_0 \subset f_0 M \subset N$ is an open embedded C^1 m-submanifold of N. (It is not closed in general, but it does not accumulate on itself.) Let $\tilde{G}_0$ be the same point set as G_0, but with an abstract, artificial C^∞ structure that is C^1 compatible with its C^1 structure as a C^1 submanifold of N. See [5, Chap. 2] for the existence of such smoothings. By C^1 compatibility, the inclusion

$$i_0 \colon G_0 \hookrightarrow N$$

defines a C^1 embedding $\tilde{G}_0 \to N$. Any C^1 embedding can be strongly C^1 approximated by a C^∞ embedding; say $\tilde{i}_0 : \tilde{G}_0 \to N$ is such an embedding. A strong C^1 perturbation of a C^1 embedding extends to an ambient C^1 diffeomorphism, say $i \colon N \to N$, where $i|_{G_0} = \tilde{i}_0$. In fact, we can make i the identity map off a sharply tapered neighborhood V of G_0; see Figure 6. The image of G_0 under $\tilde{i}_0$ is a C^∞ submanifold of N.

Define a map $f \colon M \to N$ as $f = i \circ f_0$. Then f is a proper C^1 immersion whose good set G is a C^∞ submanifold of N, $G = \tilde{i}_0(G_0)$. By construction, $f^{-1}G = f_0^{-1}(G_0)$ is an open set $U \subset M$. The map $f|_U$ is a C^1 submersion of U onto G. Since any C^1 map from one smooth manifold to another can be strongly C^1 approximated by a C^∞ map, we strongly C^1 approximate $f|_U$ by a C^∞ map $\tilde{f} \colon U \to G$.

By the implicit function theorem and the global rank theorem, every strong C^1 small perturbation of a submersion $f \colon U \to G$ is a submersion $U \to G$ of the

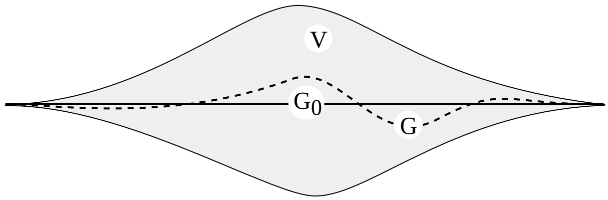

Figure 6 $\quad i$ is the identity map off V

form $f \circ j$, where j is a C^1 diffeomorphism $U \to U$ that strongly C^1 approximates the identity map $U \to U$. Thus j extends to a C^1 diffeomorphism $M \to M$, still called j, such that $j(x) = x$ for all $x \in M \setminus U$. Define $F: M \to N$ as $F = f \circ j$. It is a proper C^1 immersion that C^1 approximates f_0, and since $F|_U = \tilde{f}$ is C^∞, it follows that F is C^∞ on its good set. Commutativity of the diagram

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "j"'] & N \arrow[d, "\mathrm{id}"] \\ M \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & N \arrow[d, "i^{-1}"] \\ M \arrow[r, "f_0"] & N \end{tikzcd}$$

now shows that F is C^1 diffeomorphic to f_0. $\square$

LEMMA 2. *Let f be a C^1 immersion $M \to N$ and let G be any open-dense subset of fM. Then, for any smooth manifold W, the generic smooth map $g: W \to N$ is transverse to f, and $g^{-1}G$ is open-dense in $g^{-1}(fM)$.*

Proof. Since f is a C^1 immersion, fM consists of a countable collection of overlapping embedded C^1 m-discs D_i. The generic smooth map $g: W \to N$ (or the generic C^r map, $1 \le r < \infty$) is transverse to D_i. (Note that this fact does not rely on high smoothness of D_i.) Since W is second countable, the generic smooth map $g: W \to N$ is transverse to all the D_i; that is, g is transverse to f. (If f is merely a smooth map, not an immersion, the question of whether the generic g is transverse to f is a three-star problem in [5, p. 84].)

