Michigan Math. J. 49 (2001)

An Analog of the Classical Invariant Theory for Lie Superalgebras

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This is a detailed version of my 1992 short announcement [S3]. For prerequisites on Lie superalgebras see Appendix 0 and Appendix 1, which are mainly borrowed from Leites's book [L3]. A draft of this paper was put on the net (math.RT/9810113) "earlier and independently", as Cheng and Wang referred to it in their papers [CW1; CW2], where they elucidate some of the results given here and also give an interpretation of a formula for projective symmetric functions. Still, a further elucidation will not hurt, and I intend to return to it elsewhere. Meanwhile, recall that Howe suggested a unified approach to the first and second theorems of the classical invariant theory: compare [Wy] with [H]. This approach becomes even more unified in [LSh], where Lie superalgebras that more or less implicitly linger in the background of [H] become the main characters. In this and a subsequent paper I consider analogs of these theorems for "classical" Lie superalgebras.

Related are problems on description of the centers of $U(\mathfrak{g})$ (cf. [LS; S1; S5]). The pioneer here was Berezin [B1; B2; B3], who somewhat differently considered, to an extent, $\mathfrak{g} = \mathfrak{gl}$, $\mathfrak{sl}$, and $\mathfrak{osp}$. Scheunert [Sch1; Sch2; Sch3; Sch4] has reproduced some of my results.

The reader should be aware of a totally different approach to invariant theory due to Shander [Sd1; Sd2], who justly observes that for Lie superalgebras it is possible not to restrict oneself to the study of polynomial functions and makes a step in this purely super direction.

1. Setting of the Problem. Formulation of the Results

1.0. Let V be a finite-dimensional superspace over $\mathbb{C}$ and let $\mathfrak{g}$ be an arbitrary *matrix* Lie superalgebra, that is, a Lie subsuperalgebra in $\mathfrak{gl}(V)$. Under the *classical invariant theory* for $\mathfrak{g}$ we mean the description of $\mathfrak{g}$-invariant elements of the algebra

$$\mathfrak{A}^{p,q}_{k,l} = S^{\boldsymbol{\cdot}}(V^p \oplus \Pi(V)^q \oplus V^{*k} \oplus \Pi(V)^{*l}),$$

where V^p denotes the direct sum of p copies of V. Clearly,

$$\mathfrak{A}^{p,q}_{k,l} = S^{\boldsymbol{\cdot}}(U \otimes V \bigoplus V^* \otimes W),$$

where $\dim U = (p, q)$ and $\dim W = (k, l)$. Therefore, Lie superalgebras $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ also act on $\mathfrak{A}^{p,q}_{k,l}$; hence, the enveloping algebra $U(\mathfrak{gl}(U \otimes W))$ also

Received August 2, 2000. Revision received January 22, 2001.

acts on $\mathfrak{A}^{p,q}_{k,l}$. The elements of $U(\mathfrak{gl}(U \otimes W))$ will be called *polarization operators.* These operators commute with the natural $\mathfrak{gl}(V)$-action.

A set $\mathfrak{M}$ of $\mathfrak{g}$-invariants will be called *basic* if the algebra of invariants coincides with the least subalgebra containing $\mathfrak{M}$ and invariant with respect to polarization operators.

Hereafter, $\mathfrak{g}$ is one of the “classical” Lie superalgebras (i.e., simple ones, their central extensions, and the Lie superalgebras of derivations of the simple ones); for such $\mathfrak{g}$, I describe a basic set of invariants $\mathfrak{M}$.

Let us introduce $\mathbb{Z}_2$-graded sets, the disjoint unions of the “even” and “odd” elements (the odd elements are barred):

$$T = T_{\bar 0} \coprod T_{\bar 1} = \{1, \dots, k, \bar 1, \dots, \bar l\},$$

$$S = S_{\bar 0} \coprod S_{\bar 1} = \{1, \dots, p, \bar 1, \dots, \bar q\},$$

$$I = I_{\bar 0} \coprod I_{\bar 1} = \{1, \dots, n, \bar 1, \dots, \bar m\}.$$

In the spaces U, W, V we select bases such that the parity of the basis vector coincides with the parity of its index:

$$\{u_t\}_{t\in T}, \quad \{w_s\}_{s\in S}, \quad \{e_i\}_{i\in I}.$$

In V^*, select the basis $\{e_i^*\}_{i\in I}$ that is left dual to $\{e_i\}_{i\in I}$. Let

$$x_{ti} = u_t \otimes e_i \quad \text{and} \quad x^*_{is} = e^*_i \otimes w_s.$$

Let v_s be the column vector with coordinates $x^*_{1s}, \dots, x^*_{\bar m s}$, and let v^*_t be the row vector with coordinates $x_{t1}, \dots, x_{t\bar m}$. Define the scalar product by setting

$$(v^*_t, v_s) = \sum_{i\in I} x_{ti}x^*_{is} \quad \text{for any } \ v_s \in V, \ v^*_t \in V^*.$$

Set

$$\Delta = \det(x_{ti})_{t,i\in I_{\bar 0}}, \quad \Delta^* = \det(x^*_{is})_{i,s\in I_{\bar 0}};$$

$$\omega = \det(x_{ti})_{t,i\in I_{\bar 1}}, \quad \omega^* = \det(x^*_{is})_{i,s\in I_{\bar 1}}.$$

1.1. THEOREM. *As a basic set of invariants of $\mathfrak{gl}(V)$ in $\mathfrak{A}^{p,q}_{k,l}$, one can take the collection of scalar products*

$$(v^*_t, v_s), \quad \textit{where } \ t \in T, \ s \in S.$$

(*This statement clearly holds for nonempty T and S; if at least one of T and S is empty then there are no invariants.*)

1.1.1. COROLLARY. *The scalar products (v^*_s, v_t), where $t \in T$ and $s \in S$, constitute a system of generators of $\mathfrak{gl}(V)$-invariants.*

1.2. THEOREM. *As a basic set of invariants of $\mathfrak{sl}(V)$ in $\mathfrak{A}^{p,q}_{k,l}$, one can take the set consisting of*

(a) *basic invariants of* $\mathfrak{gl}(V)$ *and*

(b) *the collection of the following polynomials* $f_{\pm k}$ *for* $k \in \mathbb{N}$:

$$f_k = (\Delta^*)^k \omega^k \prod_{t \in I_{\bar{1}},\, s \in I_{\bar{0}}} (v_t^*, v_s) \quad \textit{and} \quad f_{-k} = \Delta^k (\omega^*)^k \prod_{t \in I_{\bar{0}},\, s \in I_{\bar{1}}} (v_t^*, v_s).$$

Let us use $\mathfrak{osp}(V)$ to denote the *orthosymplectic* Lie superalgebra preserving the tensor

$$\sum_{i \in I_{\bar{0}}} e_i^* \otimes e_{n-i+1}^* + \sum_{j=1}^{r} \left(e^*_{\overline{m-j+1}} \otimes e^*_{\bar{j}} - e^*_{\bar{j}} \otimes e^*_{\overline{m-j+1}} \right) \quad \text{for } m = 2r.$$

Then the inner products

$$(v_s, v_t) = \sum_{i \in I_{\bar{0}}} x^*_{is} x^*_{n-i+1t} + (-1)^{p(s)} \sum_{j=1}^{r} \left(x^*_{\overline{m-j+1}s} x^*_{\bar{j}t} - x^*_{\bar{j}s} x^*_{\overline{m-j+1}t} \right)$$

are clearly $\mathfrak{osp}(V)$-invariant. In what follows we will show that there also exists an invariant polynomial Ω (*Pfaffian*) such that

$$\Omega^2 = (\det(v_s, v_t)_{s,t \in I_{\bar{0}}})^{2r+1}.$$

The existence of an even $\mathfrak{osp}(V)$-invariant form determines an isomorphism of algebras as well as of $\mathfrak{osp}(V)$-modules $\mathfrak{A}^{p,q}_{k,l} \simeq \mathfrak{A}^{p+k,q+l}$. Therefore, we may (and will) confine ourselves to the case $k = l = 0$.

1.3. THEOREM. *As a basic set of invariants of* $\mathfrak{osp}(V)$ *in* $\mathfrak{A}^{p,q}$, *one can take the set consisting of*

(a) *the scalar products* (v_s, v_t) *for* $s, t \in S$ (*for nonempty* S); *and*

(b) *the polynomial* Ω.

1.4. Let $\dim V = (n, n)$; let us use $\mathfrak{pe}(V)$ to denote the *periplectic* Lie superalgebra preserving the tensor

$$\sum_{i \in I_{\bar{0}}} (e_i^* \otimes e_{\bar{i}}^* + e_{\bar{i}}^* \otimes e_i^*).$$

The inner products

$$(v_s, v_t) = \sum_{i \in I_{\bar{0}}} ((-1)^{p(s)} x^*_{is} x^*_{\bar{i}t} + x^*_{\bar{i}s} x^*_{it})$$

are clearly $\mathfrak{pe}(V)$-invariants.

The existence of an odd $\mathfrak{pe}(V)$-invariant form determines an isomorphism of algebras as well as of $\mathfrak{pe}(V)$-modules $\mathfrak{A}^{p,q}_{k,l} \simeq \mathfrak{A}^{p+l,q+k}$. Therefore, we may (and will) assume that $k = l = 0$.

1.4.1. THEOREM. *As a basic set of invariants of* $\mathfrak{pe}(V)$ *in* $\mathfrak{A}^{p,q}$, *one can take the set of the inner products* (v_s, v_t) *for* $s, t \in S$ (*for nonempty* S).

1.4.1.1. COROLLARY. *The inner products form a system of generators of the algebra of* $\mathfrak{pe}(V)$*-invariants.*

1.4.2. THEOREM. *As a basic set of invariants of $\mathfrak{spe}(V)$ in $\mathfrak{A}^{p,q}$, one can take the set formed by*

(a) *the basic invariants for $\mathfrak{pe}(V)$ and*
(b) *the collection of the following polynomials $p_{\pm k}$ for $k \in \mathbb{N}$:*

$$p_k = \Delta^{*k} \prod_{s\le t,\ s,t\in I_{\bar 0}} (v_s, v_t) \quad \text{and} \quad p_{-k} = \omega^{*k} \prod_{s<t,\ s,t\in I_{\bar 1}} (v_s, v_t).$$

1.5. Let $\dim V = (n, n)$, and let $\mathfrak{q}(V)$ denote the Lie superalgebra preserving the tensor

$$\sum_{i\in I_{\bar 0}} (e_i \otimes e^*_{\bar i} + e_{\bar i} \otimes e^*_i).$$

The expression

$$[v^*_t, v_s] = \sum (x_{it} x^*_{\bar i s} + x_{\bar i s} x^*_{is}) \quad \text{for any } t \in T_{\bar 0},\ s \in S_{\bar 0}$$

is clearly a $\mathfrak{q}(V)$-invariant. Since there is an isomorphism of algebras $\mathfrak{A}^{p,q}_{k,l} \simeq \mathfrak{A}^{p+q}_{k+l}$ as well as of $\mathfrak{q}(V)$-modules, we may (and will) assume that $q = l = 0$.

1.5.1. THEOREM. *As a basic set of invariants of $\mathfrak{q}(V)$ in $\mathfrak{A}^p_k$, one can take the collection of inner products*

$$(v^*_t, v_s),\ [v^*_t, v_s] \quad \textit{for any } t \in T_{\bar 0},\ s \in S_{\bar 0}.$$

(*This statement clearly holds for nonempty $T_{\bar 0}$ and $S_{\bar 0}$; if at least one of them is empty then there are no invariants.*)

COROLLARY. *The inner products form a system of generators of the algebra of $\mathfrak{q}(V)$-invariants.*

Let Z be a matrix of the form

$$Z = \begin{pmatrix} Z_0 & Z_1 \\ Z_1 & Z_0 \end{pmatrix}, \quad \text{where } Z_0 = \{(v^*_t, v_s)\}_{t,s\in I_{\bar 0}},\ Z_1 = \{[v^*_t, v_s]\}_{t,s\in I_{\bar 0}},$$

and let Y be a matrix of the form

$$Y = \begin{pmatrix} Y_0 & Y_1 \\ Y_1 & Y_0 \end{pmatrix}, \quad \text{where } Y_0 = \{x^*_{is}\}_{i,s\in I_{\bar 0}},\ Y_1 = \{x^*_{it}\}_{i\in I_{\bar 0},\, t\in I_{\bar 1}}. \tag{1.5.0}$$

In what follows we will prove that, for any partition $\lambda = (\lambda_1, \dots, \lambda_n)$, where

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0, \tag{1.5.1}$$

the expression composed of queer traces and the queer determinant

$$q_\lambda = \operatorname{qtr} Z^{\lambda_1} \cdots \operatorname{qtr} Z^{\lambda_n} \cdot \operatorname{qet} Y$$

is a polynomial.

1.5.2. THEOREM. *As a basic set of invariants of $\mathfrak{sq}(V)$, one can take*

(a) *the basic invariants for $\mathfrak{q}(V)$ and*
(b) *the polynomials q_λ, where λ runs over all partitions of the form* (1.5.1).

2. Preparatory Theorems

2.1. THEOREM. *Let* $\dim V = (n, m)$ *and* $\dim U = (k, l)$ *with* $k \geq n$ *and* $l \geq m$. *Then the algebra* $S^{\bullet}(U \otimes V)$ *considered as a* $\mathfrak{gl}(U) \oplus \mathfrak{gl}(V)$*-module can be represented in the form*

$$S^{\bullet}(U \otimes V) = \bigoplus_{\lambda} U^{\lambda} \otimes V^{\lambda},$$

where λ *runs over the set of Young tableaux such that* $\lambda_{n+1} \leq m$ *and* U^{λ} *and* V^{λ} *are irreducible* $\mathfrak{gl}(U)$*- and* $\mathfrak{gl}(V)$*-modules corresponding to the tableau* λ.

Proof. By [S2] we have the following decompositions:

$$V^{*\otimes k} = \bigoplus_{\lambda}(V^{*\lambda} \otimes S^{\lambda}), \qquad U^{\otimes k} = \bigoplus_{\mu}(U^{\mu} \otimes S^{\mu}).$$

Here S^{λ} and S^{μ} are irreducible $\mathfrak{S}_k$-modules corresponding to the tableaux λ and μ, respectively.

We then have the following isomorphisms:

$$\begin{aligned} S^k(U \otimes V) &= S^k(U \otimes (V^*)^*) = S^k(\mathrm{Hom}(V^*, U)) \\ &= \mathrm{Hom}_{\mathfrak{S}_k}(V^{*\otimes k}, U^{\otimes k}) = \mathrm{Hom}_{\mathfrak{S}_k}\Big(\bigoplus_{\lambda}(V^{*\lambda} \otimes S^{\lambda}), \bigoplus_{\mu}(U^{\mu} \otimes S^{\mu})\Big) \\ &= \bigoplus_{\lambda,\mu} \mathrm{Hom}(V^{*\lambda}, U^{\mu}) \otimes \mathrm{Hom}_{\mathfrak{S}_k}(S^{\lambda}, S^{\mu}) = \bigoplus_{\lambda} U^{\lambda} \otimes (V^{*\lambda})^* \\ &= \bigoplus_{\lambda}(U^{\lambda} \otimes V^{\lambda}). \end{aligned}$$

All these isomorphisms are $\mathfrak{gl}(U)$- and $\mathfrak{gl}(V)$-isomorphisms. □

The following theorem is similar to Theorem II.5.A from [Wy].

