Michigan Math. J. 49 (2001)

An Analog of the Classical Invariant Theory for Lie Superalgebras, II

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This paper is a detailed exposition of [S3] with several new results added. It also complements and refines the results of [S2]. Meanwhile there has appeared a paper [J1] where a particular case is considered but in more detail and where other references are offered; see also [J2] and [Y].

1. Preliminaries

In what follows, $\mathfrak{S}_k$ stands for the symmetric group on k elements. Let λ be a partition of the number k and let t be a λ-tableau. Recall that t is called *standard* if the numbers in its rows and columns grow from left to right and downward. Denote by C_t the column stabilizer of t, and let R_t be its row stabilizer. We further set

$$e_t = \sum_{\tau \in C_t;\, \sigma \in R_t} \varepsilon(\tau)\sigma\tau, \qquad \tilde{e}_t = \sum_{\tau \in C_t;\, \sigma \in R_t} \varepsilon(\tau)\tau\sigma. \tag{0.1}$$

Let $\mathbb{N}$ be the set of positive integers, let $\bar{\mathbb{N}}$ be another, "odd", copy of $\mathbb{N}$, and let $\mathbb{M} = \mathbb{N} \coprod \bar{\mathbb{N}}$ be ordered so that each element of the "even" copy ($\mathbb{N}$) is smaller than any element from the "odd" copy; inside of each copy, the order is the natural one. We will call the elements from $\mathbb{N}$ "even" and those from $\bar{\mathbb{N}}$ "odd", so we can encounter an "even" odd element and so forth.

Let I be the sequence of elements from $\mathbb{M}$ of length k. We fill in the tableau t with elements from I, replacing element α with i_α. The sequence I is called *t-semistandard* if the elements of t do not decrease from left to right and downward, the "even" elements strictly increase along columns, and the "odd" elements strictly increase along rows.

The group $\mathfrak{S}_k$ naturally acts on sequences I. Let $\mathfrak{A}$ be the free supercommutative superalgebra with unit generated by $\{x_i\}_{i\in I}$. For any $\sigma \in \mathfrak{S}_k$, define $c(I,\sigma) = \pm 1$ from the equation

$$c(I,\sigma)x_I = x_{\sigma^{-1}I}, \quad \text{where } x_I = x_{i_1}\dots x_{i_k}. \tag{0.2}$$

Clearly, $c(I,\sigma)$ is a cocycle, that is,

$$c(I,\sigma\tau) = c(\sigma^{-1}I,\tau)c(I,\sigma).$$

With the help of this cocycle, a representation of $\mathfrak{S}_k$ in $T^k(V) = V^{\otimes k}$ for any superspace V may be defined as

Received August 2, 2000. Revision received January 22, 2001.

$$\begin{gathered}\sigma v_I = c(I, \sigma^{-1}) v_{\sigma I}, \quad \text{where} \\ v_I = v_{i_1} \otimes \cdots \otimes v_{i_k} \ \text{ and } \ v_{i_\alpha} \in V \ \text{ for each } \alpha.\end{gathered} \tag{0.3}$$

Let $\{v_1, \ldots, v_n; v_{\bar{1}}, \ldots, v_{\bar{m}}\}$ be a basis of V in the standard format (the even elements come first, followed by the odd ones). Then the elements v_I for all possible sequences I of length k and with elements from

$$R_V = \{1, \ldots, n; \bar{1}, \ldots, \bar{m}\} \tag{0.4}$$

form a basis of $T^k(V)$.

The following theorem describes the decomposition of $T^k(V)$ into irreducible $(\mathfrak{S}_k \times \mathfrak{gl}(V))$-modules.

1.1. THEOREM (cf. [S1]). *The commutant of the natural $\mathfrak{gl}(V)$-action on $T^k(V)$ is isomorphic to $\mathbb{C}[\mathfrak{S}_k]$ and*

$$T^k(V) = \bigoplus_{\lambda : \lambda_{n+1} \leq m} S^\lambda \otimes V^\lambda,$$

where S^λ is an irreducible $\mathfrak{S}_k$-module and V^λ is an irreducible $\mathfrak{gl}(V)$-module.

The following refinement of Theorem 1.1 holds.

1.2. THEOREM. *If t runs over the standard tableaux of type λ and if I runs over semistandard t-sequences, then the family $\{e_t(v_I)\}$ (resp., $\{\tilde{e}_t(v_I)\}$) is a basis in $S^\lambda \otimes V^\lambda$. Moreover, for a fixed t the families $\{e_t(v_I)\}$ and $\{\tilde{e}_t(v_I)\}$ span V^λ.*

The proof follows from results of [S1].

Let U and W be two superspaces with bases u_i and w_j for $i \in R_U$ and $j \in R_W$ and where

$$R_U = \{1, \ldots, k; \bar{1}, \ldots, \bar{l}\}, \qquad R_W = \{1, \ldots, p; \bar{1}, \ldots, \bar{q}\}.$$

The symmetric algebra $S^\bullet(U \otimes W)$ is generated by $z_{ij} = u_i \otimes w_j$ for $i \in R_U$ and $j \in R_W$. Let I be a sequence of length N with elements from R_U and let J be a sequence of the same length with elements from R_W. Let $p(i_\alpha)$ and $p(j_\beta)$ be the parities of the corresponding elements of the sequence. Set $\alpha(I, J) = \sum_{\alpha > \beta} p(i_\alpha) p(j_\beta)$ and define an element of $S^\bullet(U \otimes W)$ by setting

$$Z(I, J) = (-1)^{\alpha(I,J)} \prod_{\alpha=1}^{N} Z_{i_\alpha j_\beta}. \tag{1.1}$$

For a given tableau t of order N, we define polynomials

$$\begin{aligned} P_t(I, J) &= \sum_{\sigma \in R_t,\, \tau \in C_t} \varepsilon(\tau) c(I, (\sigma\tau)^{-1}) Z(\sigma\tau I, J), \\ \tilde{P}_t(I, J) &= \sum_{\sigma \in R_t,\, \tau \in C_t} \varepsilon(\tau) c(I, (\tau\sigma)^{-1}) Z(\tau\sigma I, J). \end{aligned}$$

The Lie superalgebras $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ act naturally on $S^\bullet(U \otimes W)$, and their actions commute.

1.3. THEOREM. *$S^{\bullet}(U \otimes W) = \bigoplus_{\lambda} U^{\lambda} \otimes W^{\lambda}$, where U^{λ} and W^{λ} are irreducible $\mathfrak{gl}(U)$- and $\mathfrak{gl}(W)$-modules (respectively) corresponding to the partition λ and where the sum runs over partitions such that $\lambda_{\alpha+1} \leq \beta$ for $\alpha = \min(k, p)$ and $\beta = \min(l, q)$.*

Proof. By Theorem 1.1,

$$W^{\otimes N} = \bigoplus W^{\lambda} \otimes S^{\lambda} \quad \text{and} \quad U^{\otimes N} = \bigoplus U^{\mu} \otimes S^{\mu}.$$

Hence,

$$\begin{aligned} S^N(U \otimes W) &= ((U \otimes W)^{\otimes N})^{\mathfrak{S}_N} = (U^{\otimes N} \otimes W^{\otimes N})^{\mathfrak{S}_N} \\ &= \bigoplus_{\lambda, \mu} (U^{\lambda} \otimes W^{\mu} \otimes S^{\lambda} \otimes S^{\mu})^{\mathfrak{S}_N} \\ &= \bigoplus_{\lambda, \mu} (U^{\lambda} \otimes W^{\mu}) \otimes (S^{\lambda} \otimes S^{\mu})^{\mathfrak{S}_N}. \end{aligned}$$

Since $(S^{\lambda})^* \simeq S^{\lambda}$ and since S^{λ} and S^{μ} are irreducible, we have

$$(S^{\lambda} \otimes S^{\mu})^{\mathfrak{S}_N} = \mathrm{Hom}_{\mathfrak{S}_N}(S^{\lambda}, S^{\mu}) = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

The theorem is proved. □

1.4. THEOREM. *Let t be a standard tableau of type λ and let I and J be t-semistandard sequences. Then the family $P_t(I, J)$, as well as the similar family $\tilde{P}_t(I, J)$, forms a basis in the module $U^{\lambda} \otimes W^{\lambda}$.*

Proof. The natural homomorphism

$$\phi_N \colon U^{\otimes N} \otimes W^{\otimes N} \to S^N(U \otimes W)$$

is clearly a homomorphism of $(\mathfrak{gl}(U) \oplus \mathfrak{gl}(W))$-modules. It is not difficult to verify that

$$\phi_N(e_t(v_I) \otimes \tilde{e}_t(w_J)) = c \cdot P_t(I, J) \quad \text{for a constant } c.$$

Let t be a fixed λ-tableau and let I, J be two t-semistandard sequences with elements from R_U and R_W, respectively. Then, by Theorem 1.2, the vectors $e_t(v_I) \otimes \tilde{e}_t(w_J)$ form a basis of a subspace $L \subset U^{\otimes N} \otimes W^{\otimes N}$ which is also a $(\mathfrak{gl}(U) \oplus \mathfrak{gl}(W))$-submodule. By the same theorem, $L \simeq U^{\lambda} \otimes W^{\lambda}$ and it remains to establish $\phi_N(L) \neq 0$. For this it suffices to show that there exists an $l \in L$ such that $\phi(l) \neq 0$. Since $\phi_N(\sigma v_i \otimes \sigma w_J) = \phi_N(v_I \otimes w_J)$, it follows that

$$\begin{aligned} \phi_N(e_t(v_I) \otimes \tilde{e}_t(w_J)) &= c\phi_N(e_t(v_I) \otimes w_J) \\ &= c\phi_N(\sigma e_t(v_I) \otimes \sigma w_J) = c\phi_N(e_{\sigma t}(\sigma v_I) \otimes \sigma w_J) \\ &= \pm c\phi_N(e_{\sigma t}(v_{\sigma I}) \otimes w_{\sigma J}). \end{aligned}$$

We may therefore assume that the tableau t is consecutively filled in along the rows with the numbers $1, 2, \ldots$. Observe that the sequences σI and σJ remain σt-semistandard.