By continuity, $g^{-1}G$ is an open subset of $g^{-1}(fM)$. The proof that it is generically dense in $g^{-1}(fM)$ has nothing to do with smoothness of fM. Let $g_0: W \to N$ be given, and take any $\delta > 0$ and any compact set $K \subset W$. Choose points $w_1, \ldots, w_\ell \in g_0^{-1}(fM)$ that are $\delta/2$-dense in $K \cap g_0^{-1}(fM)$. Their images $y_i = g_0(w_i)$ are "independently mobile" in the sense that small perturbations g of g_0 exist that simultaneously move the y_i to any prescribed points y_i' near y_i, $g(w_i) = y_i'$. Since G is dense in fM, this means that we can perturb g_0 to g so that $g(w_i) \in G$. Therefore, $g^{-1}G$ includes $\{w_1, \ldots, w_\ell\}$, which is $\delta/2$-dense in $K \cap g_0^{-1}(fM)$.

On the other hand, compactness of K implies that the set $K \cap g^{-1}(fM)$ is not much larger than $K \cap g_0^{-1}(fM)$. In fact, for g near enough to g_0, the former lies in the $\delta/2$-neighborhood of the latter. It follows that $g^{-1}(G)$ is δ-dense in $K \cap g^{-1}(fM)$. This condition of δ-density is open in the space of maps g, even under a weaker topology like the compact open topology. Thus

$$\mathcal{G}(\delta, K) = \{g : g^{-1}G \text{ is } \delta\text{-dense in } K \cap g^{-1}(fM)\}$$

is open-dense in $C^\infty(W, N)$. Taking the intersection of $\mathcal{G}(\delta_i, K_i)$ as $K_i \uparrow W$ and $\delta_i \downarrow 0$ we see that, for the generic g, $g^{-1}G$ is dense in $g^{-1}(fM)$. $\square$

THEOREM A (in codimension 1). *If $f : M^m \to N^{m+1}$ is a proper C^1 immersion then fM 1-separates N.*

Proof. By Lemma 1 it is no loss of generality to assume that f is C^∞ on its good set. Suppose the theorem is false and that fM does not 1-separate N homologically. Following Feighn's construction, we find a smooth map of a compact surface W into N, $g : W \to N$, which is transverse to f such that g embeds ∂W onto a smooth loop γ meeting fM only at the point y_0. (The difference between not 1-separating homologically versus homotopically boils down to whether W can have handles or is the disc.) We choose a good y_0 with least even multiplicity $s_0 = 2^{\alpha_0}\ell$, where ℓ is odd and the minimization is done over the good points. The immersion f has multiplicity divisible by 2^{α_0} at all the good points.

By Lemma 2, we may assume that g is transverse to f and that $g^{-1}G$ is open-dense in $g^{-1}(fM)$. Now $g^{-1}(fM)$ is the immersed image of P, the joint pullback, under the immersion $\pi_W : P \to W$ that projects (x, w) to w. As in Step 3 of Feighn's proof (explained in Section 2), $\chi(P) = s_0/2$.

Near w_0, $g^{-1}(fM)$ is an arc $\eta \subset W$ that ends at w_0. For y_0 is a good point of fM and gD meets fM at y_0 transversely, as shown in Figure 4. Let $\tau_0 : W \to \mathbb{R}$ be a smooth function such that $\tau_0|_\eta$ has a noncritical maximum at w_0. Any C^1 small perturbation of τ_0 still has a noncritical local maximum along η at τ_0. The rest of $g^{-1}(fM)$ is a finite collection of C^1 arcs A_i in W, $i = 1, \ldots, L$. The good set $\tilde{A}_i = A_i \cap g^{-1}G$ is open-dense in A_i and it is smooth. Each A_i has a C^1 tubular neighborhood V_i in which it appears to be a line segment. Take $i = 1$ and change τ_0 in V_1 so that (a) it becomes an apparently C^∞ function τ_1 defined on V_1 and (b) on the apparently C^∞ arc A_1, τ_1 is Morse. Only finitely many critical points occur on A_1, and we push them into the good set $\tilde{A}_1$. *Small enough subsequent perturbations do not destroy the fact that the critical points lie in the good sets $\tilde{A}_i$, although Morseness may be lost.* See Section 5.