2.2. THEOREM. *Let* $\mathfrak{g}$ *be a Lie subsuperalgebra in* $\mathfrak{gl}(V)$. *If* $\mathfrak{M}$ *is a basic system of* $\mathfrak{g}$*-invariants in* $\mathfrak{A}^{n,m}_{n,m}$, *then* $\mathfrak{M}$ *is also a basic system of invariants in the algebra* $\mathfrak{A}^{p,q}_{k,l}$ *for any* k, $p \geq n$ *and* q, $l \geq m$.

Proof. Let $U_1 \subset U$, $W_1 \subset W$, and $\dim U_1 = \dim W_1 = (n, m)$. By Theorem 2.1 we have

$$\begin{aligned} S^{\bullet}(U_1 \otimes V \oplus V^* \otimes W_1) &= S^{\bullet}(U_1 \otimes V) \otimes S^{\bullet}(V^* \otimes W_1) \\ &= \bigoplus_{\lambda,\mu}(U_1^{\lambda} \otimes V^{\lambda} \otimes V^{*\mu} \otimes W_1^{\mu}), \end{aligned}$$

where λ, μ are Young tableaux such that $\lambda_{n+1} \leq m$ and $\mu_{n+1} \leq m$.

Similarly,

$$S^{\bullet}(U \otimes V \oplus V^* \otimes W) = \bigoplus_{\lambda,\mu}(U^{\lambda} \otimes V^{\lambda} \otimes V^{*\mu} \otimes W^{\mu}),$$

where λ and μ are the same here as in the previous expansion.

The embeddings $U_1 \hookrightarrow U$ and $W_1 \hookrightarrow W$ induce an embedding $\varphi\colon \mathfrak{A}^{n,m}_{n,m} \hookrightarrow \mathfrak{A}^{p,q}_{k,l}$ that is a $(\mathfrak{gl}(U_1) \oplus \mathfrak{gl}(W_1))$-homomorphism. Set

$$(\mathfrak{A}^{p,q}_{k,l})_{\lambda,\mu} = U^{\lambda} \otimes V^{\lambda} \otimes V^{*\mu} \otimes W^{\mu}.$$

Hence $\varphi((\mathfrak{A}^{n,m}_{n,m})_{\lambda,\mu}) \subset (\mathfrak{A}^{p,q}_{k,l})_{\lambda,\mu}$, and since φ is also a $\mathfrak{gl}(V)$-homomorphism it follows that $\varphi((\mathfrak{A}^{n,m}_{n,m})^{\mathfrak{g}}) \subset (\mathfrak{A}^{p,q}_{k,l})^{\mathfrak{g}}$. The elements of the space $(\mathfrak{A}^{p,q}_{k,l})^{\mathfrak{g}}_{\lambda,\mu}$ will be called $\mathfrak{g}$-*invariants of type* (λ, μ).

Let f be a $\mathfrak{g}$-invariant highest vector of type (λ, μ) with respect to a Borel subalgebra in $\mathfrak{gl}(U_1) \oplus \mathfrak{gl}(W_1)$. If we select a Borel subalgebra in $\mathfrak{gl}(U) \oplus \mathfrak{gl}(W)$ that preserves U_1 and W_1, then f is still a highest vector for such a subalgebra. This proves that, as a $(\mathfrak{gl}(U) \oplus \mathfrak{gl}(W))$-module, $(\mathfrak{A}^{k,l}_{p,q})^{\mathfrak{g}}_{\lambda,\mu}$ is generated by the subspace $\varphi((\mathfrak{A}^{n,m}_{n,m})^{\mathfrak{g}}_{\lambda,\mu})$ and the theorem is proved. □

2.2.1. REMARK. One can similarly show that:

(a) if $\mathfrak{M}$ is a basic system of invariants for $\mathfrak{A}^{n,m}$ then it is also a basic system of invariants for $\mathfrak{A}^{p,q}$, where $p \geq n$ and $q \geq m$;

(b) if $\mathfrak{M}$ is a basic system of invariants for $\mathfrak{A}^{n}_{n}$ then it is also a basic system for any algebra $\mathfrak{A}^{p}_{k}$, where $p, k \geq n$.

Let A be a supercommutative superalgebra over $\mathbb{C}$ and let L be a $\mathfrak{g}$-module; set $L_A = (L \otimes A)_{\bar{0}}$ and $\mathfrak{g}_A = (\mathfrak{g} \otimes A)_{\bar{0}}$. Then the elements of $S^{\boldsymbol{\cdot}}(L^*)$ can be considered as functions on L_A with values in A. Let $l \in L_A = (L \otimes A)_{\bar{0}} = (\mathrm{Hom}(L^*, A))_{\bar{0}}$. Hence, l determines a homomorphism $\varphi_l\colon S^{\boldsymbol{\cdot}}(L^*) \to A$. For $f \in S^{\boldsymbol{\cdot}}(L^*)$, set $f(l) = \varphi_l(f)$. Notice that $\mathfrak{g}_A$ naturally acts on L_A and on the algebra of functions on L_A.

2.3. HOW TO DESCRIBE $\mathfrak{g}$-INVARIANTS IN TERMS OF THE POINT FUNCTOR. The following result from [S1; S3] essentially means that, if A is a Grassmann superalgebra with "sufficiently large" number of generators, then for the description of invariants of $\mathfrak{g}$ or the corresponding Lie supergroup G it suffices to confine ourselves to A-points. We recall the language of points (see Appendix 1). Observe that, instead of a functor in A or a tower of sets of A-points, Berezin [Be1; Be2; Be3] considered just one Grassmann superalgebra A with an infinite (countably many) number of generators; however, such an approach may lead to complications occasioned by the infinite number of generators.

STATEMENT [S3]. *Let A be a Grassmann superalgebra with the number of generators greater than* $\dim L_{\bar{1}}$. *An element of $S^{\boldsymbol{\cdot}}(L^*)$ is a $\mathfrak{g}$-invariant if and only if, considered as a function on L_A, it is invariant with respect to $\mathfrak{g}_A$.*

COROLLARY. *Let G_A be the connected Lie group corresponding to the Lie algebra $\mathfrak{g}_A$. Then an element of $S^{\boldsymbol{\cdot}}(L^*)$ is a $\mathfrak{g}$-invariant if and only if, as a function on L_A, this element is G_A-invariant.*

Let

$$L = V^p \oplus \Pi(V)^q \oplus V^{*k} \oplus \Pi(V)^{*l}.$$

Then $S^\bullet(L^*) = \mathfrak{A}^{p,q}_{k,l}$ and the space L_A can be considered as the set of collections

$$\mathfrak{L} = (v_1, \dots, v_p, v_{\bar 1}, \dots, v_{\bar q}, v^*_1, \dots, v^*_k, v^*_{\bar 1}, \dots, v^*_{\bar l}); \tag{2.3}$$

here $v_s \in V \otimes A$ and $v^*_t \in \mathrm{Hom}_A(V \otimes A, A)$, and their parities coincide with parities of their indices. The vectors will be expressed by means of right coordinates and covectors by means of left coordinates:

$$v_s = \sum_i e_i a^*_{is}, \qquad v^*_t = \sum_i a_{ti} e^*_i.$$

If we consider the elements of the algebra $\mathfrak{A}^{p,q}_{k,l}$ as functions on $\mathfrak{L}$, then

$$x^*_{is}(\mathfrak{L}) = a^*_{is} \quad \text{and} \quad x_{ti}(\mathfrak{L}) = a_{ti}.$$

In $\mathfrak{gl}(V)$, introduce a $\mathbb{Z}$-grading by setting

$$\begin{aligned}
\mathfrak{gl}(V)_+ &= \{\mathcal{A} \in \mathfrak{gl}(V) : \mathcal{A}V_{\bar 0} = 0,\ \mathcal{A}V_{\bar 1} \subset V_{\bar 0}\},\\
\mathfrak{gl}(V)_0 &= \mathfrak{gl}(V)_{\bar 0},\\
\mathfrak{gl}(V)_- &= \{\mathcal{A} \in \mathfrak{gl}(V) : \mathcal{A}V_{\bar 1} = 0,\ \mathcal{A}V_{\bar 0} \subset V_{\bar 1}\}.
\end{aligned}$$

Denote by $\mathfrak{b}_+(V)$ the Borel subalgebra that consists of even upper triangular matrices in the basis $\{e_i\}_{i\in I}$, and let $\mathfrak{b}_-(V)$ be the set of even lower triangular matrices.

We will apply similar notations to $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$.

3. Invariants for the Lie Superalgebra $\mathfrak{gl}(V)$

PROOF OF THEOREM 1.1. By Theorem 2.2, it suffices to consider the case of the algebra $\mathfrak{A}^{n,m}_{n,m}$. By the corollary to Theorem 2.3, we must consider functions on collections

$$\mathfrak{L} = (v_1, \dots, v_n, v_{\bar 1}, \dots, v_{\bar m}, v^*_1, \dots, v^*_n, v^*_{\bar 1}, \dots, v^*_{\bar m}) \tag{3.1}$$

contained in the algebra generated by coordinate functions and invariant with respect to the Lie group $\mathrm{GL}(V \otimes A)$. Denote by M the set of collections such that the vectors

$$(v_1, \dots, v_n, v_{\bar 1}, \dots, v_{\bar m})$$

form a basis in $V \otimes A$. If we consider M as an algebraic variety, then in Zariski topology it is dense in the space of all collections. If f is a $\mathrm{GL}(V \otimes A)$-invariant and $\mathfrak{L} \in M$, then there exists $g \in \mathrm{GL}(V \otimes A)$ such that $gv_s = e_s$ for $s \in I$; therefore,

$$f(\mathfrak{L}) = f(g\mathfrak{L}) = f(e_1, \dots, e_{\bar m}, gv^*_1, \dots, gv^*_{\bar m})$$

and $f(\mathfrak{L})$ is a polynomial in coordinates of the vectors gv^*_t. But

$$(gv^*_t, e_s) = (v^*_t, g^{-1}e_s) = (v^*_t, v_i) = \left(v_s = \sum_i e_i a^*_{is},\ v^*_t = \sum_i a_{ti} e^*_i\right).$$

Hence, the theorem is proved. Its corollary is true because the polarization operators turn inner products into inner products. $\square$

4. Invariants of the Lie Superalgebra $\mathfrak{sl}(V)$

By the same reasons as given for $\mathfrak{gl}(V)$, it suffices to confine ourselves to the case of the algebra $\mathfrak{A}^{n,m}_{n,m}$. First, let us find out for which λ, μ there exist invariants in $(\mathfrak{A}^{n,m}_{n,m})_{\lambda,\mu}$; then we construct an invariant of type (λ, μ). The tableaux λ and μ are called *equivalent* if the modules V^{λ} and V^{μ} have the same highest weight as $\mathfrak{sl}(V)$-modules.

4.1. LEMMA. *The tableaux λ and μ are equivalent if and only if one of the following two cases holds*:

(a) $\lambda = \mu$; *or*

(b) $\lambda \neq \mu$ *and both λ and μ contain a rectangle of size $n \times m$ such that there exists a $k \in \mathbb{Z}_+$ that yields μ when we delete k cells from the first m columns of λ and add these k cells to each of the first n rows of λ. If $k < 0$, then we delete the cells from the rows and add them to the columns.*

Proof. The case (a) is obvious.

For case (b), let $\lambda \neq \mu$ and let χ_λ, χ_μ be highest weights of modules V^{λ}, V^{μ} with respect to $\mathfrak{b}_+(V)$. We take the coordinates of the highest weight with respect to the Cartan subalgebra consisting of diagonal matrix units.

If $\gamma = (1, \ldots, 1, -1, \ldots, -1)$ then $\chi_\lambda - \chi_\mu = k\gamma$, where $k \in \mathbb{Z}$. Let $k > 0$; then $(\chi_\mu)_{\bar{m}} > 0$ and $\mu_n \geq m$. It follows that $\lambda_n = k + \mu_n > m$, that is, both tableaux contain an $n \times m$ rectangle. The case $k < 0$ is treated similarly. The statement of the lemma is now completely proved. □

4.2. LEMMA.

$$\dim(V^{\lambda} \otimes V^{*\mu})^{\mathfrak{sl}(V)} = \begin{cases} 1 & \textit{if } \lambda \textit{ and } \mu \textit{ are equivalent,} \\ 0 & \textit{otherwise.} \end{cases}$$

In the proof of this lemma we need basics of the notion of a typical representation (typical module and highest weight). In this paper it suffices to know that, roughly speaking, the highest weight of the irreducible $\mathfrak{g}$-module is *typical* if is induced from a representation of $\mathfrak{g}_{\bar{0}}$.

For the reader interested in further representation theory, recall that Kac [K1; K2] termed the generic highest weights of irreducible finite-dimensional modules over simple Lie superalgebras with Cartan matrix *typical* weights and described conditions for the coordinates of the highest weight to be typical in certain "simplest" system of simple roots. Using "odd reflections", Penkov [P2] and Serganova [Se] were able to extend Kac's conditions to any system of simple roots. The term "typical" became popular, and the description of typical highest weights was generalized on Lie superalgebras without Cartan matrix. For the periplectic and queer series, the conditions for typicality were established by Leites [L2] and Penkov [P1], respectively.

Proof of Lemma 4.2. Since $(V^{\lambda} \otimes V^{*\mu})^{\mathfrak{sl}(V)} = \mathrm{Hom}_{\mathfrak{sl}(V)}(V^{\mu}, V^{\lambda})$, it is clear that the $\mathfrak{sl}(V)$-invariants of type (λ, μ) that are distinct from $\mathfrak{gl}(V)$-invariants exist only if λ and μ correspond to typical modules and are equivalent. □

4.3. LEMMA. *Let M and N be finite-dimensional $\mathfrak{gl}(V)_0$-modules. Set*

$$\mathfrak{gl}(V)_+ M = 0 \quad \textit{and} \quad \mathfrak{gl}(V)_- N = 0.$$

Then

$$\operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_+}(M) \otimes \operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_-}(N) = \operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_0}(M \otimes N). \qquad (*)$$

Proof. Since the dimensions of both modules in $(*)$ are the same, it suffices to show that the submodule generated by $M \otimes N$ in the LHS coincides with the whole module.