Let $I = J$ be the sequence

$$\underbrace{1\dots1}_{\lambda_1}\underbrace{2\dots2}_{\lambda_2}\dots\underbrace{\alpha\dots\alpha}_{\lambda_\alpha}\bar{1}\dots\bar{\lambda}_{\alpha+1}\bar{1}\dots\bar{\lambda}_{\alpha+2}\dots\bar{1}\dots\bar{\lambda}_{\gamma},$$

where $(\lambda_1, \lambda_2, \dots, \lambda_\alpha, \dots, \lambda_\gamma)$ is the partition corresponding to t with $\alpha = \min(\dim U_{\bar{0}}, \dim W_{\bar{0}})$ and $\beta = \min(\dim U_{\bar{1}}, \dim W_{\bar{1}})$. It is not difficult to verify that $\phi_N(e_t(v_I) \otimes w_I) \neq 0$.

Since ϕ_N is a homomorphism of $(\mathfrak{gl}(U) \oplus \mathfrak{gl}(W))$-modules, its restriction onto L is an isomorphism. This implies the statement of Theorem 1.4 for the family $P_t(I, J)$; for the family $\tilde{P}_t(I, J)$, the proof is similar. □

Let us elucidate how the results obtained can be applied to invariant theory.

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie superalgebra. By "the invariant theory of $\mathfrak{g}$" we mean the description of $\mathfrak{g}$-invariants in the superalgebra

$$\mathfrak{A}^{p,q}_{k,l} = S((U \otimes V) \oplus (V^* \otimes W)).$$

On $\mathfrak{A}^{p,q}_{k,l}$, the Lie superalgebras $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ act naturally. By Theorem 1.3 we have

$$\mathfrak{A}^{p,q}_{k,l} = \bigoplus_{\lambda,\mu} U^\lambda \otimes V^\lambda \otimes V^{*\mu} \otimes W^\mu.$$

Therefore, to describe $\mathfrak{g}$-invariant elements, it suffices to describe the $\mathfrak{g}$-invariants in $V^\lambda \otimes V^{*\mu} = \mathrm{Hom}(V^\mu, V^\lambda)$. But $(V^\lambda \otimes V^{*\mu})^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V^\mu, V^\lambda)$; that is, the description of $\mathfrak{g}$-invariants is equivalent to the description of $\mathfrak{g}$-homomorphisms of $\mathfrak{g}$-modules V^μ.

Let us consider how the method works in the simplest example: $\mathfrak{g} = \mathfrak{gl}(V)$. Let $\{e_i : i \in R_V\}$ be a basis of V in a standard format, with $\{e_i^*\}$ the left dual basis. Set

$$\theta = \sum_{i \in T} e_i \otimes e_i^*, \qquad \hat{\theta} = \sum_{i \in T} (-1)^{p(i)} e_i^* \otimes e_i.$$

It is not difficult to verify that θ and $\hat{\theta}$ are $\mathfrak{g}$-invariants.

Set

$$T^{p,q}(V) = V^{\otimes p} \otimes V^{*\otimes q}, \qquad \hat{T}^{p,q}(V) = V^{*\otimes p} \otimes V^{\otimes q}.$$

On $T^{p,q}(V)$ and $\hat{T}^{p,q}(V)$, the group $\mathfrak{S}_p \times \mathfrak{S}_q$ acts and its action commutes with that of $\mathfrak{gl}(V)$. Hence, $\mathfrak{S}_p \times \mathfrak{S}_q$ also acts on the space of $\mathfrak{gl}(V)$-invariants in $T^{p,q}(V)$ and $\hat{T}^{p,q}(V)$.

2. Invariants of $\mathfrak{gl}(V)$

Set $v_r{}^* = (x_{r1}, \dots, x_{rn}; x_{r\bar{1}}, \dots, x_{r\bar{m}})$ and $v_s = (x_{1s}{}^*, \dots, x_{ns}{}^*; x_{\bar{1}s}{}^*, \dots, x_{\bar{m}s}{}^*)^t$, where $x_{ri} = u_r \otimes e_i$ and $x_{is} = e_i{}^* \otimes w_s$. That is, $v_r{}^*$ is a row vector and v_s is a column vector, so their scalar product is equal to $(v_r{}^*, v_s) = \sum_i x_{ri} x_{is}{}^*$.

2.1. THEOREM. *The algebra of $\mathfrak{gl}(V)$-invariant elements in $\mathfrak{A}^{p,q}_{k,l}$ is generated by the elements $(v_r{}^*, v_s)$ for all $r \in R_U$ and $s \in R_W$.*

Proof. Let A be a supercommutative superalgebra and L a $\mathfrak{g}$-module. Let $L_A = (L \otimes A)_{\bar{0}}$ and $\mathfrak{g}_A = (\mathfrak{g} \otimes A)_{\bar{0}}$.

The elements of $S^{\cdot}(L^*)$ may be considered as functions on L_A with values in A. Let $l \in L_A = (L \otimes A)_{\bar{0}} = (\mathrm{Hom}(L^*, A))_{\bar{0}}$. Then l determines a homomorphism $\phi_l \colon S(L^*) \to A$. Set

$$f(l) = \phi_l(f) \quad \text{for any} \quad f \in S^{\cdot}(L^*).$$

Observe that $\mathfrak{g}_A$ acts naturally on L_A and on the algebra of functions on L_A.

Let V^p denote $V \oplus \cdots \oplus V$ (p summands), and set $L = V^p \oplus \Pi(V)^q \oplus (V^*)^k \oplus \Pi(V^*)^l$. Then $S(L^*) = \mathfrak{A}^{p,q}_{k,l}$ and we can consider L_A as the set of collections

$$\mathcal{L} = (v_1, \ldots, v_p, v_{\bar{1}}, \ldots, v_{\bar{q}}, v_1^*, \ldots, v_k^*, v_{\bar{1}}^*, \ldots, v_{\bar{l}}^*),$$

where $v_s \in V \otimes A$ and $v_t^* \in \mathrm{Hom}_A(V \otimes A, A)$ and where the parities of these vectors coincide with the parities of their indices.

Let us write the vectors with right coordinates and the covectors with left ones:

$$v_s = \sum_i e_i a_{is}^*, \qquad v_t^* = \sum_i a_{ti} e_i^*.$$

Consider now the elements of $\mathfrak{A}^{p,q}_{k,l}$ as functions on $\mathcal{L}$ by setting

$$x_{is}^*(\mathcal{L}) = a_{is}^*, \qquad x_{ti}^*(\mathcal{L}) = a_{ti}.$$

Therefore, thanks to Statement 2.3 from [S2], it suffices to describe the functions on $\mathcal{L}$ contained in the subalgebra generated by the coordinate functions x_{is}^* and x_{ti}^* and by invariants with respect to $\mathrm{GL}(V \otimes A)$. Because the scalar products turn into scalar products under the $(\mathfrak{gl}(U) \oplus \mathfrak{gl}(W))$-action, it is sufficient to confine ourselves to the invariants in $\mathfrak{A}^{n,m}_{n,m}$.

Denote by M the set of collections $(v_1, \ldots, v_n, v_{\bar{1}}, \ldots, v_{\bar{m}})$ that form bases of $V \otimes A$. In Zariski topology, the set M is dense in the space of all collections. If f is an invariant and $\mathcal{L} \in M$ then there exists a $g \in \mathrm{GL}(V \otimes A)$ such that $gv_i = e_i$ for each $i \in T$. Therefore, $f(\mathcal{L}) = f(g\mathcal{L}) = f(e_1, \ldots, e_n, gv_{\bar{1}}^*, \ldots, gv_{\bar{m}}^*)$ and $f(\mathcal{L})$ is a polynomial in coordinates of the $gv_{\bar{t}}$. But $(gv_{\bar{t}}^*, e_i) = (v_{\bar{t}}^*, g^{-1}e_i) = (v_{\bar{t}}^*, v_i)$, which proves the theorem. $\square$

COROLLARY. *The nonzero $\mathfrak{gl}(V)$-invariants in $T^{p,q}$ exist only if $p = q$. In this case the $(\mathfrak{S}_p \times \mathfrak{S}_p)$-module of invariants is generated by the images of the canonical elements $\theta^{\otimes p}$ in $T^{p,p}$ and $\hat{\theta}$ in $\hat{T}^{p,p}$.*

Consider now the algebra homomorphism

$$S^{\cdot}(U \otimes W) \to (\mathfrak{A}^{p,q}_{k,l})^{\mathfrak{gl}(V)}, \quad u_r \otimes w_s \mapsto (v_r^*, v_s). \tag{2.1}$$

The kernel of this homomorphism is the ideal of relations between the scalar products.

2.2. THEOREM. *The ideal of relations between scalar products* (v_r^*, v_s) *is generated by the polynomials* $P_t(I, J)$*, where* t *is a fixed standard rectangular* $(n+1) \times (m+1)$ *tableau and where* I *and* J *are* t*-semistandard sequences with elements from* R_U *and* R_W*, respectively.*

Proof. By Theorem 1.3, $S^\bullet(U \otimes W) = \bigoplus_\lambda U^\lambda \otimes W^\lambda$ and

$$\begin{aligned}\mathfrak{A}_{k,l}^{p,q} &= S^\bullet(U \otimes V \oplus V^* \otimes W) = S^\bullet(U \otimes V) \otimes S^\bullet(V^* \otimes W)\\ &= \Big(\bigoplus_\mu U^\mu \otimes V^\mu\Big) \otimes \Big(\bigoplus_\nu (V^*)^\nu \otimes W^\nu\Big);\end{aligned}$$

hence

$$(\mathfrak{A}_{k,l}^{p,q})^{\mathfrak{gl}(V)} = \bigoplus_{\mu:\,\mu_{n+1} \le m} U^\mu \otimes W^\mu.$$

Since homomorphism (2.1) is a homomorphism of $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(W))$-modules, its kernel coincides with $\bigoplus_{\lambda:\,\lambda_{n+1} \ge m+1} U^\lambda \otimes W^\lambda$.