After L progressively smaller modifications we obtain a function τ_L that is really just C^1, but the critical points of $\tau_L \circ \pi_W$ are in the good set and this fact is permanent under C^1 small changes of τ_L. Let τ be a generic smooth perturbation of τ_L. By genericity of τ and smoothness of $\tilde{A}_i$, the restriction of τ to $\tilde{A}_i$ is Morse, $1 \le i \le L$. All critical points of $\tau \circ \pi_W$ are good. That is, τ satisfies the conditions in Step 4 of Feighn's proof, as outlined in Section 2. The rest of the proof of Theorem A is the same as the C^2 case: $\mu = \tau \circ \pi_W$ is a Morse function on P and the Euler characteristic calculated using μ is divisible by 2^{α_0}, which contradicts the fact that it equals $s_0/2$. $\square$

It is interesting to see how the preceding proof fails when $f\colon S^2 \to \mathbb{R}^3$ is the house with two rooms. Each sheet of fM has multiplicity 2 at good points; at bad points, it is 3. Even so, we might draw a disc gD as shown in Figure 7, transverse to f at a typical point y_0. Then we could consider the pullback P and compute its Euler characteristic in two ways. Since f has multiplicity 2 at y_0, there are two endpoints in $f^{-1}(y_0)$ and hence there is just one arc in P, so $\chi(P) = 1$.

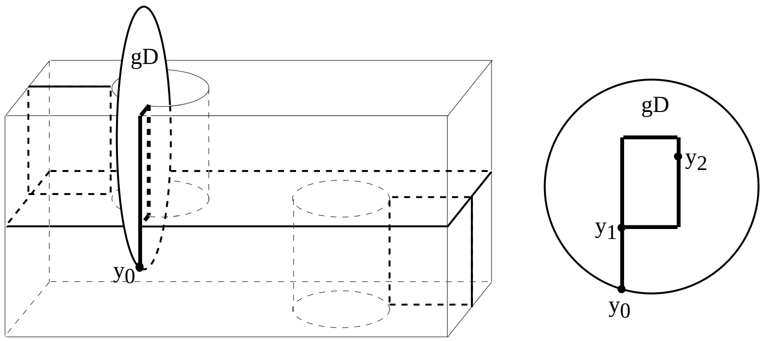

Figure 7 The map $g\colon D \to \mathbb{R}^3$ embeds the disc across fS^2 and meets it in the curve shown, where y_1 is the tee-junction point

Piecewise linear Morse theory on 1-dimensional manifolds is easy. A point p is noncritical for μ if μ is monotone in a neighborhood of p and is critical otherwise. If μ achieves a local minimum or a local maximum at each of its critical points, then it is Morse. Suppose that $\mu = \tau \circ \pi_D$ is Morse on P and that it increases toward ∂P.

Consider τ and the rectangular loop Λ of $fS^2 \cap gD$ at y_1 shown in Figure 7. The perturbation τ must have a minimum or a maximum on Λ, and one of the critical values must be different from $\tau(y_1)$. Let y_2 be such a critical point. For simplicity suppose there are no other critical points. In particular, τ decreases on the segment $[y_0, y_1]$.

If $\tau(y_1) < \tau(y_2)$ then τ achieves a minimum along all three branches of fM that pass through y_1. Thus y_1 accounts for three minima of μ on P, while y_2 accounts for two maxima. This agrees with the previous calculation

$$\chi(P) = 1 = 3 - 2.$$

On the other hand, if $\tau(y_2) < \tau(y_1)$ then τ is noncritical along two of the three branches through y_1 but achieves a local maximum along the L-shaped branch. Thus y_1 accounts for one maximum of μ while y_2 accounts for two minima. Again, $\chi(P) = 1 = 2 - 1$. In any case, both calculations of the Euler characteristic agree and we have no contradiction. When there are more critical points, the same type of reasoning carries through.

It may be worth mentioning explicitly what we noticed about an arbitrary continuous function $\tau : T \to \mathbb{R}$, where T is the letter T. It explains why the bad point y_1 inevitably pulls back to a critical point of μ.