Select bases $\{X_\alpha\}_{\alpha>0}$ in $\mathfrak{gl}(V)_+$ and $\{Y_\beta\}_{\beta<0}$ in $\mathfrak{gl}(V)_-$, and let L be the $\mathfrak{gl}(V)$-submodule generated by $M \otimes N$. Consider an element

$$u = Y_{\beta_1} \dots Y_{\beta_l} m \otimes X_{\alpha_1} \dots X_{\alpha_k} n, \quad \text{where } m \in M,\ n \in N.$$

We shall prove by induction on $k + l$ that $u \in L$. For $k + l = 0$, the statement is obvious. Let $k + l > 0$ and

$$\tilde{u} = Y_{\beta_2} \dots Y_{\beta_l} m \otimes X_{\alpha_1} \dots X_{\alpha_k} n.$$

By inductive hypothesis, $\tilde{u} \in L$; hence $Y_{\beta_1}\tilde{u} \in L$. Furthermore,

$$u = Y_{\beta_1}\tilde{u} \pm Y_{\beta_2} \dots Y_{\beta_l} m \otimes Y_{\beta_1} X_{\alpha_1} \dots X_{\alpha_k} n$$

and

$$\begin{aligned} Y_{\beta_1} X_{\alpha_1} \dots X_{\alpha_k} n = {} & [Y_{\beta_1}, X_{\alpha_1}] X_{\alpha_2} \dots X_{\alpha_k} n - X_{\alpha_1} [Y_{\beta_1}, X_{\alpha_2}] X_{\alpha_3} \dots X_{\alpha_k} n \\ & + \dots \pm X_{\alpha_1} \dots X_{\alpha_{k-1}} [Y_{\beta_1}, X_{\alpha_k}] n. \end{aligned}$$

By induction we have $Y_{\beta_2} \dots Y_{\beta_l} m \otimes Y_{\beta_1} X_{\alpha_1} \dots X_{\alpha_k} n \in L$. □

4.4. LEMMA. *Let $\mathfrak{g} = \mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$ and let L be a $\mathfrak{g}$-module. If $u \in L$ is a $\mathfrak{g}_0$-invariant, then*

$$\prod_\alpha X_\alpha \prod_\beta Y_\beta u \Big(\prod_\beta Y_\beta \prod_\alpha X_\alpha u \Big)$$

is a $\mathfrak{g}$-invariant (*perhaps equal to zero*).

Proof. Straightforward verification with the help of the multiplication table for $\mathfrak{g}$. □

Let $\lambda = (m^{n+k})$ and $\mu = ((m+k)^n)$; then, in $\mathfrak{A}^{n,0}_{0,m} = S^{\bullet}(V^{*n} \oplus \Pi(V)^m)$, by Lemma 4.2 there exists an invariant of type (λ, μ). It is not difficult to see that this invariant is unique up to a constant factor. The submodule generated by this invariant is isomorphic to $(\operatorname{Vol} \Pi(V))^{\otimes k}$. Recall that $\operatorname{Vol}(V)$ is the $\mathfrak{gl}(V)$-module determined by the supertrace (or, on the supergroup level, by the Berezinian or the superdeterminant).

Similarly, in the algebra $\mathfrak{A}^{0,m}_{n,0} = S^{\bullet}(V^n \oplus \Pi(V)^{*m})$ there exists a unique $\mathfrak{sl}(V)$-invariant of type (μ, λ), and the module generated by this invariant is isomorphic to $(\operatorname{Vol} V)^{\otimes k}$.

An explicit description of these invariants (as polynomials $f_{\pm k}$) was given in Theorem 1.2.

4.5. LEMMA. *For $k \in \mathbb{N}$, the polynomials*

$$f_k = (\Delta^*)^k \omega^k \prod_{t \in I_{\bar 1},\, s \in I_{\bar 0}} (v_t^*, v_s) \quad \textit{and} \quad f_{-k} = \Delta^k (\omega^*)^k \prod_{t \in I_{\bar 0},\, s \in I_{\bar 1}} (v_t^*, v_s)$$

are $\mathfrak{sl}(V)$-invariant.

Proof. Consider f_k for $k > 0$ (the case $k < 0$ is similar). Select bases $\{X_\alpha\}_{\alpha>0}$ in $\mathfrak{gl}(V)_+$ and $\{Y_\beta\}_{\beta<0}$ in $\mathfrak{gl}(V)_-$, and set $X = \prod_\alpha X_\alpha$ and $Y = \prod_\beta Y_\beta$. We introduce the polynomials

$$\Pi_{10}^* = \prod_{i \in I_{\bar 1},\, s \in I_{\bar 0}} x_{is}^* \quad \text{and} \quad \Pi_{10} = \prod_{t \in I_{\bar 1},\, i \in I_{\bar 0}} x_{ti}. \tag{4.5.0}$$

This yields

$$X(\Delta^{*m+k}) = c_1 \Delta^{*k} \Pi_{10}^*, \tag{4.5.1}$$

$$YX(\Delta^{*m}\Pi_{10}) = c_2 \prod_{t \in I_{\bar 1},\, s \in I_{\bar 0}} (v_t^*, v_s), \tag{4.5.2}$$

$$YX(\Delta^{*m+k}\omega^k \Pi_{10}) = c_3 \Delta^{*k} \omega^k \prod_{t \in I_{\bar 1},\, s \in I_{\bar 0}} (v_t^*, v_s), \tag{4.5.3}$$

where c_1, c_2, c_3 are nonzero constants.

Indeed, consider $S^\bullet(V^{*n})_\mu$ with $\mu = ((m+k)^n)$. The elements Δ^{*m+k} and $\Delta^{*k}\Pi_{10}^*$ belong to $S^\bullet(V^{*n})_\mu$, and

$$\mathfrak{gl}(V)_-(\Delta^{*m+k}) = \mathfrak{gl}(V)_+(\Delta^{*k}\Pi_{10}^*) = 0.$$

Since the module corresponding to μ is a typical one, the equality (4.5.1) holds.

Now consider $S^\bullet(V^{*n} \oplus \Pi(V)^m)_{\lambda,\lambda}$ with $\lambda = (m^n)$. We have $\dim U^\lambda = \dim W^\lambda = 1$ for this λ and hence there exists only one invariant of type (λ, λ):

$$\prod_{t \in I_{\bar 1},\, s \in I_{\bar 0}} (v_t^*, v_s).$$

On the other hand, $\Delta^{*m}\Pi_{10}$ belongs to $V^\lambda \otimes V^{*\lambda}$ and is a $\mathfrak{gl}(V)_0$-invariant. Lemmas 4.3 and 4.4 then imply (4.5.2).

Finally,

$$\begin{aligned} YX(\Delta^{*m+k}\omega_k \Pi_{10}) &= Y(c_1 \Delta^{*k}\Pi_{10}^* \omega^k \Pi_{10}) \\ &= c_1 \Delta^{*k}\omega^k Y(\Pi_{10}^*\Pi_{10}) = c_1 c_2 \Delta^{*k}\omega^k \prod_{t \in I_{\bar 1},\, s \in I_{\bar 0}} (v_t^*, v_s); \end{aligned}$$

by Lemma 4.4, this expression is an $\mathfrak{sl}(V)$-invariant. The proof is complete. □

4.6. PROOF OF THEOREM 1.2. It suffices to construct an invariant polynomial in $(\mathfrak{A}^{n,m}_{n,m})_{\lambda,\mu}$ for λ and μ as described in Lemma 4.1 that depends only on the polynomials f_k and the inner products. Let $k > 0$ and let λ and μ be chosen as in Figure 4.6.

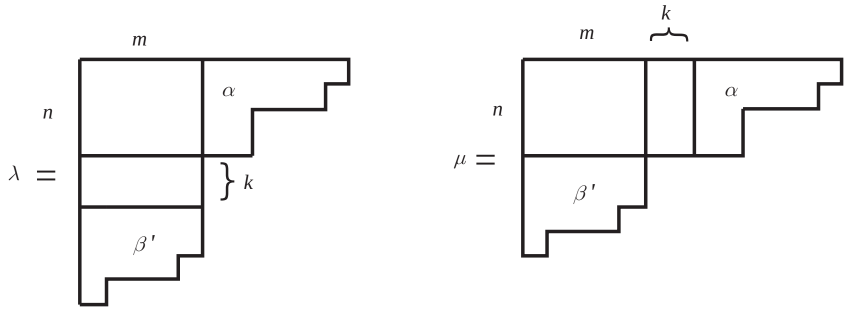

Figure 4.6

Let φ_α be a $(\mathfrak{b}_+(U) \oplus \mathfrak{b}_+(W))$-highest invariant of type (α, α) in the algebra $S^\bullet(V^n \oplus V^{*n})$, and let ψ_β be a similar invariant of type (β, β) in the algebra $S^\bullet(\Pi(V)^m \oplus \Pi(V)^{*m})$. Then one can verify that $f_k \varphi_\alpha \psi_\beta$ is a highest-weight vector with respect to $(\mathfrak{b}_+(U) \oplus \mathfrak{gl}(U)_+) \oplus (\mathfrak{b}_+(W) \oplus \mathfrak{gl}(W)_-)$. Clearly, it is an $\mathfrak{sl}(V)$-invariant and its weight corresponds to the pair of tableaux (λ, μ).

The case $k < 0$ is treated similarly. □

5. Invariants of the Lie Superalgebra $\mathfrak{osp}(V)$

5.0. Let $\dim V = (n, 2r)$ and $\mathfrak{osp}(V)$ be the Lie superalgebra described in Section 1. By Remark 2.2.1, it suffices to confine ourselves to the algebra $\mathfrak{A}^{n,2r}$. From the point of view of Theorem 2.3 we must describe the polynomials that depend on the set

$$\mathfrak{L} = (v_1, \ldots, v_n, v_{\bar{1}}, \ldots, v_{\overline{2r}}) \tag{5.1}$$

and that are invariant with respect to the simply connected Lie group G_A whose Lie algebra is $(\mathfrak{osp}(V) \otimes A)_{\bar{0}}$, where A is the Grassmann superalgebra with a sufficiently large number of generators. Denote by $\mathrm{OSp}(V \otimes A)$ the subgroup of $\mathrm{GL}(V \otimes A)$ whose elements preserve the inner product

$$(v_s, v_t) = \sum_{i=1}^{n} x^*_{is} x^*_{n-i+1,t} + (-1)^{p(s)} \sum_{j=1}^{2r} \left(x^*_{\overline{m-j+1},s} x^*_{\bar{j},t} - x^*_{\bar{j},s} x^*_{\overline{m-j+1},t} \right)$$

and by $\mathrm{SOSp}(V \otimes A)$ the subgroup of $\mathrm{OSp}(V \otimes A)$ consisting of transformations with Berezinian $= 1$. It is not difficult to verify that $\mathrm{SOSp}(V \otimes A)$ is precisely the group G_A discussed previously.

Denote by $\mathrm{O}(V_{\bar{0}})$ the orthogonal group that preserves the form

$$\sum_{i=1}^{n} x^*_i x^*_{n-i+1}.$$

It is not difficult to verify that the invariance of an element of $\mathfrak{A}^{n,m}$ with respect to $\mathrm{OSp}(V \otimes A)$ is equivalent to the simultaneous invariance with respect to $\mathrm{OSp}(V)$ and $\mathrm{O}(V_{\bar{0}})$.

First, let us prove several lemmas.

5.1. LEMMA. *Let M be a $\mathfrak{gl}(V)_{\bar 0}$-module, and set $\mathfrak{gl}(V)_+(M) = 0$. Then we have an isomorphism of $\mathfrak{osp}(V)$ and $\mathrm{O}(V_{\bar 0})$-bimodules*:

$$\mathrm{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_{\bar 0}\oplus\mathfrak{gl}(V)_+}(M) \simeq \mathrm{ind}^{\mathfrak{osp}(V)}_{\mathfrak{osp}(V)_{\bar 0}}(M).$$

Proof. We describe a basis in $\mathfrak{osp}(V)_1$. A nondegenerate form determines a map $\mathcal{A} \mapsto \bar{\mathcal{A}}$ in $\mathfrak{gl}(V)$ such that $\bar{\bar{\mathcal{A}}} = (-1)^{p(\mathcal{A})}\mathcal{A}$. If, in a basis, S is the matrix of the form preserved by $\mathfrak{osp}(V)$ and P is the matrix of $\mathcal{A}$ in the same basis, then

$$\bar{P} = S^{-1}P^{\mathrm{st}}S,$$

where P^{st} is the supertransposed matrix. If $\{X_\alpha\}$ is a basis in $\mathfrak{gl}(V)_-$, then $\{X_\alpha - \bar{X}_\alpha\}$ is a basis in $\mathfrak{osp}(V)_{\bar 1}$ such that $\bar{X}_\alpha \in \mathfrak{gl}(V)_+$. Let

$$\varphi\colon \mathrm{ind}^{\mathfrak{osp}(V)}_{\mathfrak{osp}(V)_{\bar 0}}(M) \to \mathrm{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_{\bar 0}\oplus\mathfrak{gl}(V)_+} = L$$

be a homomorphism induced by the natural embedding $M \hookrightarrow L$. On L, there exists a filtration $L_0 \subset L_1 \subset \cdots \subset L_N$, where L_k is the linear hull of $f(X_\alpha)m$, $m \in M$, and $\deg f \le k$.

Let us prove by induction that $L_k \subset \mathrm{Im}\,\varphi$. The case $k = 0$ is obvious. Let $L_k \subset \mathrm{Im}\,\varphi$; then $(X_\alpha - \bar{X}_\alpha)L_k \subset \mathrm{Im}\,\varphi$ but $\bar{X}_\alpha L_k \subset L_{k-1}$. Therefore, $X_\alpha L_k \subset \mathrm{Im}\,\varphi$ or $L_{k+1} \subset \mathrm{Im}\,\varphi$.

The statement on $\mathrm{O}(V_{\bar 0})$-modules is obvious, so the lemma is proved. □

5.2. LEMMA. *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and let the representation of $\mathfrak{g}_{\bar 0}$ in $\Lambda^{\dim \mathfrak{g}_{\bar 1}}(\mathfrak{g}_{\bar 1})$ be trivial. Then, for a finite-dimensional $\mathfrak{g}_{\bar 0}$-module M, there exists an isomorphism of vector spaces*:

$$(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M))^{\mathfrak{g}} \simeq M^{\mathfrak{g}_{\bar 0}}.$$

Proof. We will show that $(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M))^* \simeq \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M^*)$ as $\mathfrak{g}$-modules. Let $\mathfrak{g}_{\bar 1} = \mathrm{Span}(\xi_1, \dots, \xi_p)$. Then $L = \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M)$ has a natural filtration with $\mathfrak{g}_{\bar 0}$-modules and, as in Lemma 5.2, $L_0 \subset L_1 \subset \cdots \subset L_p = L$.