Let ν be a $(n+1) \times (m+1)$ rectangle. The condition $\lambda_{n+1} \ge m+1$ means that $\lambda \supset \nu$ and so by Theorem 1.3 it suffices to demonstrate that $P_t(I, J)$, where t is a fixed standard rectangular tableau of size λ, belongs to the ideal generated by $U^\nu \otimes W^\nu$.

Let e_t be the corresponding minimal idempotent and let e_s be the minimal idempotent for a standard tableau s of size ν. Decomposing R_t into the right cosets relative to R_s and decomposing C_t into the left cosets relative to C_s, we obtain a representation of e_t in the form $\sum \tau_i e_s \sigma_j$. This implies that $P_t(I, J)$ is the sum of polynomials of the form $f_i P_{t_i}(I_i, J_j)\phi_j$, that is, it belongs to the ideal generated by the $P_t(I, J)$. □

3. Invariants of $\mathfrak{sl}(V)$

First, let us describe certain tensor invariants. Obviously, all $\mathfrak{gl}(V)$-invariants are also $\mathfrak{sl}(V)$-invariants; we will thus describe only the $\mathfrak{sl}(V)$-invariants that are not $\mathfrak{gl}(V)$-invariants. Denote by $\theta_k = \theta^{\otimes k}$ the invariant in $T^{k,k}$ and by $\hat\theta_k = \hat\theta^{\otimes k}$ the invariant in $\hat T^{k,k}$, and for a given sequence I with elements from R_V set

$$v_I = e_{i_1} \otimes \cdots \otimes e_{i_k}, \qquad v_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*.$$

Let us represent $\mathfrak{sl}(V)$ in the form $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar 0}$ and where $\mathfrak{g}_\pm$ are the $\mathfrak{g}_0$-modules generated by the positive and negative root vectors.

Let $\{X_\alpha\}_{\alpha \in R^-}$ and $\{X_\beta\}_{\beta \in R^+}$, where $R^\pm$ are the sets of positive (negative) roots, be some bases of $\mathfrak{g}_-$ and $\mathfrak{g}_+$, respectively; set $X_- = \prod X_\alpha$ and $X_+ = \prod X_\beta$. The elements $X_\pm$ are uniquely determined up to a constant factor because the subalgebras $\mathfrak{g}_\pm$ are commutative.

3.1. LEMMA. *Let* M *be a* $\mathfrak{g}_0$*-module and let* $\tilde M = \mathrm{ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(M)$ *be the induced* $\mathfrak{g}$*-module. Then each of the correspondences* $m \mapsto X_+X_-m$ *and* $m \mapsto X_-X_+m$ *is a bijection of* $M^{\mathfrak{g}_0}$ *onto* $\tilde M^{\mathfrak{g}}$.

Proof. As follows from Lemma 4.2, $\dim M^{\mathfrak{g}_0} = \dim \tilde{M}^{\mathfrak{g}}$. Hence, it suffices to show that the correspondence $m \mapsto n = X_+X_-m$ is an injective map of $M^{\mathfrak{g}_0}$ to $\tilde{M}^{\mathfrak{g}}$. The injectivity is manifest, so we need only check that the image is $\mathfrak{g}$-invariant. Clearly, $\mathfrak{g}_+n = \mathfrak{g}_0n = 0$. It therefore suffices to verify that $X_{-\alpha}n = 0$ for every simple root α. This is subject to a direct check with the help of the multiplication table in $\mathfrak{gl}(V)$. □

3.2. LEMMA. *Let V_1 and V_2 be finite-dimensional $\mathfrak{g}_0$-modules. Set $\mathfrak{g}_+V_1 = \mathfrak{g}_-V_2 = 0$. Then*

$$\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_+} V_1 \otimes \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_-}(V_2) \simeq \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0}(V_1 \otimes V_2). \tag{3.2.1}$$

is an isomorphism of $\mathfrak{g}$-modules.

Proof. Since the dimensions of both modules are equal, it suffices to show that the natural homomorphism

$$\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0}(V_1 \otimes V_2) \to \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_+}(V_1) \otimes \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_-}(V_2) \tag{3.2.2}$$

is surjective—in other words, that the module generated by $V_1 \otimes V_2$ coincides with the whole module.

The module on the right-hand side has a natural filtration induced by filtrations of the modules $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_+}(V_1)$ and $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_-}(V_2)$. Let the X_α be a basis of $\mathfrak{g}_+$ and the $X_{-\alpha}$ a basis of $\mathfrak{g}_-$. Consider the module W generated by $V_1 \otimes V_2$, that is, by the elements of filtration zero and the element

$$w = X_{-\alpha_1} \dots X_{-\alpha_k} v_1 \otimes X_{\beta_1} \dots X_{\beta_l} v_2.$$

We have

$$\begin{aligned} w &= X_{-\alpha_1}(X_{-\alpha_2} \dots X_{-\alpha_k} v_1 \otimes X_{\beta_1} \dots X_{\beta_l} v_2) \\ &\quad \pm X_{-\alpha_2} \dots X_{-\alpha_k} v_1 \otimes X_{-\alpha_1} X_{\beta_1} \dots X_{\beta_l} v_2 \\ &= X_{-\alpha_1}(X_{-\alpha_2} \dots X_{-\alpha_k} v_1 \otimes X_{\beta_1} \dots X_{\beta_l} v_2) \\ &\quad \pm X_{-\alpha_2} \dots X_{-\alpha_k} v_1 \otimes \left(\sum_i X_{\beta_1} \dots X_{\beta_{i-1}} [X_{-\alpha_1}, X_{\beta_i}] X_{\beta_{i+1}} \dots X_{\beta_l} v_2 \right). \end{aligned}$$

Since each summand is of filtration $< k + l$, they all belong (by the inductive hypothesis) to W; hence, so is $w \in W$. □

Let t be a tableau consisting of m columns and $n+k$ rows, filled in as follows: first we fill in the tableau t_1 that occupies the first n rows and next the tableau t_2 that occupies the remaining rows; both tableaux are filled in consecutively columnwise.

Let s be a tableau consisting of n rows and $k+m$ columns, filled in as follows: first we fill in the tableau s_1 that occupies the first k columns and next the tableau t_2 that occupies the remaining columns; both tableaux are filled in consecutively columnwise.

Let I_k be the sequence obtained by k-fold repetition of the sequence $1, 2, \dots, n$. Let J_k be the sequence consisting of k copies of $\bar{1}$ in a row, k copies of $\bar{2}$ in a row, $\dots$, k copies of $\bar{m}$ in a row.

3.3. THEOREM. *In $\hat{T}^{(m+k)n,(n+k)m}(V)$, the element*

$$e_s \times \tilde{e}_t(v^*_{I_k} \otimes \hat{\theta}_{nm} \otimes v_{J_k})$$

is an $\mathfrak{sl}(V)$-invariant.

Proof. Let N be any positive integer, and consider the map $\phi\colon T^{N,N}(V) \to T^{N,N}(V_{\bar{0}})$ such that $\phi(V_{\bar{1}}) = \phi(V^*_{\bar{1}}) = 0$. Clearly, ϕ is an $(\mathfrak{S}_N \times \mathfrak{S}_N)$-module homomorphism because it is induced by projections of V and V^* onto their even parts.

Take X_+ and X_- from Lemma 3.1 and consider the map

$$\psi\colon T^{N,N}(V_{\bar{0}}) \to T^{N,N}(V), \quad v_0 \mapsto X_+X_-v_0.$$

Clearly, ψ is an $(\mathfrak{S}_N \times \mathfrak{S}_N)$-module homomorphism.

Let us now consider the restrictions of the maps ϕ and ψ onto $T^{N,N}(V)^{\mathfrak{gl}(V)}$ and $T^{N,N}(V_{\bar{0}})^{\mathfrak{gl}(V_{\bar{0}})}$, respectively. It is evident that ϕ sends the first of these spaces into the second one, whereas (by Lemma 3.1) ψ sends the second of these spaces into the first one. Theorem 1.1 implies that, as $(\mathfrak{S}_N \times \mathfrak{S}_N)$-modules, the spaces $T^{N,N}(V)^{\mathfrak{gl}(V)}$ and $T^{N,N}(V_{\bar{0}})^{\mathfrak{gl}(V_{\bar{0}})}$ have simple spectra.

Let $S^\lambda \otimes S^\lambda \subset T^{N,N}(V)^{\mathfrak{gl}(V)}$ and $S^\lambda_0 \otimes S^\lambda_0 \subset T^{N,N}(V_{\bar{0}})^{\mathfrak{gl}(V_{\bar{0}})}$ correspond to a typical diagram λ (both are nonzero; i.e., $\lambda_n \geq m$ and $\lambda_{n+1} = 0$). Then the simplicity of the spectrum, together with Lemma 3.1, implies that ϕ and ψ are (up to a constant factor) mutually inverse isomorphisms of the modules $S^\lambda \otimes S^\lambda$ and $S^\lambda_0 \otimes S^\lambda_0$.