PROPOSITION. *Along at least one of the three maximal arcs in T (two are* L-*shaped and one is straight), τ is critical at t, the tee-junction point. If τ is Morse then τ is critical at t along one or three of these arcs, but not just two.*

4. C^1 Immersions in Higher Codimension

The extension from codimension 1 to codimension k is a straightforward modification of what appears in [4]. Let $f : M^m \to N^{m+k}$ be a proper C^1 immersion. The situation with good and bad points is the same in all codimensions: the good points form open-dense sets of fM and M. Let D be a small smooth k-disc in N that meets fM transversely at a good point y_0 of minimal least-even multiplicity $s_0 = 2^{\alpha_0}\ell$. Here, y_0 is at the center of D, not in the image of its boundary, and in fact transversality implies that the boundary is disjoint from fM. The pair $(D, \partial D)$ defines a homology class $h \in H_k(N, N \setminus fM; \mathbb{Z})$ and a homotopy class $p \in \pi_k(N, N \setminus fM)$. They could be called the homology and homotopy *k-separating classes* of fM.

THEOREM A (in codimension ≥ 2). *Both h and p are nontrivial. Furthermore, h is not of odd order, nor is it divisible by 2. The same is true for p; it is not of odd order and does not have a square root.*

Proof. As in the codimension-1 case, if p is trivial then D is homotopic (relative to its boundary) to a disc D' in $N \setminus fM$, and the union $D \cup D'$ bounds a singular $(k+1)$-disc $g : B \to N$, where B is the $(k+1)$-ball and g sends the northern hemisphere of $S^k = \partial B$ to D and the southern hemisphere to D'. Following the proof of Feighn's theorem in Section 2, we modify g so that:

(a) g is smooth and in general position with respect to f;
(b) $gB \cap \partial N = \emptyset$, and g sends the northern hemisphere of ∂B to D and the southern hemisphere into $N \setminus fM$; and
(c) $g^{-1}(y_0) \cap \partial B$ is a single point z_0.

Thus $g(\partial B)$ crosses fM at y_0 with multiplicity s.

Exactly as before, the pullback $\{(x, z) : fx = gz\}$ is a 1-manifold $P \subset M \times B$. On one hand its Euler characteristic is half the number of pre-images of y_0, namely $s_0/2$, while on the other hand it is the sum of the Morse indices of $\mu = \tau \circ \pi_B$. The latter sum is divisible by 2^{α_0} since critical points occur along whole good fibers $P(y)$, as all critical points in a given fiber are of the same type, and 2^{α_0} divides the cardinality of each good fiber, which contradicts the fact that $\chi(P) = s_0/2$, so p is nontrivial after all.

The proofs of the other assertions in Theorem A in codimension ≥ 2 are also the same as those in [4], once the existence of a Morse function like μ is accepted.

The only obstruction in the homology case is the need for a transversality perturbation when W is no longer a $(k+1)$-disc but is a $(k+1)$-dimensional simplicial complex. A result of Fenn [3] that "any W^{k+1} supports a C^∞ structure off its $(k-1)$-skeleton" permits this, and then the previous homotopy analysis carries over to homology. $\square$

5. Low Differentiability Phenomena

When looking for a proof of Feighn's theorem in the C^1 category, there is a natural trick to try: given a C^1 immersion, make C^1 changes of variables in the domain and target space so that the new immersion is C^2. If the trick can be done, Feighn's C^2 theorem implies Theorem A at once. In our proof of Theorem A, we skirted this issue by making the immersion smooth on its good points rather than everywhere.

THEOREM. *There are C^1 immersions that are not equivalent to C^2 immersions by C^1 changes of variables.*

Proof. The example is simple. Take an immersion $f\colon S^1 \to \mathbb{R}^2$ with two branches through the origin. One branch contains $X = [-1, 1] \times 0$ on the x-axis, and the other contains the curve Y,

$$y = |x|^{3/2}, \quad -1 \le x \le 1.$$

We show there is no C^1 equivalence of f to a C^2 immersion. In fact, we show there is no Lipeomorphic (i.e., bi-Lipschitz) equivalence.