The map

$$M \to L_p/L_{p-1}, \quad m \mapsto \xi_1 \dots \xi_p m \pmod{L_{p-1}}$$

induces an isomorphism of $\mathfrak{g}_{\bar 0}$-modules: $M^* \simeq (L_p/L_{p-1})^*$. Therefore, we have an embedding of $\mathfrak{g}_{\bar 0}$-modules

$$M^* \to (L_p/L_{p-1})^* \to L^*.$$

This map induces a homomorphism $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M^*) \to L^*$. Consider the $\mathfrak{g}$-invariant bilinear form corresponding to this homomorphism:

$$\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M^*) \times \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M) \to \mathbb{C},$$
$$(m^*, \xi_1 \dots \xi_p m) = m^*(m) \quad \text{for } m^* \in M^*,\ m \in M.$$

Let u be a nonzero element from the left kernel of the form. Then there exists a filtration on the module $T = \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M^*)$, and the same is true on L. Let $u \in T_k$ but $u \notin T_{k-1}$. Then

$$u = \sum \xi_{i_1} \dots \xi_{i_k} m^*_{i_1 \dots i_k} + u_{k-1}.$$

Set

$$v = \xi_{j_1}, \dots, \xi_{j_l} m, \quad \text{where } \{j_1 \dots j_l\} = [1, \dots, p] \setminus \{i_1, \dots, i_k\}.$$

Then

$$(u, v) = (\xi_{i_1} \dots \xi_{i_k} m^*_{i_1 \dots i_k}, v) = \pm m^*_{i_1 \dots i_k}(m) = 0.$$

Because m is arbitrary, $m^*_{i_1 \dots i_k} = 0$ and thus $u = u_{k-1} \in T_{k-1}$, a contradiction. Hence, $u = 0$. Furthermore,

$$(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M))^{\mathfrak{g}} = (\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}(M^*))^{*\mathfrak{g}} = (M^*)^{*\mathfrak{g}_{\bar 0}} = M^{\mathfrak{g}_{\bar 0}}. \qquad \square$$

REMARK. (1) If $\mathfrak{g}_{\bar 0} \supset \mathfrak{o}(n)$ and M is such that the $\mathfrak{g}_{\bar 1}$ are $\mathrm{O}(n)$-modules and the $\mathrm{O}(n)$-action in $\Lambda^p(\mathfrak{g}_{\bar 1})$ is trivial, then the statement of the lemma remains valid for the mutual $\mathfrak{g}$- and $\mathrm{O}(n)$-invariants and the mutual $\mathfrak{g}_{\bar 0}$- and $\mathrm{O}(n)$-invariants.

(2) The following refinement of Lemma 5.2 can be obtained: *If $m \in M$ is a $\mathfrak{g}_{\bar 0}$-invariant, then the corresponding $\mathfrak{g}$-invariant vector u is of the form*

$$u = \xi_1 \dots \xi_p m + u_{p-1}, \quad \textit{where } u_{p-1} \in L_{p-1}.$$

Description of $\mathrm{OSp}(V \otimes A)$*-Invariants*

5.3. THEOREM. *Any* $\mathrm{OSp}(V \otimes A)$*-invariant element from* $\mathfrak{A}^{p,q}$ *is a polynomial in inner products* (v_s, v_t), *where* $s, t \in S$.

Proof. The proof proceeds by induction on $\dim V_{\bar 0}$. If $\dim V_{\bar 0} = 0$ then the theorem is proved in [Wy], so let $\dim V_{\bar 0} = n > 0$. It suffices to show that any invariant of type λ with $\lambda_{n+1} \le 2r$ can be expressed in terms of inner products.

Let λ satisfy the condition $\lambda_n \le 2r$. Consider the algebra

$$\mathfrak{A}^{n-1,2r} = \bigoplus_{\lambda_n \le 2r} V^{*\lambda} \otimes W^{\lambda}.$$

If $\{v_1, \dots, v_{n-1}, v_{\bar 1}, \dots, v_{\overline{2r}}\}$ is a collection of vectors in general position from $V \otimes A$, then after an orthogonalization we may assume that there exists a $g \in \mathrm{OSp}(V \otimes A)$ such that

$$g\,\mathrm{Span}(v_1, \dots, v_{n-1}, v_{\bar 1}, \dots, v_{\overline{2r}}) = H = \mathrm{Span}(e_1, \dots, e_{n-1}, e_{\bar 1}, \dots, e_{\overline{2r}}).$$

Let $f \in \mathfrak{A}^{n-1,2r}$ be an invariant with respect to $\mathrm{OSp}(V \otimes A)$, and let $\bar f$ denote the restriction of f onto H. By the inductive hypothesis, $\bar f$ is a function in inner products $(\bar v_s, \bar v_t)$ with $\bar v_s, \bar v_t \in H$. Hence,

$$\begin{aligned} f(v_1, \dots, v_{n-1}, v_{\bar 1}, \dots, v_{\overline{2r}}) &= \bar f(gv_1, \dots, gv_{\overline{2r}}) \\ &= F((gv_s, gv_t)) = F((v_s, v_t)) \quad \text{for } s, t \in I \setminus \{n\}. \end{aligned}$$

Now, let $\lambda_n > 2r$ but $\lambda_{n+1} \le 2r$. Then the $\mathfrak{gl}(V)$-module $V^{*\lambda}$ is a typical one; that is,

$$V^{*\lambda} = \mathrm{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_+}(M),$$

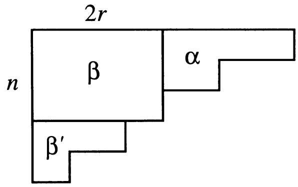

Figure 5.3

where M is an irreducible $\mathfrak{gl}(V)_0$-module. If λ is of the form shown in Figure 5.3, then $M = V_{\bar 0}^{*\alpha} \otimes V_{\bar 1}^{*\beta+\delta'}$.

It is not difficult to verify, for the orthogonal case and (similarly) for the symplectic case, that

$$\dim(V_{\bar 0}^{*\alpha})^{\mathfrak{o}(V_{\bar 0})} = \begin{cases} 1 & \text{if } \alpha \text{ is even,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\dim(V_{\bar 1}^{*\beta+\delta'})^{\mathfrak{sp}(V_{\bar 1})} = \begin{cases} 1 & \text{if } (\beta+\delta')' \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

These conditions are equivalent to the fact that λ is even (all rows are of an even length). Lemmas 5.1 and 5.2 imply that, if λ is typical, then

$$\dim(V^{*\lambda})^{\mathrm{OSp}(V\otimes A)} = \begin{cases} 1 & \text{if } \lambda \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Further, for $s, t \in I$ the inner products (v_s, v_t) are algebraically independent. If we consider the algebra $\mathbb{C}[(v_s, v_t)_{s,t\in I}]$ as a $\mathfrak{gl}(W)$-module, then

$$\mathbb{C}[(v_s, v_t)_{s,t\in I}] = \bigoplus_{\lambda_{n+1}\leq 2r} W^\lambda.$$

This is a corollary of a general identity for λ-rings (see [M, Sec. 5]). This shows that if λ is typical and even then there exists an invariant of type λ depending on inner products. The induction is completed and thus Theorem 5.3 is proved. □

5.4. LEMMA. *There exists an $\mathfrak{osp}(V)$-invariant Ω such that*

$$\Omega^2 = [\det(v_s, v_t)_{s,t\in I_0}]^{2r+1}.$$

Proof. We have

$$S^{\cdot}(V^{*n}) = \bigoplus_{\lambda_{n+1}=0} V^{*\lambda} \otimes W^\lambda.$$

Let $\lambda = ((2r+1)^n)$; then $\dim W^\lambda = 1$, the module $V^{*\lambda}$ is typical, and $V^{*\lambda} = \operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_0\oplus\mathfrak{gl}(V)_-}(M)$, where $\dim M = 1$ and $M = \operatorname{Span}(\Delta^*\Pi^*_{10})$. By Lemmas 5.1 and 5.2 there exists an $\mathfrak{osp}(V)$-invariant

$$\Omega = \prod (X_\alpha - \bar X_\alpha)\Delta^*\Pi^*_{10} + \tilde\Omega = \prod X_\alpha \Delta^*\Pi^*_{10} + \tilde\Omega_1 = e\cdot \Delta^{*2r+1} + \tilde\Omega_1.$$

Therefore, $\Omega^2 \neq 0$ but Ω^2 is an $\mathrm{OSp}(V \otimes A)$-invariant and its type is equal to $((4r+2)^n)$. However, as is easy to see, the algebra $S^{\bullet}(V^{*n})$ has only one $\mathrm{OSp}(V \otimes A)$-invariant of such type—namely, $[\det(v_s, v_t)_{s,t \in I_0}]^{2r+1}$. The lemma is proved. $\square$

5.5. Proof of Theorem 1.3. Let f be an $\mathfrak{osp}(V)$-invariant but not an $\mathrm{O}(V_{\bar{0}})$-invariant. Let f depend on $n-1$ even and $2r$ odd vectors, and let these vectors be in general position. Then, as in Theorem 5.3, there exists a $g \in \mathrm{OSp}(V \otimes A)$ such that

$$g\,\mathrm{Span}(v_1, \dots, v_{\overline{2r}}) = \mathrm{Span}(e_1, \dots, e_{\overline{2r}}).$$

Let $he_n = -e_n$ and $he_i = e_i$ $(i \neq n)$. Then $\mathrm{ber}(h) = -1$ (see Appendix 0) and $f(hg\mathfrak{L}) = -f(g\mathfrak{L})$ for $\mathfrak{L}$ as defined in (5.1). On the other hand, $f(hg\mathfrak{L}) = f(g\mathfrak{L})$ and thus $f = 0$. This means that $\mathfrak{osp}(V)$-invariants distinct from inner products can only be of type λ, corresponding to a typical module.

We can thus apply Lemmas 5.1 and 5.2. The same arguments as in Theorem 5.3 yield that $\dim(V^{*\lambda})^{\mathfrak{osp}(V)} = 1$ if (a) λ is typical, (b) its first n rows are of odd length, and (c) the remaining rows are of even length; otherwise, $\dim(V^{*\lambda})^{\mathfrak{osp}(V)} = 0$. (We do not take $\mathrm{OSp}(V \otimes A)$-invariants into account.) Let λ be as in Figure 5.5.

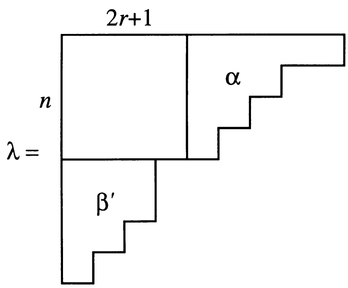

Figure 5.5

Let us construct an invariant of type λ. Denote by φ_α the invariant of type α that is highest with respect to $\mathfrak{b}_+(W)$ in $S^{\bullet}(V^{*n})$, and let ψ_β be the invariant of type β that is highest with respect to $\mathfrak{b}_+(W)$ in $S^{\bullet}(\Pi(V)^{*2r})$. If $\{D_j\}$ is a basis of $\mathfrak{gl}(W)_-$, then it is not difficult to verify that $\psi_\beta \prod D_j(\Omega\varphi_\alpha)$ is $(\mathfrak{b}_+(W) \oplus \mathfrak{gl}(W)_-)$-highest and is of type λ. The theorem is proved. $\square$

6. Invariants of the Lie Superalgebra $\mathfrak{pe}(V)$

Let $\dim V = (n, n)$. By Remark 2.2.1, it suffices to consider algebras $\mathfrak{A}^{n,n}$. Denote by $\mathrm{Pe}(V \otimes A)$ the Lie subgroup in $\mathrm{GL}(V \otimes A)$ whose elements preserve the inner products

$$(v_s, v_t) = \sum_{i=1}^{n}((-1)^{p(s)} x^*_{is} x^*_{\bar{i}t} + x^*_{\bar{i}s} x^*_{it}) \quad \text{for } s, t \in I_0.$$

This is the connected Lie group corresponding to the Lie algebra of A-points of the Lie superalgebra $\mathfrak{pe}(V)$.

6.1. LEMMA. *For any irreducible typical $\mathfrak{gl}(V)$-module*

$$L = \operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_{\bar{0}} \oplus \mathfrak{gl}(V)_+}(M) \simeq \operatorname{ind}^{\mathfrak{gl}(V)}_{\mathfrak{gl}(V)_{\bar{0}} \oplus \mathfrak{gl}(V)_-}(N),$$

there is an isomorphism of vector spaces $L^{\mathfrak{spe}(V)} = M^{\mathfrak{spe}(V)_{\bar{0}}} = N^{\mathfrak{spe}(V)_{\bar{0}}}$.

Proof. The decomposition (and also the $\mathbb{Z}$-grading)

$$\mathfrak{gl}(V) = \mathfrak{gl}(V)_- \oplus \mathfrak{gl}(V)_{\bar{0}} \oplus \mathfrak{gl}(V)_+$$

induces the decomposition (and also the $\mathbb{Z}$-grading)

$$\mathfrak{spe}(V) = \mathfrak{spe}(V)_- \oplus \mathfrak{spe}(V)_{\bar{0}} \oplus \mathfrak{spe}(V)_+.$$

Select a basis $\{Y_\beta\}_{1\le\beta\le n^2}$ in $\mathfrak{gl}(V)_-$ so that $\{Y_\beta\}_{1\le\beta\le\frac{1}{2}n(n-1)}$ is a basis in $\mathfrak{spe}(V)_-$. Similarly, select a basis $\{X_\alpha\}_{1\le\alpha\le n^2}$ in $\mathfrak{gl}(V)_+$ so that $\{X_\alpha\}_{1\le\alpha\le\frac{1}{2}n(n+1)}$ is a basis in $\mathfrak{spe}(V)_+$. Consider two gradings of the module L:

$$L^+_k = \operatorname{Span}(f(X_\alpha)n : n \in N,\ \deg f = k \text{ for all } \alpha);$$

$$L^-_k = \operatorname{Span}(f(Y_\beta)m : m \in M,\ \deg f = k \text{ for all } \beta).$$

It is not difficult to verify that $L^+_k = L^-_{n^2-k}$. Let l be an $\mathfrak{spe}(V)$-invariant; then $X_\alpha l = 0$ for $1 \le \alpha \le \frac{1}{2}n(n+1)$. Let $X^+ = \prod X_\alpha$ for $\alpha \le \frac{1}{2}n(n+1)$; hence, $l = X^+ f(X_\alpha)n$ for $n \in N$ and therefore $l = \sum_{r\ge\frac{1}{2}n(n+1)} l^+_r$, where $l^+_r \in L^+_r$.

We can similarly verify that $l = \sum_{s\ge\frac{1}{2}n(n-1)} l^-_s$ for $l^-_s \in L^-_s$. This implies that $\sum_{r\ge\frac{1}{2}n(n+1)} l^+_r = \sum_{s\ge\frac{1}{2}n(n-1)} l^-_s$. Since $L^-_s = L^+_{n^2-s}$, we obtain $l \in L^+_{\frac{1}{2}n(n+1)} = L^-_{\frac{1}{2}n(n-1)}$ and $l = X^+ n = X^- m$, where $m \in M$ and $X^- = \prod Y_\beta$ for $\beta \le \frac{1}{2}n(n-1)$.