Let

$$C_t = \bigcup_\tau \tau(C_{t_1} \times C_{t_2}) \quad \text{and} \quad R_s = \bigcup_\sigma \sigma(R_{s_1} \times R_{s_2})$$

be the decomposition of the column stabilizer C_t of the tableau t into the left cosets relative to the product of the column stabilizers of t_1 and t_2, and likewise for the row stabilizer R_s. Then

$$\tilde{e}_t = \sum_\tau \varepsilon(\tau)\tau\tilde{e}_{t_1}\tilde{e}_{t_2}, \qquad e_s = \sum_\sigma \sigma e_{s_1}e_{s_2}.$$

It is easy to verify that

$$\mathfrak{g}_+(V_{\bar{0}}) = \mathfrak{g}_-(V^*_{\bar{0}}) = \mathfrak{g}_+(V^*_{\bar{1}}) = \mathfrak{g}_-(V_{\bar{1}}) = 0. \tag{3.3}$$

The vector $X_+e_s(v^*_{I_k} \otimes v^*_{I_m})$ is nonzero, belongs to a typical module, and is a highest one with respect to $\mathfrak{g}_+ \oplus (\mathfrak{g}_0)_+$, where $(\mathfrak{g}_0)_+$ is the set of strictly upper triangular matrices with respect to the fixed basis of $V_{\bar{0}}$. But (3.3) implies that $e_s(v^*_{I_k} \otimes v^*_{J_n})$ is also highest with respect to $\mathfrak{g}_+ \oplus (\mathfrak{g}_0)_+$ and lies in the same module. This shows that

$$X_+e_s(v^*_{I_k} \otimes v^*_{I_m}) = c \cdot e_s(v^*_{I_k} \otimes v^*_{J_n}), \quad \text{where } c \neq 0.$$

Further, from Lemmas 3.1 and 3.2 it follows that the vector

$$X_- X_+[e_s(v^*_{I_k} \otimes v^*_{I_m}) \otimes \tilde{e}_t(v_{I_m} \otimes v_{J_k})]$$

is $\mathfrak{g}$-invariant because $e_s(v^*_{I_k} \otimes v^*_{I_m}) \otimes \tilde{e}_t(v_{I_m} \otimes v_{J_k})$ is $\mathfrak{g}_{\bar{0}}$-invariant. We make use of the fact that $X_+\tilde{e}_t(v^*_{I_m} \otimes v^*_{J_k}) = 0$ to deduce that

$$\begin{aligned} w &= X_- X_+[e_s(v^*_{I_k} \otimes v^*_{I_m}) \otimes \tilde{e}_t(v_{I_m} \otimes v_{J_k})] \\ &= X_-\big[[X_+ e_s(v^*_{I_k} \otimes v^*_{I_m})] \otimes \tilde{e}_t(v_{I_m} \otimes v_{J_k})\big] \\ &= \text{const} \cdot X_-[e_s(v^*_{I_k} \otimes v^*_{J_n}) \otimes \tilde{e}_t(v_{I_m} \otimes v_{J_k})] \\ &= \text{const} \cdot \sum_{\sigma,\tau} \varepsilon(\tau)\sigma \times \tau\big(X_-[e_{s_1}e_{s_2}(v^*_{I_k} \otimes v^*_{I_m}) \otimes \tilde{e}_{t_1}\tilde{e}_{t_2}(v_{I_m} \otimes v_{J_k})]\big) \\ &= \text{const} \cdot \sum_{\sigma,\tau} \varepsilon(\tau)\sigma \times \tau\big(e_{s_1}(v^*_{I_k}) \otimes X_-[e_{s_2}(v^*_{I_m}) \otimes \tilde{e}_{t_1}(v_{I_m})]\tilde{e}_{t_2}(v_{J_k})\big) \\ &= \text{const} \cdot \sum_{\sigma,\tau} \varepsilon(\tau)\sigma \times \tau\big(e_{s_1}(v^*_{I_k}) \otimes X_- X_+[e_{s_2}(v^*_{I_m}) \otimes \tilde{e}_{t_1}(v_{I_m})]\tilde{e}_{t_2}(v_{J_k})\big). \end{aligned}$$

Moreover,

$$\phi(e_{s_2} \times \tilde{e}_{t_1}(\hat{\theta}_{nm})) = e_{s_2} \times \tilde{e}_{t_1}(\phi(\hat{\theta}_{nm})) = e_{s_2} \times \tilde{e}_{t_1}\Big(\sum v^*_L \otimes v_L\Big),$$

where L runs over all the sequences of length nm composed from the integers 1 to n. But, as is not difficult to see,

$$e_{s_2} \times \tilde{e}_{t_1}\Big(\sum v^*_L \otimes v_L\Big) = \text{const} \cdot e_{s_2}(v^*_{I_m}) \otimes \tilde{e}_{t_1}(v_{I_m})$$

and hence

$$X_- X_+ e_{s_2}(v^*_{I_m}) \otimes \tilde{e}_{t_1}(v_{I_m}) = \text{const} \cdot e_{s_2} \times e_{s_1}(\hat{\theta}_{nm}).$$

Therefore,

$$\begin{aligned} w &= \text{const} \cdot \sum_{\sigma,\tau} \varepsilon(\tau)\sigma \times \tau\big(e_{s_1}(v^*_{I_k}) \otimes e_{s_2} \times \tilde{e}_{t_1}(\hat{\theta}_{nm}) \otimes e_{t_2}(v_J)\big) \\ &= e_s \times \tilde{e}_t(v^*_{I_k} \otimes \hat{\theta}_{nm} \otimes v_J), \end{aligned}$$

which proves the theorem. □

Proof of the following theorem is similar.

3.4. Theorem. *The element* $e_s \times \tilde{e}_t(v_{I_k} \otimes \theta_{nm} \otimes v^*_{J_k})$ *in* $T^{nm+kn,nm+km}(V)$ *is* $\mathfrak{sl}(V)$*-invariant.*

3.5. Corollary. *Let* L *be the sequence with elements from* $\mathbb{M}$*. Set*

$$p(L) = \sum p(l_i) \quad \textit{and} \quad \alpha(L, L) = \sum_{i<j} p(l_i)p(l_j).$$

In the notation of Theorems 3.3 and 3.4, the invariant elements can be expressed in the form

$$e_s \times \tilde{e}_t(v^*_{I_k} \otimes \theta_{nm} \otimes v_{J_k}) = \sum_L (-1)^{p(L)+\alpha(L,L)} e_s(v^*_{I_k} \otimes v^*_L) \otimes \tilde{e}_t(v_L \otimes v_{J_k}) \quad (3.5.1)$$

and

$$e_s \times \tilde{e}_t(v_{I_k} \otimes \theta^*_{nm} \otimes v^*_{J_k}) = \sum_L (-1)^{\alpha(L,L)} e_s(v_{I_k} \otimes v_L) \otimes \tilde{e}_t(v^*_L \otimes v^*_{J_k}), \qquad (3.5.2)$$

where the sums run over all the sequences L of length nm with elements from R_V.

Proof. It is easy to verify that $\hat{\theta}_{nm} = \sum_L (-1)^{\alpha(L,L)+p(L)} v^*_L \otimes v_L$, which immediately implies (3.5.1). Formula (3.5.2) is similarly proved. □

Recall from (0.4) and (0.5) the definition of R_U, R_V, R_W. For any sequences I and J, denote by $I * J$ the sequence obtained by appending J to the end of I. Let now I be the sequence of length $(k+m)n$ with elements from R_U and J the sequence of length $(k+n)m$ with elements from R_W; let $\hat{I}$ be the sequence of length $(k+n)m$ with elements from R_U and $\hat{J}$ the sequence of length $(k+m)n$ with elements from R_W. For any sequence L of length nm with elements from R_V, we define:

$$\tilde{P}_s(I, I_k * L) \in S^{\bullet}(U \otimes V), \quad \tilde{P}_t(L * J_k, J) \in S^{\bullet}(V^* \otimes W);$$

$$P_t(\hat{I}, L * J_k) \in S^{\bullet}(U \otimes V), \quad P_s(I_k * L, \hat{J}) \in S^{\bullet}(V^* \otimes W).$$

3.6. THEOREM. *The algebra of $\mathfrak{sl}(V)$-invariant elements in $\mathfrak{A}^{p,q}_{k,l}$ is generated by the elements*

(i) (v^*_r, v_s), *where $r \in R_U$ and $s \in R_W$;*
(ii) $F_k(I, J) = \sum_L (-1)^{\alpha(L,L)} \tilde{P}_s(I, I_k * L) \tilde{P}_t(L * J_k, J)$, *where I is an s-semistandard sequence, J is a t-semistandard one, and L runs over all the sequences of length nm with elements from R_V; and*
(iii) $F_{-k}(\hat{I}, \hat{J}) = \sum_L (-1)^{\alpha(L,L)+p(L)(p(\hat{I})+p(\hat{J}))} P_s(I_k * L, \hat{J}) P_t(\hat{I}, L * J_k)$, *where $\hat{I}$ is an s-semistandard sequence, $\hat{J}$ is a t-semistandard one, and L runs over all the sequences of length nm with elements from R_V.*

Proof. For Young tableaux λ and μ, we have

$$(V^\lambda \otimes V^{*\mu})^{\mathfrak{sl}(V)} = \mathrm{Hom}_{\mathfrak{sl}(V)}(V^\mu, V^\lambda).$$

The dimension of this space is equal to either 0 or 1. It is equal to 1 only if (a) $\lambda = \mu$ or (b) both λ and μ contain a $n \times m$ rectangle and, for any $k \in \mathbb{Z}$, $\lambda_i = \mu_i + k$ for $i = 1, \dots, n$ and $\lambda'_j = \mu'_j + k$ for $j = 1, \dots, m$.