Suppose there is one, $\psi\colon S^1 \to S^1$ is a Lipeomorphism, $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipeomorphism, and $\phi \circ f \circ \psi\colon S^1 \to \mathbb{R}^2$ is C^2. The map ψ is irrelevant; the Lipeomorphism ϕ carries the curves X, Y to a pair of C^2 curves X', Y'. A Lipeomorphism cannot carry a pair of tangent C^1 curves to a pair of transverse C^1 curves. Thus X', Y' are tangent at $\phi(0)$. A subsequent C^2 diffeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ carries X' to X and carries Y' to the graph of a C^2 function $h\colon [-1, 1] \to \mathbb{R}$. Hence $\lambda = \varphi \circ \phi$ is a Lipeomorphism that carries the pair X, Y to the pair X, H, where $H = \operatorname{graph} h$. Since h is C^2 and $h(0) = 0 = h'(0)$, there is a constant K such that

$$|hx| \le K|x|^2.$$

Let μ be the inverse Lipeomorphism, $\mu = \lambda^{-1}$, and write it in coordinates as

$$\mu(x, y) = (\mu_1(x, y), \mu_2(x, y)).$$

Then $\mu(H) = Y$. Write

$$\mu(x, hx) = (x_1, |x_1|^{3/2}),$$

where $x_1 = \mu_1(x, hx)$. Let L be the Lipschitz constant of μ. Then, for small x,

$$\begin{aligned} |x_1| &= |\mu_1(x, hx)| = |\mu_1(x, hx) - \mu_1(x, 0) + \mu_1(x, 0) - \mu_1(0, 0)| \\ &\ge |\mu_1(x, 0) - \mu_1(0, 0)| - |\mu_1(x, hx) - \mu_1(x, 0)| \\ &\ge \frac{1}{L}|x| - L|hx| \ge \left(\frac{1}{L} - KL|x|\right)|x| \ge \frac{1}{2L}|x|; \end{aligned}$$

that is,

$$|x| \leq 2L|x_1|.$$

Since $\mu_2(x, 0) = 0$, we also have

$$\begin{aligned} |x_1|^{3/2} &= |\mu_2(x, hx) - \mu_2(x, 0)| \leq L|hx| \\ &\leq LK|x|^2 \leq LK(2L|x_1|)^2 = 4L^3K|x_1|^2, \end{aligned}$$

which implies that $1 \leq 4L^3K|x_1|^{1/2}$, an impossibility for small x_1 (i.e., for small x). $\square$

The same phenomenon appears to persist in higher dimensions and higher degrees of differentiability—a C^r change of variables can not always convert a C^r immersion to a C^{r+1} immersion. In contrast, here is a theorem whose proof is easy enough once you remember that every closed subset of $\mathbb{R}$ is the zero locus of some smooth function.

THEOREM. *If $h\colon \mathbb{R} \to \mathbb{R}$ is continuous, H is its graph, and X is the x-axis, then there is a homeomorphism of the plane to itself that carries $X \cup H$ to $X \cup G$, where G is the graph of a smooth function.*

In the same vein, it is interesting to note that, in the proof of Theorem A, we made a generic smooth perturbation of τ_L in a special C^1 coordinate system where τ_L appears to be smooth, even though τ_L is intrinsically just C^1. The resulting critical points of τ occur only at good points and are of Morse type (strict maxima and minima). Had we made the perturbation in coordinates where τ_L is merely C^1 and the perturbation is merely C^1 generic, then degenerate critical points would have appeared. In fact, it is also easy to prove the following theorem.

THEOREM. *The critical points of the generic $h \in C^1(\mathbb{R}, \mathbb{R})$ are all degenerate, and they form a Cantor set of Hausdorff dimension zero.*

We close by posing a question that arose in the course of our initial attempts to prove a C^1 Feighn's theorem. Recall that Whitney [8] constructed a C^1 function $w\colon \mathbb{R}^2 \to \mathbb{R}$ that is nonconstant on a connected set of critical points. (His example shows that the differentiability hypothesis in the Morse–Sard theorem is sharp.) The graph of w over a path in the critical set is like a mountain road with so many switchbacks that its tangent is everywhere horizontal although it climbs steadily. We ask whether this Whitney phenomenon can occur simultaneously in many directions, not merely in the vertical direction. More precisely, we ask whether there exists a C^1 embedded surface $W \subset \mathbb{R}^3$ such that, for every direction v in $\mathbb{R}^3$, there is a nontrivial path $\sigma\colon [0, 1] \to W$ along which all the tangent spaces to W are perpendicular to v, $\langle v, T_{\sigma(t)}W\rangle \equiv 0$.

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