It is clear that m and n are $\mathfrak{spe}(V)_{\bar{0}} = \mathfrak{sl}(V_{\bar{0}})$-invariants. Conversely, if m and n are $\mathfrak{sl}(V_{\bar{0}})$-invariants then a straightforward verification shows that $X^+ n$ and $X^- m$ are $\mathfrak{spe}(V)$-invariants. The statement on bijection is obvious, so the lemma is proved. □

6.2. PROOF OF THEOREM 1.4.1. The proof is via induction on $\dim V_{\bar{0}}$.

The case $n = 1$ is straightforward, so suppose the theorem holds for $\dim V_{\bar{0}} = n - 1$. Let us consider $f \in S^\bullet(V^{*n-1} \oplus \Pi(V)^{*n-1})^{\mathrm{Pe}(V)}$, where $\dim V = (n, n)$. For the generic vectors

$$\big(v_1, \dots, v_{n-1}, v_{\bar{1}}, \dots, v_{\overline{n-1}}\big),$$

there exists some $g \in \mathrm{Pe}(V \otimes A)$ such that

$$g v_i \in \operatorname{Span}\big(e_1, \dots, e_{n-1}, e_{\bar{1}}, \dots, e_{\overline{n-1}}\big) = H.$$

Let $\bar{f}$ be the restriction of f to H; by the inductive hypothesis, f is a polynomial in inner products yet

$$f\big(v_1, \dots, v_{\overline{n-1}}\big) = f\big(gv_1, \dots, gv_{\overline{n-1}}\big) = F((gv_s, gv_t)_{s,t\in I_0}) = F((v_s, v_t)_{s,t\in I_0}).$$

Since

$$S^{\bullet}(V^{*n-1} \oplus \Pi(V)^{*n-1}) = \bigoplus_{\lambda_n \le n-1} V^{*\lambda} \otimes W^{\lambda},$$

it remains to demonstrate that the invariants in typical modules can be expressed in terms of inner products. Let λ be a tableau of the form shown in Figure 6.2a. Applying Lemma 6.1, we see that an invariant of type λ exists if and only if there exists an $\mathfrak{sl}(V_{\bar{0}})$-invariant in $M = V_{\bar{0}}^{*\alpha} \otimes V_{\bar{1}}^{*\beta+\delta'}$. We have an isomorphism of $\mathfrak{spe}(V)_{\bar{0}}$-modules

$$M \cong V_{\bar{0}}^{*\alpha} \otimes V_{\bar{0}}^{\beta+\delta'} = \mathrm{Hom}(V_{\bar{0}}^{\alpha}, V_{\bar{0}}^{\beta+\delta'})$$

and thus M contains an $\mathfrak{sl}(V_{\bar{0}})$-invariant if $\beta+\delta'-\alpha$ is a multiple of $\gamma = (1, \dots, 1)$.

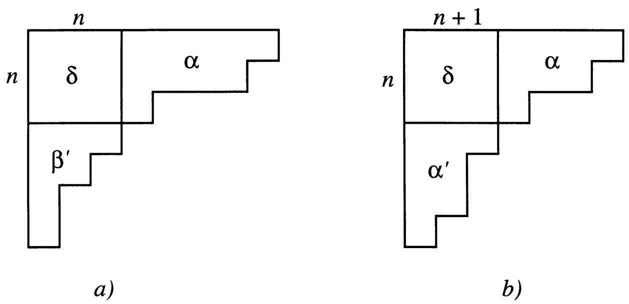

Figure 6.2

Again by Lemma 6.1, the invariant is of the form $X^{-}m$ and—since we wish it to be $\mathfrak{pe}(V)$-invariant—we need it to be a $\mathfrak{gl}(V_{\bar{0}})$-invariant. Its weight is equal to

$$\beta + \delta' - \alpha - (n-1)\gamma = \beta + n\gamma - \alpha - (n-1)\gamma = \beta - \alpha + \gamma = 0;$$

that is, $\alpha = \beta + \gamma$ and the tableau λ should be of the form shown in Figure 6.2b.

Let us explicitly indicate an invariant of such type λ. The Lie algebra $\mathfrak{gl}(W_{\bar{0}}) \oplus \mathfrak{gl}(W_{\bar{1}})$ acts on the algebra

$$\tilde{\mathfrak{A}} = \mathbb{C}[(v_i, v_j)_{i\in I_{\bar{0}},\, j\in I_{\bar{1}}}],$$

and with respect to this action $\tilde{\mathfrak{A}} = \bigoplus_{\alpha} W_{\bar{0}}^{\alpha} \otimes W_{\bar{1}}^{\alpha}$. Let φ_α be a vector from $\tilde{\mathfrak{A}}$ of type α that is highest with respect to $\mathfrak{b}_{+}(W)$. Then we can verify that $\prod_{1\le i\le j\le n}(v_i, v_j)\varphi_\alpha$ is a $(\mathfrak{b}_{+}(W) \oplus \mathfrak{gl}(W)_{-})$-highest vector of type λ, proving the theorem. □

7. Invariants of the Lie Superalgebra $\mathfrak{spe}(V)$

First, let us construct certain $\mathfrak{spe}(V)$-invariant elements in the algebra $\mathfrak{A}^{n,n}$.

7.1. LEMMA. *The polynomials* $\Delta^{*k}\prod_{s\leq t,\, s,t\in I_{\bar{0}}}(v_s, v_t)$ *are* $\mathfrak{spe}(V)$*-invariant for* $k = 1, 2, 3, \ldots$.

Proof. Consider $\mathfrak{A} = S^{\bullet}(V^{*n})$ and let $\lambda = ((n+1)^n)$. Then, in $\mathfrak{A}$, there exists only one invariant of type λ:

$$\Pi^+ = \prod_{s\leq t,\, s,t\in I_{\bar{0}}}(v_s, v_t).$$

On the other hand, by Lemma 6.1 we have

$$X^-(\Delta^*\Pi^*_{10}) = \Pi^+$$

(see (4.5.0) for the definition of Π^*_{10}). Furthermore, the vector $\Delta^{*k+1}\Pi^*_{10}$ is a $\mathfrak{spe}(V)_{\bar{0}}$-invariant and $\mathfrak{gl}(V)_+(\Delta^{*k+1}\Pi^*_{10}) = 0$; hence by Lemma 6.1 we obtain an invariant $X^-(\Delta^{*k+1}\Pi^*_{10})$. It is not difficult to verify that $X^-(\Delta^*) = 0$; therefore,

$$X^-(\Delta^{*k+1}\Pi^*_{10}) = \Delta^{*k}X^-(\Delta^*\Pi^*_{10}) = \Delta^{*k}\Pi^+$$

and the lemma is proved. □

7.2. LEMMA. *The polynomials* $\omega^{*k}\prod_{s\leq t,\, s,t\in I_{\bar{0}}}(v_{\bar{s}}, v_{\bar{t}})$ *are* $\mathfrak{spe}(V)$*-invariant for* $k = 1, 2, 3, \ldots$.

Proof. As in Lemma 7.1, let $\lambda = ((n+1)^n)$. In $\mathfrak{A}^{n,n}$, consider the vector

$$n = \Delta^* \prod_{i\in I_{\bar{0}},\, s\in I_{\bar{1}}} X^*_{is} = \Delta^*\Pi^*_{10}.$$

It is not difficult to verify that $\mathfrak{gl}(W)_-(\Delta^*\Pi^*_{10}) = 0$ and $\Delta^*\Pi^*_{10}$ is a $\mathfrak{spe}(V)_{\bar{0}}$-invariant that is highest with respect to $\mathfrak{b}_+(W)\oplus\mathfrak{gl}(W)_-$ and of type λ. By Lemma 6.1, X^+n is a $\mathfrak{spe}(V)$-invariant (and even $\mathfrak{pe}(V)$-invariant), and clearly it is highest with respect to $\mathfrak{b}_+(W)\oplus\mathfrak{gl}(W)_-$. But such is also the invariant

$$\det(v_i, v_{\bar{j}})_{i,j\in I_{\bar{0}}} \prod_{s\leq t,\, s,t\in I_{\bar{0}}}(v_{\bar{s}}, v_{\bar{t}}).$$

Using the notation $d = \det(v_i, v_j)_{i,j\in I_{\bar{0}}}$ and $\Pi^- = \prod_{s\leq t,\, s,t\in I_{\bar{0}}}(v_{\bar{s}}, v_{\bar{t}})$, we have $X^+n = cd\Pi^-$ for $c\neq 0$. Now consider the vector $\omega^{*k}n$. By similar arguments, the expression

$$X^+(\omega^{*k}n) = \omega^{*k}X^+n = \omega^{*k}d\Pi^-c$$

is an $\mathfrak{spe}(V)$-invariant. Dividing by d yields the statement of the lemma. □

Lemma 7.1 implies that, on the group $\mathrm{Pe}(V\otimes A)$, there exists a multiplicative function

$$B : B^2(g) = \mathrm{ber}(g) \quad \text{for } g\in\mathrm{Pe}(V\otimes A).$$

Let us denote this function by $\sqrt{\mathrm{ber}}(g)$; denote by $\mathrm{SPe}(V \otimes A)$ the subgroup of $\mathrm{Pe}(V \otimes A)$ consisting of matrices g such that $\mathrm{ber}(g) = 1$ and denote by $\mathrm{SSPe}(V \otimes A)$ the subgroup consisting of g such that $\sqrt{\mathrm{ber}}(g) = 1$. We observe that $\mathrm{SSPe}(V \otimes A)$ is the connected Lie group corresponding to the A-points of the Lie superalgebra $\mathfrak{spe}(V)$.

7.3. Proof of Theorem 1.4.2. Let us find λ for which there exists an $\mathfrak{spe}(V)$-invariant of type λ that is distinct from inner products. Let λ be atypical. Then an invariant of type λ, if any, belongs to the algebra $S^{\cdot}(V^{*n-1} \oplus \Pi(V)^{*n-1})$ and there exists a $k \in \mathbb{Z}$ such that

$$f(g\mathfrak{L}) = \left(\sqrt{\mathrm{ber}}(g)\right)^k f(\mathfrak{L}) \quad \text{for } \mathfrak{L} \text{ as in (2.3).}$$

For generic vectors there exists some $g \in \mathrm{Pe}(V \otimes A)$ such that

$$gv_i \in \mathrm{Span}\big(e_1, \ldots, e_{n-1}, e_{\bar{1}}, \ldots, e_{\overline{n-1}}\big).$$

Then $\mathrm{Span}(g^{-1}e_n, g^{-1}e_{\bar{n}})$ and $\mathrm{Span}\big(e_1, \ldots, e_{n-1}, e_{\bar{1}}, \ldots, e_{\overline{n-1}}\big)$ are orthogonal. By applying an appropriate transformation from $\mathrm{Pe}(1)$,

$$\langle g^{-1}e_n, g^{-1}e_{\bar{n}}\rangle \mapsto \langle e_n, e_{\bar{n}}\rangle,$$

we may assume that $g \in \mathrm{SSPe}(V \otimes A)$.

Let $he_n = ae_n$ and $he_{\bar{n}} = a^{-1}e_{\bar{n}}$ with the other vectors fixed; then

$$h \in \mathrm{Pe}(V \otimes A) \quad \text{and} \quad \sqrt{\mathrm{ber}}(h) = a.$$

Besides, $f(hg\mathfrak{L}) = a^k f(g\mathfrak{L}) = f(g\mathfrak{L})$ and thus $f = 0$ if $k \neq 0$. Therefore, λ should be typical. By applying Lemma 6.1 we deduce that there exists a $k > 0$ and that λ is of the form shown in Figure 4.6, where β should be read as α' and $m = n$.

Finally, we construct invariants of type λ. Let φ_α be as in Theorem 6.1. Then $\omega^{*k}\Pi^-\varphi_\alpha$ and $\Delta^{*k}\Pi^+\varphi_\alpha$ are the invariants desired. The theorem is proved. □

8. Invariants of the Lie Superalgebra $\mathfrak{q}(V)$

Denote by $\mathrm{GQ}(V \otimes A)$ the subgroup in $\mathrm{GL}(V \otimes A)$ that preserves the inner product

$$[v_t^*, v_s] = \sum_{i=1}^{n} (x_{ti}x_{is}^* + x_{t\bar{i}}x_{\bar{i}s}^*) \quad \text{for } s, t \in I_{\bar{0}}.$$

As mentioned in Remark 2.2.1, it suffices to confine ourselves to the algebra $\mathfrak{A}_n^n = S^{\cdot}(V^n \oplus V^{*n})$.

8.1. Proof of Theorem 1.5.1. Let us consider a generic collection

$$\mathfrak{L} = (v_1, \ldots, v_n, v_{\bar{1}}^*, \ldots, v_{\bar{n}}^*).$$

There exists $g \in \mathrm{GQ}(V \otimes A)$ such that $gv_i = l_i$ $(i \in I_{\bar{0}})$. If f is a $\mathrm{GQ}(V \otimes A)$-invariant, then

$$f(\mathfrak{L}) = f(g\mathfrak{L}) = f(e_1, \dots, e_n, gv_1^*, \dots, gv_n^*)$$

is a polynomial in coordinates of gv_t^*, but $(gv_t^*, e_s) = (v_t^*, v_s)$ and $[gv_t^*, e_s] = [v_t^*, v_s]$. Hence, f is a polynomial in inner products. $\square$

9. Invariants of the Lie Superalgebra $\mathfrak{sq}(V)$

First, let us prove a theorem which for the Lie superalgebra $\mathfrak{q}(V)$ plays the same role as Theorem 2.1 plays for $\mathfrak{gl}(V)$.

9.1. THEOREM. *Let* $\dim V = (n, n)$ *and* $\dim U = (l, l)$. *Then we have an isomorphism of* $(\mathfrak{q}(U) \oplus \mathfrak{q}(V))$*-modules*:

$$S^{\bullet}(2^{-1}U \otimes V) \simeq \bigoplus_{\lambda_{n+1}=0} 2^{-\delta(|\lambda|)} U^{\lambda} \otimes V^{\lambda}.$$

Here U^λ *and* V^λ *are irreducible* $\mathfrak{q}(U)$*- and* $\mathfrak{q}(V)$*-modules corresponding to* λ, *where* λ *is a strict partition such that*

$$\delta(|\lambda|) = \begin{cases} 0 & \text{if } |\lambda| \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

(For the definition of the module $2^{-1}U \otimes V$, see Appendix 1.)