To prove the theorem it suffices to show that, for these λ and μ, the module $V^\lambda \otimes V^{*\mu}$ contains an invariant that can be expressed via the invariants listed in the theorem. By [S2], such an invariant exists. Under the canonical homomorphism of the tensor algebra onto the symmetric one, the invariants of the form (i)–(iii) turn into a system of generators. The theorem is proved. □

To the invariant element in $T^{n(m+k),m(n+k)}(V)$ there corresponds an invariant operator $T^{m(n+k)}(V) \to T^{n(m+k)}(V)$. To describe it, observe that C_t can be represented as $C_t = \coprod_{\pi \in Z} (C_{t_1} \times C_{t_2})\pi$, the decomposition into right cosets relative to the product of the column stabilizers of tableaux t_1 and t_2; let Z be a collection of their representatives. Define

$$D_{J_k}\colon T^{m(n+k)}(V) \to T^{nm}(V), \qquad D_{J_k}(v_1 \otimes v_2) = (-1)^{p(J_k)p(v_1)} v_1 \cdot v^*_{J_k}(v_2).$$

3.7. LEMMA. *Let $\mathcal{L}$ correspond to $e_s \times \tilde{e}_t(v_{I_k} \otimes \theta_{nm} \otimes v^*_{J_k})$ as an invariant operator. Then*

$$\mathcal{L}(e_t(v_L)) = \text{const} \cdot \mathcal{L}(v_L) = e_s\left(v_{I_k} \otimes D^*_{J_k} e_{t_2} \sum_{\pi \in Z} \varepsilon(\pi)\pi v_L\right). \tag{3.7}$$

Proof. To θ_{nm} there corresponds the identity operator id$\colon V^{\otimes nm} \to V^{\otimes nm}$. Hence, to $\theta_{nm} \otimes v^*_{J_k}$ there corresponds the operator $D_{J_k}\colon V^{\otimes m(n+k)} \to V^{\otimes nm}$ and to $v_{I_k} \otimes \theta_{nm} \otimes v^*_{J_k}$ there corresponds the operator $v_{I_k} \otimes D_{J_k}$; finally, to $e_s \times \tilde{e}_t(v_{I_k} \otimes \theta_{nm} \otimes v^*_{J_k})$ there corresponds the operator $e_s(v_{I_k} \otimes D_{J_k})e_t$. Therefore,

$$\begin{aligned}
&\mathcal{L}(e_t(v_L)) \\
&\quad = e_s(v_{I_k} \otimes D_{J_k})e_t^2(v_L) = c_1 e_s(v_{I_k} \otimes D_{J_k})e_t(v_L) = c_1 \cdot \mathcal{L}(v_L) \\
&\quad = c_1 e_s\left(v_{I_k} \otimes D_{J_k} \sum_{\pi} e_{t_1}e_{t_2}\varepsilon(\pi)\pi v_L\right) = c_1 e_s\left(v_{I_k} \otimes e_{t_1} D_{J_k} e_{t_2} \sum_{\pi} \varepsilon(\pi)\pi v_L\right) \\
&\quad = c_1 e_s e_{t_1}\left(v_{I_k} \otimes D_{J_k} e_{t_2} \sum_{\pi} \varepsilon(\pi)\pi v_L\right) = c_1 c_2 e_s\left(v_{I_k} \otimes D_{J_k} e_{t_2} \sum_{\pi} \varepsilon(\pi)\pi v_L\right).
\end{aligned}$$

The last equality follows from $e_s e_{t_1} = c_2 e_s$. □

Let us consider the case $k = 1$ in more detail. Let L be a sequence of length $nm + m$ with elements from R_V, considered as a t-tableau.

In each column L, mark an "odd" element so that all the elements marked—say, $l = (l_1, \dots, l_m)$—are distinct. The pair (L, l) will be called a *marked* tableau. We introduce the following notation: c_i for the parity of the ith column; d_i for the parity of the last element in the ith column; b_i for the parity of the column under the ith marked element; $|b_i|$ for the number of elements in the ith column under the ith marked element; and $\varepsilon(l)$ for the sign of the permutation $l = (l_1, \dots, l_m)$. Set $\varepsilon(L, l) = (-1)^{q(L)}\varepsilon(l)$ and set

$$\varepsilon(L) = c_2 + c_4 + \cdots + d_2 + d_4 + \cdots, \qquad q(L) = b_1 + |b_1| + b_2 + |b_2| + \cdots.$$

3.8. THEOREM. *The invariant operator is of the form*

$$\mathcal{L}(e_t(v_L)) = \text{const} \cdot \mathcal{L}(v_L) = \text{const} \cdot \varepsilon(L) \sum_{(L,l)} \varepsilon(L, l) e_s(v_{I_1} \otimes v_{L \setminus l}), \tag{3.8}$$

where the constant factor does not depend on L.

Proof. Since for the representatives of the cosets of $\mathfrak{S}_{n+1}/\mathfrak{S}_n$ we can take a collection of cycles, we may assume in (3.7) that

$$\pi = \pi_1 \dots \pi_m, \qquad \pi_i \pi_j = \pi_j \pi_i \quad \text{for any } i, j.$$

Hence, $\pi^2 = 1$. Furthermore, $D_{J_1} e_{t_2} \sum \varepsilon(\pi)\pi v_L \neq 0$ if and only if the last row of πL for some π is, up to a permutation, a permutation of $\{\bar{1}, \dots, \bar{m}\}$.

The set of marked tableau (L, l) is in one-to-one correspondence with the set of pairs (L, π) such that the last row of πL is, up to a permutation, $\{\bar{1}, \dots, \bar{m}\}$. Indeed, from the pair (L, l) determine $\pi = \pi_1 \dots \pi_m$, where π_i is the cycle that shifts the elements under the ith marked element one cell up along the column and places the marked one at the bottom. If the marked element lies in the last row, we set $\pi_i = 1$.

Conversely, given π we mark $\pi(k_1), \dots, \pi(k_m)$, where $(k_1, \dots, k_m)$ is the last row of L. Hence, (3.7) implies that

$$\mathcal{L}(v_L) = \sum_{(L,l)} \delta(L, l) e_s(v_{I_1} \otimes v_{L \setminus l}),$$

where $\delta(L, l)$ is a sign depending on (L, l). Direct calculation of this sign leads us to (3.8), proving the theorem. □

4. Absolute Invariants of $\mathfrak{osp}(V)$

Let $A = U(\mathfrak{osp}(V))[\varepsilon]$ be the central extension with the only extra relation $\varepsilon^2 = 1$. Then, introduce on A the co-algebra structure on $U(\mathfrak{osp}(V))$ and setting $\varepsilon \mapsto \varepsilon \otimes \varepsilon$. Assuming that ε acts on V as the scalar operator of multiplication by -1, we may consider V as an A-module. Using the co-algebra structure on A, one can determine a natural A-action in $T^{p,q}(V)$ and $\mathfrak{A}^{p,q}_{k,l}$. We can therefore speak about A-invariants in these modules.

4.1. LEMMA. *Let $\mathfrak{gl}(V) = \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ (as in Section 3) and let M be a $\mathfrak{g}_0$-module. Set $\mathfrak{g}_+ M = 0$. Then there is an isomorphism of $\mathfrak{osp}(V)$-modules*

$$\operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{g}_0 \oplus \mathfrak{g}_+}(M) \simeq \operatorname{ind}^{\mathfrak{osp}(V)}_{\mathfrak{osp}(V)_{\bar{0}}}(M). \tag{4.1}$$

Proof. See [S2, Lemma 5.1]. □

4.2. LEMMA. *Let $\mathfrak{g}$ be a Lie superalgebra, and let the representation of $\mathfrak{g}_{\bar{0}}$ in the maximal exterior power of $\mathfrak{g}_{\bar{1}}$ be trivial. Then there is an isomorphism of vector spaces*

$$\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}(M)^{\mathfrak{g}} \simeq M^{\mathfrak{g}_{\bar{0}}}. \tag{4.2}$$

Proof. See [S2, Lemma 5.2]. □

REMARK. Statements similar to Lemmas 4.1 and 4.2 hold also for $U(\mathfrak{osp}(V))[\varepsilon]$-modules. One can refine Lemma 4.2 and prove that, if $v_0 \in M$ is $\mathfrak{g}_{\bar{0}}$-invariant, then the corresponding $\mathfrak{g}$-invariant vector is of the form $\xi_1 \dots \xi_n v_0 +$ terms of lesser degree.

The presence of an even $\mathfrak{osp}(V)$- and A-invariant form on V determines an isomorphism of A-modules and algebras $\mathfrak{A}^{p,q}_{k,l} = \mathfrak{A}^{p+k,q+l}$. Therefore, we may assume that $k = l = 0$. By definition, the Lie superalgebra $\mathfrak{osp}(V)$ preserves the vector

$$\sum_{i=1}^{n} e_i^* \otimes e_{n-i+1}^* + \sum_{j=1}^{r} \left(e^*_{\overline{m-j+1}} \otimes e^*_{\bar{j}} - e^*_{\bar{j}} \otimes e^*_{\overline{m-j+1}}\right),$$

where $\dim V = (n|2r)$. Therefore, the scalar products

$$(v_s, v_t) = \sum_{i=1}^{n} x^*_{is} x^*_{n-i+1,t} + (-1)^{p(s)} \sum_{j=1}^{r} \left(x^*_{\overline{m-j+1},s} x^*_{\bar{j},t} - x^*_{\bar{j},s} x^*_{\overline{m-j+1},t}\right), \quad (4.3)$$

where $s, t \in R_W$ are $\mathfrak{osp}(V)$- and A-invariants.

4.3. THEOREM. *The algebra of A-invariant elements in $\mathfrak{A}^{p,q} = S^{\boldsymbol{\cdot}}(V^* \otimes W)$ is generated by the elements (v_s, v_t) for $s, t \in R_W$.*

Proof. See [S2, Thm. 5.3]. □

Let I be a sequence of length $2k$ with elements from R_W. Determine an element $X(I) \in S^{\boldsymbol{\cdot}}(S^2(W))$ by setting

$$X(I) = x_{i_1 i_2} \dots x_{i_{2k-1} i_{2k}},$$

where x_{ij} is the canonical image of the element $w_i \otimes w_j \in S^2(W)$.

Let t be a tableau of order $2k$ with rows of even lengths. Then the *even Pfaffian* is defined:

$$\mathrm{Pf}_t(I) = \sum_{\tau \in C_t,\, \sigma \in R_t} \varepsilon(\tau) c(I, (\sigma\tau)^{-1}) X(\sigma\tau I). \quad (4.4)$$

4.4. THEOREM. (a) $S^{\boldsymbol{\cdot}}(S^2(W)) = \oplus W^\lambda$, *where the length of each row of λ is even*.

(b) *Let t be a λ-tableau filled in along rows with the numbers* $1, 2, \dots$. *Then the family* $\mathrm{Pf}_t(I)$ *for the t-standard sequences I is a basis of W^λ.*

Proof. (a) On $T^{2k}(W) = W^{\otimes 2k}$, the group $\mathfrak{S}_{2k}$ and its subgroup $G_k = \mathfrak{S}_k \circ \mathbb{Z}_2^k$ act naturally; that is, $\mathfrak{S}_k$ permutes pairs $(2i-1, 2i)$ whereas $\mathbb{Z}_2^k$ permutes inside each pair. Clearly, $S^k(S^2(W)) = T^{2k}(W)^{G_k}$.

On the other hand, $T^{2k}(W) = \bigoplus S^\lambda \otimes W^\lambda$, so $T^{2k}(W)^{G_k} = \bigoplus (S^\lambda)^{G_k} \otimes W^\lambda$. Hence, in the decomposition of $S^k(S^2(W))$ we enter only W^λ for which $(S^\lambda)^{G_k} \neq 0$ and their multiplicity equals $\dim(S^\lambda)^{G_k}$. However,

$$(S^\lambda)^{G_k} = \mathrm{Hom}_{G_k}(\mathrm{ind}_{G_k}^{\mathfrak{S}_k}(\mathrm{id}), S^\lambda),$$

so the multiplicity of W^λ in $S^k(S^2(W))$ is equal to that of S^λ in $\mathrm{ind}_{G_k}^{\mathfrak{S}_k}(\mathrm{id})$. By [H] this multiplicity is equal to 1 if the lengths of all rows of λ are even and to 0 otherwise. This proves (a).

(b) Consider now the natural map $T^{2k}(W) \to S^k(S^2(W))$. For the tableau t from the conditions of the theorem and the sequence I, the vectors $e_t(w_I)$ form a basis of W^λ. Hence the images of these vectors (which are exactly the $\mathrm{Pf}_t(I)$) form a basis of $W^\lambda \subset S^k(S^2(W))$. $\square$

Consider the algebra homomorphism

$$S^\bullet(S^2(W)) \to S^\bullet(V^* \otimes W), \quad x_{st} \mapsto (v_s, v_t). \tag{4.5}$$

Its kernel is the ideal of relations between scalar products.

4.5. THEOREM. *The ideal of relations between scalar products is generated by polynomials* $\mathrm{Pf}_t(I)$, *where* t *is a* $(2r+2) \times (n+1)$ *rectangle filled in along rows and* I *is a* t*-standard sequence with elements from* R_W.

Proof. By Theorem 4.3,

$$S(V^* \otimes W)^A = \bigoplus_\lambda (V^{*\lambda})^A \otimes W^\lambda = \bigoplus_{\lambda_{n+1} \le 2r} W^\lambda.$$

The kernel of homomorphism (4.5) is therefore equal to $\bigoplus_{\lambda_{n+1} \ge 2r+2} W^\lambda$. We show that this kernel is contained in the ideal generated by W^λ, where λ is a $(2r+2) \times (n+1)$ rectangle.

Let $\mu \supset \lambda$ and let e_s be the corresponding idempotent; then $e_s = \sum \tau_i e_t \sigma_j$. Hence,

$$e_s(J) = \sum \tau_i e_t(\sigma_j J) = \sum e_{\tau_i t}(\tau_i \sigma_j J).$$

Thus, $\mathrm{Pf}_s(J) = \sum_{i,j} f_{ij} \, \mathrm{Pf}_{\tau_i t}(J_{ij})$ and we are done. $\square$

5. Relative Invariants $\mathfrak{osp}(V)$

The invariants of $\mathfrak{osp}(V)$ are, first of all, the ones generated by scalar products. To describe the other invariants, let us describe a certain invariant in the tensor algebra. Let $\dim V = n|m$. For $i \in R_V$, define $\tilde{i}$ by setting

$$\tilde{i} = \begin{cases} n - i + 1 & \text{if } i \text{ is “even”}, \\ \overline{m - i + 1} & \text{if } i \text{ is “odd”}. \end{cases}$$

Let $I = i_1 i_2 \dots i_{2p}$ be a sequence of even length with elements from R_V, and let I^* be the set consisting of the pairs $(i_{2\alpha-1}, i_{2\alpha})$ for $\alpha \le p$ such that $\tilde{i}_{2\alpha-1} \ne i_{2\alpha}$. Let t be a rectangular $n \times m$ tableau consecutively filled in along columns from left to right and let I be a sequence with elements from R_V. We fill in the tableau t with elements from I as follows. Replace α with i_α. Let $\mathcal{T}$ be the set of sequences I such that all rows of t (except the last row) are of the form

$$i_1 \tilde{i}_1 \dots i_r \tilde{i}_r \quad \text{for } r = \tfrac{1}{2} m;$$

the last row J should be such that if $j \in \hat{J}$ then $\tilde{j} \in \hat{J}$ and $\hat{J}$ consist of pairwise distinct "odd" elements.

Let $I \in \mathcal{T}$. Set $r = \frac{1}{2}m$ and let ν be the total amount of marked pairs from the last row consisting of pairwise conjugate "odd" elements that do not belong to $N(L)$. Let $n_1, \dots, n_\nu$ be the multiplicities with which these pairs enter the last row, and set $N = n_1 + \dots + n_\nu$. Let σ_l be the lth elementary symmetric function. Set

$$K(I) = \sum_{q=s}^{s+\nu} (N+1)^r 2^{r-q}(r-q)!\, N^q \sigma_{q-s}(n_1, \dots, n_\nu)$$

and $d(I) = d(I_1)d(I_3)\dots d(I_{2r-1})$, where $d(J) = (-1)^{\alpha(J,J)}$ (cf. (1.1)).

5.1. THEOREM. *In $V^{\otimes n(m+1)}$ lies an $\mathfrak{osp}(V)$-invariant element*

$$\nabla_{m+1} = \sum_{I \in \mathcal{T}} d(I)K(I)e_s(v_{I_1} \otimes v_I). \tag{5.1.1}$$

Proof. Set

$$c(i, \tilde{i}) = \begin{cases} 1 & \text{if } p(i) = 0 \text{ or } i < \tilde{i} \text{ and } p(i) = 1, \\ -1 & \text{if } i > \tilde{i} \text{ and } p(i) = 1. \end{cases}$$

The map

$$V \to V^*, \quad e_i \mapsto c(i, \tilde{i})e^*_{\tilde{i}}$$

is an isomorphism induced by the invariant bilinear form, and

$$\tilde{\theta}_2 = \sum_{i \in R_V} c(i, \tilde{i})e_i \otimes e_{\tilde{i}}$$

is an $\mathfrak{osp}(V)$-invariant.

Let t be a rectangular $(n+1) \times m$ tableau as in Theorem 3.8 and let J be a t-sequence such that, after being filled, each row J is of the form $j_1\tilde{j}_1 \dots j_r\tilde{j}_r$. Denote by $\mathcal{T}_1$ the set of such sequences J. Then

$$\tilde{\theta} = \theta_2^{\otimes \frac{1}{2}(n+1)m} = \sum_{J \in \mathcal{T}_1} d(J)c(J)v_J, \tag{5.1.2}$$

where $J_1, \dots, J_{2r-1}$ are the columns of the tableau t and

$$d(J) = d(J_1)d(J_3)\dots d(J_{2r-1}), \qquad c(J) = c(J_1)c(J_3)\dots c(J_{2r-1})$$

whereas $c(J_\alpha) = \prod_{i \in J_\alpha} c(i, \tilde{i})$.

The element (5.1.2) is an $\mathfrak{osp}(V)$-invariant; applying to it the operator $\mathcal{L}$ from Theorem 3.8 yields another $\mathfrak{osp}(V)$-invariant:

$$\mathcal{L}(\tilde{\theta}) = \sum_J d(J)c(J)\mathcal{L}(v_J) = \sum_{J,l} d(J)c(J)\varepsilon(J)\varepsilon(J,l)e_s(v_{I_1} \otimes v_{J\setminus l}), \tag{5.1.3}$$

where

$$\varepsilon(J) = \prod_{1 \le i \le r} \varepsilon(J_{2i-1} * J_{2i}), \qquad \varepsilon(J,l) = \operatorname{sign}(l) \prod_{1 \le i \le r} \varepsilon(J_{2i-1} * J_{2i}, l_{2i-1} * l_{2i}).$$

For the collection $(J_{2i-1}, J_{2i}, l_{2i-1}, l_{2i})$, define the sequence (I_{2i-1}, I_{2i}) as follows. If l_{2i-1} and l_{2i} lie in the same row then just strike them out; if l_{2i-1} and l_{2i} lie in distinct rows then we strike them out and place their conjugates, $\tilde{l}_{2i-1}$ and $\tilde{l}_{2i}$, in the last row in the same columns. The sequence I takes the form $(I_1, I_2, \dots, I_{2r-1}, I_{2r})$. It is not difficult to verify that

$$e_s(v_{I_1} \otimes v_{J\setminus l}) = \operatorname{sign}(l)\varepsilon(J,l)e_s(v_{I_1} \otimes v_I) \quad \text{and} \quad d(J) = (-1)^r d(I)\varepsilon(J).$$

Therefore,

$$\mathcal{L}(\tilde{\theta}) = (-1)^r \sum_{J,l} \operatorname{sign}(l)c(J)d(I)e_s(v_{I_1} \otimes v_I).$$

The constant factor (the sign) can clearly be replaced with a 1. If I is of the above form then, in the last row, for some values of i the pairs (l_{2i-1}, l_{2i}) are conjugate whereas all the remaining values of i are odd and pairwise distinct—call them $I^* = \{k_1, \dots, k_{2p}\}$. Set

$$\hat{c}(I) = \operatorname{sign}(k_1, \dots, k_{2p})(-1)^p \prod_{c(i_{2\alpha-1}, i_{2\alpha}) \neq 0} c(i_{2\alpha-1}, i_{2\alpha}),$$

where $\operatorname{sign}(k_1, \dots, k_{2p})$ is the sign of the permutation. Then $c(J)\varepsilon(l) = \hat{c}(I)$ and hence

$$\mathcal{L}(\theta_{m+1}) = \sum_{J,l} c(I)d(I)e_s(v_{I_1} \otimes v_I),$$

where the sum runs over pairs (J,l) that give the sequence I. To complete the proof, it suffices to calculate the number of such pairs; this leads to (5.1.1). □

5.2. THEOREM. *The algebra of $\mathfrak{osp}(V)$-invariants is generated by the polynomials*

(i) (v_s, v_t) *for* $s, t \in R_W$ *and*
(ii) $R(J) = \sum_I d(I)K(I)\operatorname{Pf}_s(I_1 * I, J)$ *for every* $I \in \mathcal{T}$ *and every* s*-standard sequence* J *with elements from* R_W.