Proof. According to [S1] we have

$$V^{*\otimes k} = \bigoplus_{\lambda:\lambda_{n+1}=0} V^{*\lambda} \otimes T^{\lambda} \cdot 2^{-\delta(|\lambda|)}, \qquad U^{\otimes k} = \bigoplus_{\mu:\mu_{n+1}=0} 2^{-\delta|\mu|} U^{\mu} \otimes T^{\mu}.$$

Hence,

$$\begin{aligned} S^k(2^{-1}U \otimes V) &= S^k(2^{-1}U \otimes (V^*)^*) = S^k(\mathrm{Hom}_{G_1}(V^*, U)) \\ &= \mathrm{Hom}_{G_k}(V^{*\otimes k}, U^{\otimes k}) \\ &= \bigoplus_{\lambda,\mu} 2^{-\delta(|\lambda|)} 2^{-\delta(|\mu|)} \mathrm{Hom}(V^{*\lambda}, U^{\mu}) \otimes \mathrm{Hom}_{G_k}(T^{\lambda}, T^{\mu}) \\ &= \bigoplus_{\lambda_{n+1}=0} 2^{-\delta(|\lambda|)} U^{\lambda} \otimes V^{\lambda}, \end{aligned}$$

where $G_k = \mathfrak{S}_k \circ C_k$ (see [S1]). The theorem is proved. $\square$

COROLLARY. *We have an isomorphism of* $\mathfrak{q}(V)$-$\mathfrak{q}(W)$-$\mathfrak{q}(U)$ *trimodules*:

$$S^{\bullet}(2^{-1}U \otimes V + 2^{-1}V^* \otimes W) \simeq \bigoplus_{\lambda,\mu} 2^{-\delta(|\lambda|)} U^{\lambda} \otimes V^{\lambda} \otimes 2^{-\delta(|\mu|)} V^{*\mu} \otimes W^{\mu} \simeq \mathfrak{A}_n^n.$$

Set

$$(\mathfrak{A}_n^n)_{\lambda,\mu} = 2^{-\delta(|\lambda|)} U^{\lambda} \otimes V^{\lambda} \otimes 2^{-\delta(|\mu|)} V^{*\mu} \otimes W^{\mu}$$

and call the elements of this module the *elements of type* (λ, μ). The invariants of type (λ, μ) will be called *typical* ones if $\lambda_n > 0$.

The validity of the following lemma is not difficult to establish.

9.2. LEMMA. *Let $\mathfrak{g} = \mathfrak{q}(V)$ or $\mathfrak{sq}(V)$ and let $\mathfrak{h}$ be the Cartan subalgebra in $\mathfrak{g}$. Let $\mathfrak{g}_+$ be the linear span of positive roots and L the finite-dimensional $\mathfrak{g}$-module generated by $L^{\mathfrak{g}_+}$.*

Then L is an irreducible $\mathfrak{g}$-module if and only if $L^{\mathfrak{g}_+}$ is irreducible as an $\mathfrak{h}$-module.

9.3. LEMMA. *Let λ and μ be strict partitions and $\lambda_{n+1} = \mu_{n+1} = 0$. Then*:

(a) $\dim(V^\lambda \otimes V^{*\mu})^{\mathfrak{sq}(V)} = 0$ *if* $\lambda \neq \mu$;
(b) $\dim(2^{-\delta(|\lambda|)}V^\lambda \otimes V^{*\lambda})^{\mathfrak{sq}(V)} = 1$ *if* $\lambda_n = 0$; *and*
(c) $\dim(2^{-\delta(|\lambda|)}V^\lambda \otimes V^{*\lambda})^{\mathfrak{sq}(V)} = 2$ *if* $\lambda_n > 0$.

Proof. Obviously, $(V^\lambda \otimes V^{*\mu})^{\mathfrak{sq}(V)} \simeq \mathrm{Hom}_{\mathfrak{sq}(V)}(V^\mu, V^\lambda)$. Since the even parts of Cartan subalgebras of $\mathfrak{q}(V)$ and $\mathfrak{sq}(V)$ are identical, the modules V^μ and V^λ are nonisomorphic as $\mathfrak{sq}(V)$-modules for $\lambda \neq \mu$; this proves (a).

Let $\lambda = \mu$ and $\lambda_n = 0$. Then $(V^\lambda)^{\mathfrak{q}_+(V)}$ is an irreducible module by Lemma 9.1 and by [S6] it is of the form

$$(V^\lambda)^{\mathfrak{q}_+(V)} = \mathrm{ind}_{P_\lambda}^{\mathfrak{h}}(\mathbb{C}),$$

where P_λ is the polarization subordinate to the functional λ.

For $\mathfrak{h} = \mathrm{Span}(e_1, \dots, e_n, e_{\bar{1}}, \dots, e_{\bar{n}})$, set $\mathfrak{sh} = \mathrm{Span}(e_1, \dots, e_n, e_{\bar{1}} - e_{\bar{n}}, \dots, e_{\overline{n-1}} - e_{\bar{n}})$. Since $\lambda_n = 0$, it follows that $e_{\bar{n}}$ belongs to the kernel of the form

$$b_\lambda(f_1, f_2) = \lambda([f_1, f_2]), \quad \text{where } f_1, f_2 \in \mathfrak{h}_{\bar{1}}.$$

The restriction of the form b_λ to $(\mathfrak{sh})_{\bar{1}}$ is thus of the same rank as b_λ. Therefore, the module $(V^\lambda)^{\mathfrak{q}_+(V)}$ remains irreducible as an $\mathfrak{sh}$-module and the type of its irreducibility (G or Q) is the same as that of the $\mathfrak{h}$-module. This proves (b).

Let n be even and $\lambda_n > 0$. Set $f = \sum_{i=1}^n (1/\lambda_i) e_{\bar{i}}$. Then we can verify that $b_\lambda(f, f) \neq 0$ and f is perpendicular to $(\mathfrak{sh})_{\bar{1}}$. This proves that the restriction of the form b_λ to $(\mathfrak{sh})_{\bar{1}}$ is invariant; but $\dim(\mathfrak{sh})_{\bar{1}} = n-1$ is odd and so, as the $\mathfrak{sh}$-module, $(V^\lambda)^{\mathfrak{q}_+(V)}$ is irreducible of type Q. If n is odd, then $(V^\lambda)^{\mathfrak{q}_+(V)}$ is of type Q as the $\mathfrak{h}$-module.

Since the restriction of b_λ to $(\mathfrak{sh})_{\bar{1}}$ is nondegenerate and of an even rank, it follows that $(V^\lambda)^{\mathfrak{q}_+(V)}$ is the direct sum $I \oplus \pi I$, where I is an irreducible $\mathfrak{sh}$-module of type G. In other words, for $\lambda_n > 0$ the module $2^{-\delta(|\lambda|)}V^\lambda \otimes V^{*\lambda}$ contains an additional $\mathfrak{sq}(V)$-invariant. This proves (c). □

9.4. LEMMA. *Let $\mathfrak{sq}(V) = \mathfrak{sq}(V) \otimes \mathrm{Span}(F)$. If φ is a typical $\mathfrak{q}(V)$-invariant, then there exists a unique typical $\mathfrak{sq}(V)$-invariant ψ such that $\varphi = F\psi$.*

Proof. By Lemma 9.3, for $\lambda_n > 0$ there are two invariants in the module $2^{-\delta(|\lambda|)}V^\lambda \otimes V^{*\lambda}$: one is a $\mathfrak{q}(V)$-invariant φ and the other is a $\mathfrak{sq}(V)$-invariant ψ. Hence, $F\psi \neq 0$ is clearly a $\mathfrak{q}(V)$-invariant and so $F\psi = c\varphi$ ($c \in \mathbb{C}$). Setting $\psi = \psi/c$, we obtain the statement of the lemma. □

9.5. LEMMA. *Any $\mathfrak{sq}(V)$-invariant that is not a $\mathfrak{q}(V)$-invariant is of the form $\varphi \operatorname{qet} Y$, where φ is a $\mathfrak{q}(V)$-invariant and Y is given by* (1.5.0).

Proof. Let us take a Grassmann algebra A with a sufficiently large number of generators and consider the elements of the algebra $\mathfrak{A}_n^n$ as functions on the space of collections

$$\mathfrak{L} = (v_1, \dots, v_n, v_1^*, \dots, v_n^*), \quad v_i \in (V \otimes A)_{\bar{0}},\ v_i^* \in (\operatorname{Hom}_A(V \otimes A, A))_{\bar{0}}.$$

Let f be an $\mathfrak{sq}(V)$-invariant that is not a $\mathfrak{q}(V)$-invariant and let M be the set of collections $\mathfrak{L}$ such that $\{v_1, \dots, v_n\}$ is a basis in $(V \otimes A)_{\bar{0}}$. Denote by $\mathrm{SQ}(V \otimes A)$ the subgroup of transformations from $\mathrm{GQ}(V \otimes A)$ whose queer determinant is equal to 1. Take $g \in \mathrm{GQ}(V \otimes A)$ such that $ge_i = v_i$ and

$$he_i = e_i + e_{\bar{i}}\xi, \qquad he_{\bar{i}} = e_{\bar{i}} + e_i\xi,$$

where $\xi = \frac{1}{n} \operatorname{qet} g$. Then $hg^{-1} \in \mathrm{SQ}(V \otimes A)$ and

$$f(\mathfrak{L}) = f(hg^{-1}\mathfrak{L}) = f(he_1, \dots, he_n, hg^{-1}v_1^*, \dots, hg^{-1}v_n^*)$$

is a polynomial in ξ with coordinates of $hg^{-1}v_i^*$. But $\xi = \frac{1}{n} \operatorname{qet} g = \frac{1}{n} \operatorname{qet} Y$ and

$$(hg^{-1}v_i^*, e_j) = (v_i^*, gh^{-1}e_j) = (v_i^*, ge_j - ge_{\bar{j}}\xi) = (v_i^*, v_j) - \langle v_i^*, v_j\rangle\xi.$$

The lemma is proved. □

9.6. LEMMA. *Let φ be a $\mathfrak{q}(V)$-invariant. Then $\varphi \operatorname{qet} Y$ is a polynomial if and only if φ is a typical invariant*.

Proof. First, let us prove (in the notation of Lemma 9.4) that if $\operatorname{qtr} F = 1$ then $F(\operatorname{qet} Y) = 1$. Indeed, let h be selected as in Lemma 9.5. Then

$$\begin{aligned}\operatorname{qet} Y + F\xi \operatorname{qet} Y &= \operatorname{qet}(\exp(F\xi)Y) \\ &= \operatorname{qet}(\exp(F\xi)) + \operatorname{qet} Y = \xi + \operatorname{qet} Y;\end{aligned}$$

hence, $F(\operatorname{qet} Y) = 1$. Let φ be a typical $\mathfrak{q}(V)$-invariant. Then, by Lemma 9.4, there exists a unique $\mathfrak{sq}(V)$-invariant ψ such that $\varphi = F\psi$. On the other hand, by Lemma 9.5 we have $\psi = \varphi_1 \operatorname{qet} Y$, where φ_1 is a $\mathfrak{q}(V)$-invariant. Hence

$$\varphi = F\psi = F(\varphi_1 \operatorname{qet} Y) = \pm\varphi_1 F(\operatorname{qet} Y) = \pm\varphi_1$$

and therefore $\varphi \operatorname{qet} Y = \pm\psi_1$ is a polynomial. Since it is $\mathfrak{sq}(V)$-invariant, it follows by Lemma 9.2 that it is a typical invariant and thus $\varphi = F(\varphi \operatorname{qet} Y)$ is also typical. The lemma is proved. □

The preceding arguments show that in order to construct $\mathfrak{sq}(V)$-invariants it suffices to construct typical $\mathfrak{q}(V)$-invariants. One of the ways to do so is described in the following lemma.

9.7. LEMMA (Notation from Theorem 1.5.2 and Appendix 0). *For any partition λ such that $\lambda_1 > \cdots > \lambda_n > 0$, the following polynomial is a typical $\mathfrak{q}(V)$-invariant*:

$$p_\lambda = \operatorname{qtr} Z^{\lambda_1} \dots \operatorname{qtr} Z^{\lambda_n}.$$

Sketch of Proof. Observe that $(\mathfrak{A}^n_n)^{\mathfrak{q}(V)} = S^{\bullet}(2^{-1}(U \otimes W))$ is a $(\mathfrak{q}(U) \oplus \mathfrak{q}(W))$-module such that $\dim U = \dim W = (n, n)$. Take a superspace L such that $\dim L = (n, n)$, and fix isomorphisms $L \simeq U$ and $L^* \simeq \pi(W)$ that determine isomorphisms of algebras $S^{\bullet}(\mathfrak{q}(L)^*) \simeq (\mathfrak{A}^n_n)^{\mathfrak{q}(V)}$. For an irreducible representation π, the functionals $\operatorname{str}\pi$ (and $\operatorname{qtr}\pi$ if the representation is of type Q) are $\mathfrak{q}(V)$-invariant elements of the algebra $S^{\bullet}(\mathfrak{q}(L))^*$; moreover, if π_λ corresponds to the irreducible module U^λ, then $\operatorname{qtr}\pi_\lambda$ (or $\operatorname{str}\pi_\lambda$) restricted to $S^{|\lambda|}(\mathfrak{q}(L)^*)$ is of type λ. The invariant elements are uniquely determined by their restrictions to a Cartan subalgebra in $\mathfrak{q}(L)$.

It is easy to verify that $\operatorname{str}\pi_\lambda$ (or $\operatorname{qtr}\pi_\lambda$) and p_λ have identical restrictions.

9.8. Proof of Theorem 1.5.2. Lemma 9.7 provides us with a construction of a typical $\mathfrak{q}(V)$-invariant and Lemma 9.6 with the construction of an $\mathfrak{sq}(V)$-invariant of type λ, which completes the proof of the theorem. $\square$

Appendix 0. Background

Linear Algebra in Superspaces. Generalities

A *superspace* is a $\mathbb{Z}/2$-graded space. For a superspace $V = V_{\bar 0} \oplus V_{\bar 1}$, denote by $\Pi(V)$ another copy of the same superspace but with shifted parity, that is, $(\Pi(V))_{\bar i} = V_{\bar i + \bar 1}$. The *superdimension* of V is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$, $p = \dim V_{\bar 0}$, and $q = \dim V_{\bar 1}$. (Usually $\dim V$ is shorthanded as a pair (p, q) or $p|q$; with the help of ε, the fact that $\dim V \otimes W = \dim V \cdot \dim W$ becomes lucid.)

A *superalgebra* is a superspace A with an even multiplication map $m\colon A \otimes A \to A$.

A superspace structure in V induces the superspace structure in the space $\operatorname{End}(V)$. A *basis of a superspace* always consists of *homogeneous* vectors; let $\operatorname{Par} = (p_1, \dots, p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of the basis of V. A square *supermatrix* of format (size) Par is a $\dim V \times \dim V$ matrix whose ith row and ith column are of the same parity p_i. The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$, and the bracket of supermatrices (of the same format) is defined via the “sign rule”:

> *if something of parity p moves past something of parity q then the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.*

An example application of the sign rule is setting

$$[X, Y] = XY - (-1)^{p(X)p(Y)}YX$$

to yield the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (which, in addition to superskew commutativity, satisfies the super Jacobi identity—i.e., the Jacobi identity amended with the sign rule). The *superderivation* of a superalgebra A is a linear map $D\colon A \to A$ that satisfies the Leibniz rule (and sign rule)

$$D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b).$$

Usually, Par is of the form $(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$; such a format is called *standard*. In this paper we can do without nonstandard formats, but they are vital in various questions related to the study of distinct systems of simple roots.