Proof. Let f be an $\mathfrak{osp}(V)$-invariant that is not A-invariant. Let f depend on $n-1$ even and $2r$ odd generic vectors $v_1, \dots, v_{\overline{2r}}$. Then there exists a $g \in \operatorname{OSp}(V \otimes A)$ such that $g \operatorname{Span}(v_1, \dots, v_{\overline{2r}}) = \operatorname{Span}(e_1, \dots, e_{\overline{2r}})$.

Let $he_n = -e_n$ and $he_i = e_i$ for $i \neq n$; then $\operatorname{ber}(h) = -1$ and $f(hg\mathcal{L}) = -f(g\mathcal{L})$. On the other hand, $f(hg\mathcal{L}) = f(g\mathcal{L})$ and hence $f = 0$. This means that $\mathfrak{osp}(V)$-invariants other than scalar products may only be of type λ, corresponding to a typical module. Thus, in the same vein as for A-invariants, we see that $\dim(V^{*\lambda})^{\mathfrak{osp}(V)} = 1$ if (a) λ is typical and (b) its first n rows are of odd lengths whereas the remaining rows are of even lengths. If we do not consider the scalar products, then no invariants exist for the other (atypical) λ.

Under the canonical homomorphism $T^k(V^*) \otimes T^k(W) \to S^k(V^* \otimes W)$ the module $V^{*\lambda} \otimes W^\lambda$ turns into its copy, and a basis of the first copy becomes a basis of the second one. This shows that if λ is an $n \times (2r+1)$ rectangle then the polynomials $R(J)$ from the theorem constitute a basis of $V^{*\lambda} \otimes W^{\lambda}$, a subspace of $S^k(V^* \otimes W)$.

For an arbitrary λ containing an $n \times (2r+1)$ rectangle, we now apply the same arguments as in the proof of Theorem 2.2. □

6. Invariants of $\mathfrak{pe}(V)$

Suppose that $\dim V = (n|n)$. Let the e_i^* be a basis of $V_{\bar{0}}$, and let the $e_{\bar{i}}^*$ be the dual basis of $V_{\bar{1}}$ with respect to an odd nondegenerate form on V. Then $\mathfrak{pe}(V)$ preserves the tensor $\sum (e_i^* \otimes e_{\bar{i}}^* + e_{\bar{i}}^* \otimes e_i^*)$.

Observe that the scalar products

$$(v_s, v_t) = \sum (-1)^{p(s)} (x_{is}^* \otimes e_{\bar{i},t}^* + e_{\bar{i},s}^* \otimes e_{it}^*) \quad \text{for any } s, t \in S$$

are $\mathfrak{pe}(V)$-invariants. Moreover, the presence of the odd form determines an isomorphism of algebras and $\mathfrak{pe}(V)$-modules $\mathfrak{A}_{k,l}^{p,q} = \mathfrak{A}^{p+l,q+k}$; hence, as in the orthosymplectic case, we may assume that $k = l = 0$.

The compatible $\mathbb{Z}$-grading of $\mathfrak{gl}(V)$ induces compatible $\mathbb{Z}$-gradings of $\mathfrak{pe}(V)$ and $\mathfrak{spe}(V)$:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_- = \Lambda^2(V)$, $\mathfrak{g}_+ = S^2(V^*)$ and $\mathfrak{g}_0 = \mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$. (There is another, isomorphic, representation that we will not use in this paper: $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, where $\mathfrak{g}_- = \Lambda^2(V^*)$, $\mathfrak{g}_+ = S^2(V)$, and $\mathfrak{g}_0 = \mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$.)

Let X_α, $1 \le \alpha \le \frac{1}{2}n(n+1)$, be a basis of $\mathfrak{g}_+$ and let Y_β, $1 \le \beta \le \frac{1}{2}n(n-1)$, be a basis of $\mathfrak{g}_-$. Set

$$X_+ = \prod_{1 \le \alpha \le \frac{1}{2}n(n+1)} X^+\alpha, \qquad Y_- = \prod_{1 \le \beta \le \frac{1}{2}n(n-1)} Y_\beta.$$

Observe that the weight of X_+ with respect to the Cartan subalgebra is equal to $(n+1)\sum \varepsilon_i$ and that the weight of Y_- is equal to $-(n-1)\sum \varepsilon_i$.

6.1. LEMMA. *Let* $L = \operatorname{ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{g}}(M) = \operatorname{ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_-}^{\mathfrak{g}}(N)$ *be a typical irreducible* $\mathfrak{g} = \mathfrak{gl}(V)$*-module. Then there exists an isomorphism of vector spaces*

$$L^{\mathfrak{spe}(V)} = M^{\mathfrak{spe}(V)_{\bar{0}}} = N^{\mathfrak{spe}(V)_{\bar{0}}}$$

given by the formulas

$$M \to L,\ m \mapsto Y_- m \quad \textit{and} \quad N \to L,\ n \mapsto X_+ n.$$

Proof. Consider the two gradings of L,

$$L_k^+ = \operatorname{Span}(f(X^+\alpha)n : n \in N \text{ and } \deg f = k)$$

and

$$L_k^- = \operatorname{Span}(f(Y_\beta)m : m \in M \text{ and } \deg f = k).$$

It is clear that $L_k^+ = L_{n^2-k}^-$.

If l is a $\mathfrak{spe}(V)$-invariant, then $X_\alpha l = 0$ (for $1 \le \alpha \le \frac{1}{2}n(n+1)$) and $l = X_+ f(X_\alpha)n$ for $n \in N$. Therefore, $l = \sum_{r \ge \frac{1}{2}n(n+1)} l_r^+$, where $l_r^+ \in L_r^+$. We similarly establish that $l = \sum_{1 \le s \le \frac{1}{2}n(n-1)} l_s^-$, where $l_s^- \in L_s^-$. Hence, $\sum_{r \ge \frac{1}{2}n(n+1)} l_r^+ = \sum_{1 \le s \le \frac{1}{2}n(n-1)} l_s^-$. Taking into account the equality $L_k^+ = L_{n^2-k}^-$, we deduce that $l \in L^+_{\frac{1}{2}n(n+1)} = L^-_{\frac{1}{2}n(n-1)}$ and that $l = X_+ n = Y_- m$ for some $n \in N$ and $m \in M$. Moreover, it is clear that m and n are $\mathfrak{spe}(V)_{\bar{0}} = \mathfrak{sl}(V_{\bar{0}})$-invariants.

Conversely, if m and n are $\mathfrak{sl}(V_{\bar{0}})$-invariants, then a direct check shows that $X_+ n$ and $Y_- m$ are $\mathfrak{spe}(V)$-invariants. □

6.2. THEOREM. *The algebra of* $\mathfrak{pe}(V)$*-invariants is generated by the scalar products* (v_s, v_t) *for* $s, t \in R_W$.

Proof. See [S2, Sec. 6.2]. □

6.3. Let I be a sequence of length $2k$ composed of elements from R_W. Determine the element $Y(I) \in E^\bullet(S^2(W)) = S^\bullet(\Pi(S^2(W)))$ by setting

$$Y(I) = (-1)^\beta y_{i_1 i_2} \dots y_{i_{2k-1} i_{2k}},$$

where y_{ij} is the canonical image of the element $\omega_i \otimes \omega_j$ and

$$\beta = \sum_{1 \le \alpha \le k} (k - \alpha)(i_{2\alpha-1} + i_{2\alpha}).$$

Let λ be a partition of the form $(\alpha_1, \dots, \alpha_p, \alpha_1 - 1, \dots, \alpha_p - 1)$ in the notation of Frobenius (see [M]). Let t be a tableau of the form λ filled in so that the underdiagonal columns (including the diagonal cells) are filled in consecutively with "odd" numbers while the rows to the right of the diagonal are consecutively occupied by "even" numbers. For a tableau of such a form and a sequence I, the *periplectic Pfaffian* is defined as

$$\mathrm{PPf}_t(I) = \sum_{\tau \in C_t,\, \sigma \in R_t} \varepsilon(\tau) c(I, (\sigma\tau)^{-1}) Y(\sigma\tau I). \tag{6.3.1}$$

6.3.1. THEOREM. *For the tableau t just described, the family* $\mathrm{PPf}_t(I)$ *for the* t*-standard sequences* I *is a basis in the module* $W^\lambda \subset E^\bullet(S^2(W))$.

Proof. From the theory of λ-rings it follows that $E^\bullet(S^2(W)) = \bigoplus W^\lambda$, where the sum runs over the λ of the described form. One can easily verify that, for the tableau as indicated in the formulation of the theorem and for a t-standard sequence I, the image $e_t(w_I)$ in $E^\bullet(S^2(W))$ is nonzero. Hence, for a fixed tableau t, the canonical map $T^{2k}(W) \to E^k(S^2(W))$ performs an isomorphism of $e_t(T^{2k}(W))$ with $W^\lambda \subset E^k(S^2(W))$. This implies the theorem. □

Consider now an algebra homomorphism

$$E^\bullet(S^2(W)) \to S^\bullet(V^* \otimes W), \quad y_{st} \mapsto (v_s, v_t). \tag{6.3.2}$$

6.3.2. Theorem. *The kernel of* (6.3.2) *is generated by polynomials* $\mathrm{PPf}_t(I)$, *where t is of the form of a $(n+1)\times(n+2)$ rectangle and is filled in as described in the previous section and where I is a t-standard sequence with elements from R_W.*

Proof. Clearly,

$$(S^k(V^*\otimes W))^{\mathfrak{pe}(V)}=\Bigg(\bigoplus_{\lambda:\lambda_{n+1}\leq n}V^{*\lambda}\otimes W^{*\lambda}\Bigg)^{\mathfrak{pe}(V)}=\bigoplus_{\lambda:\lambda_{n+1}\leq n}W^{*\lambda},$$

where λ is of the same form as stated in the theorem. Since (6.3.2) is a $\mathfrak{gl}(V)$-module homomorphism, its kernel is $\bigoplus_{\lambda:\lambda_{n+1}\geq n+2}W^{*\lambda}$. That this kernel is generated by the elements of the least degree is proved by the same arguments as for $\mathfrak{osp}(V)$. □

7. Invariants of $\mathfrak{spe}(V)$

First, let us describe certain tensor invariants. Let $\mathcal{T}_1$ be the set of matrices A whose entries are equal to either 1 or 0, with zeroes on the main diagonal and such that $a_{ij}+a_{ji}=1$ for all off-diagonal entries. Set $A_i=\sum a_{pq}$, where the sum runs over all the elements strictly below the ith row.

We define $|A|$ recursively as follows: for $n=2$ set $|A|=0$ and for $n>2$ set

$$|A|=|A^*|+\sum_{i=1}^{n-2}a_{in}A_i^*+\sum_{1\leq j<i<n}a_{in}a_{nj}+\sum_{i>j}a_{ij}+\frac{1}{6}n(n-1)(n-2),$$

where A^* is obtained from A by striking out the last row and the last column.

7.1. Lemma. *The element $Y_-=\prod_{i<j}(E_{\bar{i},j}-E_{\bar{j},i})$ is equal to $\sum_{A\in\mathcal{T}_1}(-1)^{|A|}E_A$, where the product runs over the lexicographically ordered set of pairs $i<j$, $E_A=\prod E_{\bar{i},j}^{a_{i,j}}$, and the last product is taken over the rows of the matrix A from left to right and downward.*

Proof. Clearly, Y_- is the product of $\frac{1}{2}n(n-1)$ factors. In each factor, select either $E_{\bar{i},j}$ or $E_{\bar{j},i}$. For $E_{\bar{i},j}$, set $a_{i,j}=1$ and $a_{j,i}=0$; for $E_{\bar{j},i}$, set $a_{i,j}=0$ and $a_{j,i}=1$. We thus obtain a matrix with the desired properties. The sign is established after reordering of the sequence of the a_{ij}:

$$a_{12}a_{21}a_{13}a_{31}\dots a_{1n}a_{n1}\dots a_{n-1,n}a_{n,n-1}$$
$$\mapsto a_{12}a_{13}\dots a_{1n}\dots a_{n-1,n}a_{n1}a_{n2}\dots a_{n,n-1}.$$

This is performed by induction. First, the pairs $a_{in}a_{ni}$ are moved to the end in increasing order; this accrues the exponent of the sign with $\frac{1}{6}n(n-1)(n-2)$. Then we reorder the elements with indices $<n$; this adds $|A^*|$. Then we rearrange the elements of the sequence $a_{1n}a_{n1}a_{2n}a_{n2}\dots a_{n-1,n}a_{n,n-1}$ into the sequence $a_{1n}\dots a_{n-1,n}a_{n1}a_{n2}\dots a_{n,n-1}$; this adds $\sum_{j<i}a_{in}a_{ni}$ to the exponent. Finally, the elements $a_{1n},\dots,a_{n-1,n}$ are placed onto the end of the ith row, adding $\sum_{i=1}^{n-2}a_{in}A_i^*$. Besides, if $i>j$ then $E_{\bar{i},j}$ enters Y_- with a negative sign; this adds $\sum_{i>j}a_{ij}$. □

The numbers i and j will be referred to as *conjugate* if $i = \bar{j}$, that is, if they are equal but belong to copies of $\mathbb{N}$ of distinct "parity".

Let $\mathcal{T}_2$ be the set of sequences of length n^2 considered as $n \times n$ tableaux filled in along columns and with the following properties: the numbers symmetric with respect to the main diagonal are conjugate; the (i, j)th position is occupied with one of the numbers i or $\bar{j}$; and the main diagonal is filled in with "odd" numbers $\bar{1}, \dots, \bar{n}$.

For every $L \in \mathcal{T}_2$, determine the matrix $A = (a_{ij})$ by setting $a_{ij} = p(l_{ij})$. Set $n(L) = \#(\text{“even” elements in } L)$ and $m(L) = m(A) = \sum_{i \text{ is “even”}} a_{ij}$. Let $\varepsilon(L) = (-1)^{|A|+n(L)}$ and

$$m_k(L) = \frac{((n+k)!)^n}{(n+k-l_1)! \dots (n+k-l_n)!}, \quad \text{where } l_i = \sum_j a_{ij}.$$

7.2. THEOREM. *The elements*

$$e_t\Big(\sum (-1)^{km(L)} \varepsilon(L) m_k(L) v_L^* \otimes v_{J_k}^*\Big) \quad \textit{for } L \in \mathcal{T}_2$$

and

$$e_t\Big(\sum \varepsilon(L) m_0(L) v_L^* \otimes v_{I_k}^*\Big) \quad \textit{for } L \in \mathcal{T}_2$$

are $\mathfrak{spe}(V)$*-invariant.*

Proof. Let r be an $(n+k) \times n$ rectangle filled in along columns, and set $w = v_{J_{n+k}}^*$. Denote by $w_{i_1}^{j_1} \dots w_{i_p}^{j_p}$ the tensor obtained from w by replacing the elements occupying positions $i_1, \dots, i_p$ with numbers $j_1, \dots, j_p$, respectively. Then

$$e_r(E_{\bar{n},j} w) = (-1)^{i-1} e_r(w_i^j)(n+k),$$

where i is any of the numbers of the positions occupied by $\bar{n}$.

If $E_{A_n} = \prod E_{\bar{n},j}^{\alpha_{n,j}}$ with the product ordered by increasing indices j, then

$$e_r(E_{A_n} w) = (-1)^{i_1-1+\dots+i_l-1} \frac{(n+k)!}{(n+k-l)!} e_r(w_{i_1,\dots,i_l}^{j_1,\dots,j_l}),$$

where $\{j_1, \dots, j_l\} = \{j \mid \alpha_{n,j} \neq 0\}$, $l = \sum_j \alpha_{n,j}$, and $i_1 < \dots < i_l$.

Assume that $\{i_1, \dots, i_l\} = \{a + j_1 - 1, \dots, a + j_l - 1\}$, where a is the number of the first element in the nth column of tableau r. We thus have

$$e_r(E_{A_n} w) = (-1)^{l \cdot a + j_1 + \dots + j_l} \frac{(n+k)!}{(n+k-l)!} e_r(w_{i_1,\dots,i_l}^{j_1,\dots,j_l}).$$

By continuing the process we obtain

$$e_r(E_A w) = (-1)^{\varepsilon(A)} \frac{[(n+k)!]^n}{(n+k-l_1)! \dots (n+k-l_n)!} e_r(v_{I_A}^*).$$

The index $\varepsilon(A) = a_1 + \dots + a_n + n(A_n) + n(A_{n-1}) + \dots + n(A_1)$, where a_i is the number of the first element in the ith column and $n(A_i)$ is equal to the sum of the numbers of the places occupied by the 1s; I_A coincides with J_{n+k} everywhere unless $a_{ij} = 1$, in which case the (ij)th entry of I_A is occupied by j.

Since

$$I(\sigma^{-1}\alpha) = t(\sigma^{-1}\alpha) = s(\alpha) = J(\alpha),$$

we deduce that

> *if I and J are two sequences and if t and s are two tableaux of the same form such that, after filling t with the elements from I and s with the elements from J, one obtains geometrically identical pictures, then $\sigma t = s$ implies $\sigma I = J$.*

Therefore, if $\sigma t = r$ then we have

$$\begin{aligned} Y_- e_t(v^*_{J_n} v^*_{J_k}) &= Y_- e_{\sigma^{-1} r}(v^*_{\sigma^{-1}(J_{n+k})}) \\ &= Y_- \sigma^{-1} e_{r\sigma}(\sigma^{-1} v^*_{J_{n+k}}) \cdot c(J_{n+k}, \sigma) = c(J_{n+k}, \sigma)\sigma^{-1} Y_- e_r(v^*_{J_{n+k}}), \end{aligned}$$

because $c(J_{n+k}, \sigma) = \mathrm{sign}(\sigma)$—thanks to the fact that J_{n+k} contains only “odd” elements. Hence,

$$\begin{aligned} &Y^- e_t(v^*_{J_n} v^*_{J_k}) \\ &= \mathrm{sign}(\sigma)\sigma^{-1} e_r \left(\sum_A (-1)^{\varepsilon(A)} \frac{[(n+k)!]^n}{(n+k-l_1)! \dots (n+k-l_n)!} v^*_{I_A} \right) \\ &= \mathrm{sign}(\sigma) e_t \left(\sum_A (-1)^{\varepsilon(A)} \frac{[(n+k)!]^n}{(n+k-l_1)! \dots (n+k-l_n)!} \sigma^{-1} v^*_{I_A} \right) \\ &= \mathrm{sign}(\sigma) e_t \left(\sum_A (-1)^{\varepsilon(A)} \frac{[(n+k)!]^n}{(n+k-l_1)! \dots (n+k-l_n)!} c(I_A, \sigma) v^*_{J_A} \otimes v^*_{J_k} \right), \end{aligned}$$

where

$$c(I_A, \sigma) = |A_2| \cdot k + |A_4| \cdot k \cdots = k \sum_{i \text{ is “even”}} a_{ij}$$

and where J_A coincides with J_n everywhere unless $a_{ij} = 1$, in which case the (i, j)th position is occupied by j. This completes the proof of Theorem 7.2. □

7.3. THEOREM. *The algebra of $\mathfrak{spe}(V)$-invariant polynomials is generated by the following elements*:

(i) (v_α, v_β) *for* $\alpha, \beta \in R_W$; *and, for* $k \geq 1$ *and sums that run over* $L \in \mathcal{T}_2$,

(ii) $\mathrm{PPf}_k(J) = \sum_L (-1)^{(k-1)m(L)} \varepsilon(L) m_{k-1}(L) P_t(L * J_k, J)$ *for any t-standard sequence J*;

(iii) $\mathrm{PPf}_{-k}(J) = \sum_L \varepsilon(L) m_0(L) P_t(L * I_{k+1}, J)$ *for any s-standard sequence J.*

The proof is similar to that of Theorem 5.2. □

ACKNOWLEDGMENT. I am thankful to D. Leites for support and help.

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