The *general linear* Lie superalgebra of all supermatrices of size Par is denoted by $\mathfrak{gl}(\mathrm{Par})$; usually, $\mathfrak{gl}(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar{0}}|\dim V_{\bar{1}})$. Any matrix from $\mathfrak{gl}(\mathrm{Par})$ can be expressed as the sum of its even and odd parts. In the standard format, this is the block expression

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar{0}, \;\; p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar{1}.$$

The *supertrace* is the map $\mathfrak{gl}(\mathrm{Par}) \to \mathbb{C}$, $(A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii}$. Since $\mathrm{str}[x, y] = 0$, the space of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(\mathrm{Par})$.

However, there exist not one but rather two super versions of $\mathfrak{gl}(n)$. The second version is called the *queer* Lie superalgebra and is defined as preserving the complex structure given by an *odd* operator J; that is, it is the centralizer $C(J)$ of J:

$$\mathfrak{q}(n) = C(J) = \{X \in \mathfrak{gl}(n|n) : [X, J] = 0\}, \quad \text{where } \; J^2 = -\mathrm{id}.$$

It is clear that, by a change of basis, we can reduce J to the form

$$J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix}.$$

In the standard format we have

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.$$

On $\mathfrak{q}(n)$, the *queer trace* is defined as

$$\mathrm{qtr}\colon \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \mathrm{tr}\, B.$$

Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices.

Observe that the identity representations of $\mathfrak{q}$ and $\mathfrak{sq}$ in V, though irreducible in supersense, are not irreducible in the nongraded sense: take homogeneous linearly independent vectors $v_1, \dots, v_n$ from V; then $\mathrm{Span}(v_1 + J(v_1), \dots, v_n + J(v_n))$ is an invariant subspace of V that is not a subsuperspace.

We will stick to the following terminology [cf. [BL; L3]). The representation of a superalgebra A in the superspace V is irreducible of *general type* (or simply of *G-type*) if it does not contain homogeneous (with respect to parity) subrepresentations distinct from 0 and V itself; otherwise, it is called *irreducible of Q-type.* Thus, an irreducible representation of Q-type has no invariant sub*super*space but does have a nontrivial invariant subspace.

Hence, there are two types of irreducible representations: those that do not contain any nontrivial subrepresentations (called of general type or of type G) and those that contain *in*homogeneous invariant subspaces (called of type Q). If V is of finite dimension, then in the first case its centralizer (as of an A-module) is isomorphic to $\mathfrak{gl}(1)$ and in the second case to $\mathfrak{q}(1)$.

Let V_1 and V_2 be finite-dimensional irreducible modules over A_1 and A_2, respectively. Then $V_1 \otimes V_2$ is an irreducible $(A_1 \otimes A_2)$-module except for the case when both V_1 and V_2 are of type Q. In the latter case, the centralizer of the $(A_1 \otimes A_2)$-module $V_1 \otimes V_2$ is isomorphic to Cl_2, the Clifford superalgebra with 2 generators.

If $e \in \mathrm{Cl}_2$ is a minimal idempotent, then $e(V_1 \otimes V_2)$ is an irreducible $(A_1 \otimes A_2)$-module of type G that we will denote by $2^{-1}(V_1 \otimes V_2)$.

More generally, we can consider matrices with the elements from a supercommutative superalgebra Λ. Then the parity of the matrix with only one nonzero (i, j)th element $X_{i,j} \in \Lambda$ is equal to $p_i + p_j + p(X_{i,j})$.

The Berezinian and the Module of Volume Forms

On $\mathrm{GL}(p|q; \Lambda)$, the group of even invertible matrices with elements from a supercommutative superalgebra Λ, we define a multiplicative function (an analog of determinant). In honor of F. Berezin, Leites [L1] baptized this function *Berezinian*. Its explicit expression in the standard format is

$$\mathrm{ber}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \det(A - BD^{-1}C)\det D^{-1}$$

or

$$\mathrm{ber}^{-1}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \det(D - CA^{-1}B)\det A^{-1}.$$

The Berezinian is a rational function, and this is one reason why the structure of the algebra of invariant polynomials on $\mathfrak{gl}(p|q)$ is much more complicated than that for the Lie algebra $\mathfrak{gl}(n)$ (see [S5]).

REMARK. For the description of other invariant polynomials see [LS], to which I should like to add that we have meanwhile proved the triviality of the center for $\mathfrak{vect}(0|n)$ $(n > 2)$ conjectured there; this was similarly but better (with other byproducts) proved in [Sho]. We also conjecture (we could not yet do this in [LS]) that the center is trivial for $\mathfrak{svect}(0|m)$ and $\widetilde{\mathfrak{svect}}(0|m)$ for $m > 3$; for the description of the center of $\mathfrak{pe}$ and $\mathfrak{spe}$, see [S1] and [Sch4].

The derivative of the Berezinian clearly is supertrace, and the relation between them is as expected: $\mathrm{ber}\, X = \exp \mathrm{str} \log X$.

The 1-dimensional representation $\mathrm{Vol}(V)$ of $\mathrm{GL}(V; \Lambda)$ corresponding both to ber and to the representation str of $\mathfrak{gl}(V)$ is called the space of *volume forms*. It can be realized in the space of tensors only as a quotient module: recall that for $\mathfrak{gl}(V)$ there is no complete reducibility (cf. [S2]).

An Odd Analog of Berezinian

On the group $\mathrm{GQ}(n; \Lambda)$ of invertible even matrices from $Q(n; \Lambda)$, the Berezinian is identically equal to 1. So on $\mathrm{GQ}(n; \Lambda)$ there is instead defined its own *queer determinant*

$$\mathrm{qet}\left(\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right) = \sum_{i \geq 0} \frac{1}{2i+1} \mathrm{tr}(A^{-1}B)^{2i+1}.$$

This strange function is $\mathrm{GQ}(n;\Lambda)$-invariant and additive; that is, $\operatorname{qet} XY = \operatorname{qet} X + \operatorname{qet} Y$ (cf. [BL]).

Superalgebras That Preserve Bilinear Forms: Two Types

To the linear map $F\colon V \to W$ of superspaces there corresponds the dual map $F^*\colon W^* \to V^*$ of the dual superspaces; if A is the supermatrix corresponding to F in a basis of the format Par, then A^{st} is the *supertransposed* matrix corresponding to F^* in the left dual basis:

$$(A^{st})_{ij} = (-1)^{(p_i+p_j)(p_i+p(A))}A_{ji}.$$

The supermatrices $X \in \mathfrak{gl}(\mathrm{Par})$ such that

$$X^{\mathrm{st}}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for a homogeneous matrix } B \in \mathfrak{gl}(\mathrm{Par})$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form on V with matrix B. Most popular is the nondegenerate supersymmetric form whose matrix in the standard format is the canonical form B_{ev} or B'_{ev}:

$$B_{\mathrm{ev}}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix};$$

or

$$B'_{\mathrm{ev}}(m|2n) = \begin{pmatrix} \operatorname{antidiag}(1,\dots,1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{\mathrm{ev}}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or $\mathfrak{osp}^{\mathrm{sy}}(m|2n)$.

Recall that the "upsetting" map $u\colon \mathrm{Bil}(V,W) \to \mathrm{Bil}(W,V)$ becomes for $V = W$ an involution $u\colon B \mapsto B^u$, which on matrices acts as follows:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \mapsto B^u = \begin{pmatrix} B_{11}^t & (-1)^{p(B)}B_{21}^t \\ (-1)^{p(B)}B_{12}^t & B_{22}^t \end{pmatrix}.$$

The forms $B = B^u$ are called *supersymmetric*, and forms $B = -B^u$ are *superskew-symmetric*. The passage from V to $\Pi(V)$ identifies the space of supersymmetric forms on V with those superskew-symmetric ones on $\Pi(V)$; these superskew-symmetric forms are preserved by the "symplectico-orthogonal" Lie superalgebra $\mathfrak{osp}^{\mathrm{sk}}(m|2n)$, which is isomorphic to $\mathfrak{osp}^{\mathrm{sy}}(m|2n)$ but has a different matrix realization. (We never use notation $\mathfrak{sp}'\mathfrak{o}(2n|m)$, in order not to confuse with the special Poisson superalgebra.)

In the standard format, the matrix realizations of these algebras are

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\},$$

and

$$\mathfrak{osp}^{\mathrm{sk}}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$

where

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m),$$

and ^{t} is the usual transposition.

A nondegenerate supersymmetric odd bilinear form $B_{\text{odd}}(n|n)$ can be reduced to the canonical form whose matrix in the standard format is J_{2n}. A canonical form of the superskew odd nondegenerate form in the standard format is

$$\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{\text{odd}}(\text{Par}))$ is $\mathfrak{pe}(\text{Par})$. The passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism $\mathfrak{pe}^{\text{sy}}(\text{Par}) \cong \mathfrak{pe}^{\text{sk}}(\text{Par})$. This Lie superalgebra is called, as suggested to Leites by A. Weil, *periplectic*. In the standard format these superalgebras are shorthanded as in the following formula, where their matrix realizations is also given:

$$\mathfrak{pe}^{\text{sy}}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, C = C^t \right\};$$

$$\mathfrak{pe}^{\text{sk}}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, C = -C^t \right\}.$$

The *special periplectic* superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \operatorname{str} X = 0\}$.

Observe that even though the Lie superalgebras $\mathfrak{osp}^{\text{sy}}(m|2n)$ and $\mathfrak{pe}^{\text{sk}}(2n|m)$, as well as $\mathfrak{pe}^{\text{sy}}(n)$ and $\mathfrak{pe}^{\text{sk}}(n)$, are isomorphic, there are sometimes crucial differences among them (cf. [Sh]).

Projectivization

If $\mathfrak{s}$ is a Lie algebra of scalar matrices and if $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing $\mathfrak{s}$, then the *projective* Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$.

Projectivization sometimes leads to new Lie superalgebras—for example: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

Appendix 1. Certain Constructions with the Point Functor

The point functor is well known in algebraic geometry since at least 1953 [Wi]. The publicity surrounding ringed spaces with nilpotents in the structure sheaf that followed the discovery of supersymmetries caused many mathematicians and physicists to realize the usefulness of the language of points. Most interesting are numerous ideas due to Witten (for some of them see [W1; W2]); for their clarification and further developments and references, see [D; Ma]. Berezin [Be1] was the first who applied the point functor to study Lie superalgebras. Here we present some of his results and their generalizations.

All superalgebras and modules are supposed to be finite-dimensional over $\mathbb{C}$.

A.0. WHAT A LIE SUPERALGEBRA IS. Lie superalgebras appeared in topology in the 1930s and earlier. So when somebody offers a "better than usual" definition of a notion that seems to have been established about 70 year ago, this might look strange, to say the least. Nevertheless, the answer to "What is a Lie superalgebra?" is still not common knowledge. Indeed, the naive definition ("apply the sign rule to the definition of the Lie algebra") is manifestly inadequate for considering the (singular) supervarieties of deformations and applying representation theory to mathematical physics—for example, in the study of the coadjoint representation of the Lie supergroup that can act on a supermanifold but never on a superspace (an object from another category). Hence, in order to deform Lie superalgebras and apply group-theoretical methods in a "super" setting, we must be able to recover a supermanifold from a superspace and vice versa.

A proper definition of Lie superalgebras is as follows (see [L3]). The *Lie superalgebra* in the category of supermanifolds corresponding to the "naive" Lie superalgebra $L = L_{\bar 0} \oplus L_{\bar 1}$ is a linear supermanifold $\mathcal{L} = (L_{\bar 0}, \mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_{\bar 0}$ with values in the Grassmann superalgebra on $L_{\bar 1}^*$; this supermanifold should be such that, for "any" (say, finitely generated, or from some other appropriate category) supercommutative superalgebra C, the space $\mathcal{L}(C) = \mathrm{Hom}(\mathrm{Spec}\, C, \mathcal{L})$, called *the space of C-points of $\mathcal{L}$*, is a Lie algebra and the correspondence $C \to \mathcal{L}(C)$ is a functor in C. (Weil introduced this approach in algebraic geometry in 1953. In the super setting it is called *the language of points* or *families*; see [D; L3].) This definition might look terribly complicated, but fortunately one can show that the correspondence $\mathcal{L} \leftrightarrow L$ is one-to-one and that the Lie algebra $\mathcal{L}(C)$, also denoted $L(C)$, admits a very simple description: $L(C) = (L \otimes C)_{\bar 0}$.

A *Lie superalgebra homomorphism* $\rho\colon L_1 \to L_2$ in these terms is a functor morphism—that is, a collection of Lie algebra homomorphisms $\rho_C\colon L_1(C) \to L_2(C)$ that is compatible with morphisms of supercommutative superalgebras $C \to C'$. In particular, a *representation* of a Lie superalgebra L in a superspace V is a homomorphism $\rho\colon L \to \mathfrak{gl}(V)$, that is, a collection of Lie algebra homomorphisms $\rho_C\colon L(C) \to (\mathfrak{gl}(V) \otimes C)_{\bar 0}$.

EXAMPLE. Consider a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. The tangent space of the moduli superspace of deformations of ρ is isomorphic to $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if $\mathfrak{g}$ is the $(0|n)$-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), then its only irreducible representations are the trivial ones $\mathbf{1}$ and $\Pi(\mathbf{1})$. Clearly $\mathbf{1} \otimes \mathbf{1}^* \simeq \Pi(\mathbf{1}) \otimes \Pi(\mathbf{1})^* \simeq \mathbf{1}$ and, because the superalgebra is commutative, the differential in the cochain complex is trivial. Therefore, $H^1(\mathfrak{g}; \mathbf{1}) = E^1(\mathfrak{g}^*) \simeq \mathfrak{g}^*$, so there are $\dim \mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ "naively" then all of the odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate and much tougher question, usually solved ad hoc.

Note that qtr is not a representation of $\mathfrak{q}(n)$ according to the naive definition ("a representation is a Lie superalgebra homomorphism" and hence an even map); however, it is a representation—in fact, an irreducible one—if we consider odd parameters.

Thus, let $\mathfrak{g}$ be a Lie superalgebra, V a $\mathfrak{g}$-module, and Λ the Grassmann superalgebra over $\mathbb{C}$ generated by q indeterminates. Define $\varphi\colon \Lambda \otimes V^* \to \mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)$ by setting

$$\varphi(\xi \otimes \alpha)(\eta \otimes v) = (-1)^{p(\alpha)(\eta)} \xi\eta\alpha(v) \quad \text{for any } \ \xi, \eta \in \Lambda,\ \alpha \in V^*.$$

Extend the ground field to Λ and consider $\Lambda \otimes V^*$ and $\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)$ as $(\Lambda \otimes \mathfrak{g})$-modules.

A1. LEMMA. *φ is a $(\Lambda \otimes \mathfrak{g})$-module isomorphism.*

Proof. Since V is finite-dimensional, φ is a vector space isomorphism over Λ; besides, it is obvious that φ is a Λ-module homomorphism. Now take $\xi_1, \xi_2, \xi_3 \in \Lambda$, $\alpha \in V^*$, $v \in V$, and $x \in \mathfrak{g}$. It is an easy exercise to prove that

$$[(\xi_1 \otimes x)\varphi(\xi_2 \otimes \alpha)](\xi_3 \otimes v) = \varphi[\xi_1 \otimes x(\xi_2 \otimes \alpha)](\xi_3 \otimes v). \qquad \square$$

Consider the composition of maps

$$V^* \xrightarrow{\varphi_1} \Lambda \otimes V^* \xrightarrow{\varphi} \mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda) \xrightarrow{\varphi_2} S_\Lambda(\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)),$$

where $\varphi_1(\alpha) = 1 \otimes \alpha$ and φ_2 is a canonical embedding of a module in its symmetric algebra. The $\mathbb{C}$-module homomorphism $\varphi_2 \circ \varphi \circ \varphi_1$ induces the algebra homomorphism

$$S(V^*) = S_{\mathbb{C}}(V^*) \to S_\Lambda(\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda))$$

and, since the latter algebra is a Λ-module, we get an algebra homomorphism

$$\Lambda \otimes S(V^*) \xrightarrow{\psi} S_\Lambda(\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)).$$

Besides, both algebras possess a natural $(\Lambda \otimes \mathfrak{g})$-module structure.

A2. LEMMA. *ψ is a $(\Lambda \otimes \mathfrak{g})$-modules and $(\Lambda \otimes \mathfrak{g})$-algebras isomorphism.*

Proof. Let us construct the inverse homomorphism. Consider the composition

$$\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda) \xrightarrow{\varphi^{-1}} \Lambda \otimes V^* \to \Lambda \otimes S(V^*).$$

Since this composition is a Λ-module homomorphism, it induces the homomorphism

$$\tilde{\psi}\colon S_\Lambda(\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)) \to \Lambda \otimes S(V^*).$$

It is not difficult to verify that

$$\psi \circ \tilde{\psi}|_{\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)} = \mathrm{id} \quad \text{for any } \ \tilde{\psi} \circ \psi|_{\Lambda \otimes S(V^*)} = \mathrm{id}.$$

Hence, ψ is an isomorphism and $\tilde{\psi}$ is its inverse. The following proposition shows that ψ is a $(\Lambda \otimes \mathfrak{g})$-module isomorphism and so completes the proof of Lemma A2. $\square$

A3. PROPOSITION. *Let A, B be Λ-superalgebras and let $\mathfrak{g}$ be a Lie superalgebra over Λ acting by differentiations on A and B. Let $M \subset A$ and $N \subset B$ be Λ-submodules that are simultaneously $\mathfrak{g}$-modules generating A and B, respectively, and let $f\colon A \to B$ be an algebra homomorphism such that $f(M) \subset N$ and $f|_M$ is a $\mathfrak{g}$-module homomorphism. Then f is a $\mathfrak{g}$-module homomorphism.*

Proof. Let $a \in A$. We may assume that $a = a_1 \dots a_n$, where the $a_i \in M$. Then for $x \in \mathfrak{g}$ we have

$$
\begin{aligned}
&f(x(a_1 \dots a_n))\\
&\quad = f\Big(\sum \pm a_1 \dots x a_i \dots a_n\Big) = \sum \pm f(a_1) \dots f(x a_i) \dots f(a_n)\\
&\quad = \sum \pm f(a_1) \dots x f(a_i) \dots f(a_n) = x[f(a_1) \dots f(a_n)] = x f(a_1 \dots a_n).
\end{aligned}
$$

This proves Proposition A3, completing the proof of Lemma A2. □

Now, let $\mathfrak{h}$ be a Lie superalgebra over Λ and let U be a Λ-module and an $\mathfrak{h}$-module. Consider $U_{\bar{0}}$ as a $\mathbb{C}$-module. Then, clearly, the natural embedding $U_{\bar{0}} \to U$ is extendable to a Λ-module homomorphism $\varphi\colon \Lambda \otimes U_{\bar{0}} \to U$.

A4. LEMMA. *The homomorphism φ is an $\mathfrak{h}_{\bar{0}}$-module homomorphism.*

Proof. Let $x \in \mathfrak{h}_{\bar{0}}$, $\xi \in \Lambda$, and $u \in U_{\bar{0}}$. Then

$$\varphi(x(\xi \otimes u)) = \varphi(\xi \otimes xu) = \xi xu$$

and

$$
\begin{aligned}
x\varphi(\xi \otimes u) &= x\xi u\\
&= \xi xu \quad \text{(by definition of a module over a superalgebra).}
\end{aligned}
$$
□

Thus, the adjoint map

$$\operatorname{Hom}_\Lambda(U, \Lambda) \to \operatorname{Hom}_\Lambda(\Lambda \otimes U_{\bar{0}}, \Lambda)$$

is also an $\mathfrak{h}_{\bar{0}}$-module homomorphism and thus it follows, by Proposition A3, that the algebra homomorphism

$$S_\Lambda(\operatorname{Hom}_\Lambda(U, \Lambda)) \to S_\Lambda(\operatorname{Hom}_\Lambda(\Lambda \otimes U_{\bar{0}}, \Lambda))$$

induced by this map is at the same time a $\mathfrak{h}_{\bar{0}}$-module morphism. Besides, by Lemma A2 the algebra $S_\Lambda(\operatorname{Hom}_\Lambda(\Lambda \otimes U_{\bar{0}}, \Lambda))$ is isomorphic as a $(\Lambda \otimes \mathfrak{h}_{\bar{0}})$-module and as an algebra to $\Lambda \otimes S(U_{\bar{0}}^*)$. In particular, they are isomorphic as $\mathfrak{h}_{\bar{0}}$-modules.

Denote by θ the composition of the homomorphisms

$$S(V^*) \to \Lambda \otimes S(V^*) \to S_\Lambda(\operatorname{Hom}_\Lambda(\Lambda \otimes V, \Lambda)) \to S_\Lambda(\operatorname{Hom}_\Lambda(\Lambda \otimes U_{\bar{0}}, \Lambda)),$$

where $U_{\bar{0}} = (\Lambda \otimes V)_{\bar{0}} = V_\Lambda$.

A5. PROPOSITION. *If $q > \dim V_{\bar{1}}$ and $\xi \in \Lambda$ with $p(\xi) = \bar{1}$, then the restriction of θ onto $\mathbb{C}[\xi] \otimes S(V^*)$ is injective.*

Proof. If $u \in V_\Lambda$ then there exists a linear form $L_u\colon \operatorname{Hom}_\Lambda(\Lambda \otimes V_\Lambda, \Lambda) \to \Lambda$ defined by the formulas $L_u(l) = l(1 \otimes u)$ and $L_u(\xi l) = \xi l(1 \otimes u) = \xi L_u(l)$. Therefore, L_u is a Λ-module homomorphism and hence is uniquely extendable to a homomorphism

$$\varphi_u\colon \mathfrak{a} = S_\Lambda(\operatorname{Hom}_\Lambda(\Lambda \otimes V_\Lambda, \Lambda)) \to \Lambda.$$

Consider the elements of $\mathfrak{a}$ as functions on V_Λ setting $f(u) = \varphi_u(f)$ for $f \in \mathfrak{a}$ and $u \in V_\Lambda$. If $f \in \Lambda \otimes S(V^*)$ then set $f(u) = \varphi_u \circ \theta(f)$. For $\alpha \in V^*$ and $\xi \in \Lambda$ we have

$$(\xi \otimes \alpha)(u) = \varphi_u \circ \theta(\xi \otimes \alpha) = L_u \circ \theta(\xi \otimes \alpha) = \theta(\xi \otimes \alpha)(1 \otimes u).$$

If $\{e_i\}_{i\in I}$ is a basis in V and $u = \sum \lambda_i \otimes e_i$, then

$$(\xi \otimes \alpha)(u) = \sum (-1)^{p(\alpha)p(e_i)} \xi \lambda_i \alpha(e_i). \tag{1}$$

On the other hand, the algebra $\mathbb{C}[\xi] \otimes S(V^*)$ is identified with the free supercommutative superalgebra generated by the e_i^* and ξ.

Let us assume that $p(e_i^*) = 0$ for $i \le n$ and $p(e_i^*) = 1$ for $i > n$. If $f \in \mathbb{C}[\xi] \otimes S(V^*)$, then

$$f = f_0 + \xi f_1,\ f_j = \sum f_{ji_1\dots i_k} e_{i_1}^* \dots e_{i_k}^*,$$

where $j = 0, 1$ and $f_{ji_1\dots i_k} \in S(V_{\bar 0}^*)$. By (1) we have

$$\begin{aligned} f(u) &= f_0(u) + \xi f_1(u) \\ &= \sum f_{0i_1\dots i_k}(u) e_{i_1}^*(u) \dots e_{i_k}^*(u) + \sum f_{1i_1\dots i_k}(u) e_{i_1}^*(u) \dots e_{i_k}^*(u). \end{aligned}$$

Set $\lambda_i = a_i$ for $i \le n$ and $\lambda_i = \xi_{i-n}$ for $i > n$. Then, since $q > \dim V_{\bar 1}$, we may assume that the family $\{\xi_i\}_{i\in I}$ freely generates $S(V_{\bar 1}^*)$ and

$$f(u) = \sum (-1)^k f_{i_1\dots i_k}(a_1 \dots a_n) \xi_{i_1 - n \dots i_k - n}. \tag{2}$$

If $\theta(f) = 0$ then $f(u) = \varphi_u \circ \theta(f)$ for any $u \in V_\Lambda$. It follows from (2) that $f_{i_1\dots i_k}(a) = 0$ for any $a \in \mathbb{C}^n$. But since $\mathbb{C}$ is algebraically closed, it follows (using [Bu, Prop. 5.3.1]) that $f_{i_1\dots i_k} = 0$; hence, $f = 0$. $\square$

A6. Lemma. *Let $q > \dim V_{\bar 1}$. Then $f \in S(V^*)$ is a $\mathfrak{g}$-invariant if and only if $\theta(f) \in \Lambda \otimes S(V_\Lambda^*)$ is $\mathfrak{g}_\Lambda$-invariant.*

Proof. Consider the factorization of θ:

$$\begin{aligned} &S(V^*) \xrightarrow{i_1} \Lambda \otimes S(V^*) \\ &\quad \xrightarrow{i_2} S_\Lambda(\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda)) \xrightarrow{i_3} S_\Lambda(\mathrm{Hom}_\Lambda(\Lambda \otimes V_\Lambda, \Lambda)) \xrightarrow{i_4} \Lambda \otimes S(V_\Lambda^*). \end{aligned}$$

Let $f \in S(V^*)^{\mathfrak{g}}$; then

$$(\xi \otimes x)(i_1(f)) = (\xi \otimes x)(1 \otimes f) = \xi \otimes xf = 0 \quad \text{for } \xi \in \Lambda,\ x \in \mathfrak{g}.$$

Conversely, let $yi_1(f) = 0$ for any $y \in \mathfrak{g}_\Lambda$. Then

$$0 = (\xi \otimes x)(1 \otimes f) = \xi \otimes xf.$$

If $p(y) = \bar 1$ then let $p(\xi) = \bar 1$. Therefore, $\xi \otimes xf = 0$ and $yf = 0$.

Thus we have

$$f \in S(V^*)^{\mathfrak{g}} \iff i_1(f) \in (\Lambda \otimes S(V^*))_\Lambda^{\mathfrak{g}}.$$

Since i_2, i_3, i_4 are $\mathfrak{g}_\Lambda$-module homomorphisms, the foregoing expression implies that if f is a $\mathfrak{g}$-invariant then $\theta(f) = i_4 \circ i_3 \circ i_2 \circ i_1(f)$ is also a $\mathfrak{g}_\Lambda$-invariant.

Conversely, let $\theta(f)$ be a $\mathfrak{g}_\Lambda$-invariant. Let $x \in \mathfrak{g}_{\bar 0}$. Then

$$\theta(1 \otimes xf) = \theta((1 \otimes x)(1 \otimes f)) = (1 \otimes x)\theta(f) = 0.$$

By Proposition A5, $1 \otimes xf = 0$ and $xf = 0$. Let $x \in \mathfrak{g}_{\bar 1}$, $\xi \in \Lambda$, and $p(\xi) = \bar 1$. Then $\theta(\xi \otimes xf) = (\xi \otimes x)\theta(1 \otimes f) = 0$ and again by Proposition A5 we have $\xi \otimes xf = 0$; hence $xf = 0$ and thus $f \in S(V^*)^{\mathfrak{g}} g$. □

A7. REMARK. The point of the preceding lemmas and propositions is that, when seeking invariant polynomials on V, we may consider them as functions on V_Λ that are invariant with respect to the Lie algebra $\mathfrak{g}_\Lambda$. This makes it possible to apply the theory of usual Lie groups and Lie algebras and their representations.

A8. REMARK. Let φ be an automorphism of the Lie algebra $\mathfrak{g}$ of the form $\varphi_\beta = \exp\bigl(\mathrm{ad}\,\frac{\lambda}{\beta(h)} Y_\beta\bigr)\exp\bigl(-\mathrm{ad}\,\frac{\mu}{\beta(h)} Y_\beta\bigr)$ for any odd isotropic root β, $h \in f$, and odd parameters λ, μ. Clearly, φ can be uniquely extended to an automorphism of the Lie superalgebra $\mathfrak{g}$. Let $\varphi(\mathfrak{h}) = \mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. If $i\colon S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{h}^*)$ is the restriction homomorphism then clearly $i(S(\mathfrak{g}^*)^{\mathfrak{g}}) \subset S(\mathfrak{h}^*)^{\varphi}$, where A^{φ} is the set of φ-invariant elements of A.

A9. PROPOSITION. *Let A be a commutative finitely generated algebra over $\mathbb{C}$ without nilpotents, and let $\mathfrak{a} = A \otimes \Lambda(p)$. Let $q \geq p$ and $f \in \mathfrak{a}$ be such that $\varphi(f) = 0$ for any $\varphi\colon \mathfrak{a} \to \Lambda(q)$. Then $f = 0$.*

Proof. Let $\psi\colon A \to \mathbb{C}$ be an arbitrary homomorphism. We extend ψ to a homomorphism $\varphi\colon \mathfrak{a} \to \Lambda(q)$, setting $\varphi = \psi \otimes 1$. If $\xi_1, \dots, \xi_p$ are generators of $\Lambda(p)$ and if $f \in \mathfrak{a}$ and $f = \sum f_{i_1\dots i_k}\xi_{i_1}\dots\xi_{i_k}$, then the condition $\varphi(f) = 0$ yields $\psi(f_{i_1\dots i_k}) = 0$. Since ψ is arbitrary, [Bu, Prop. 5.3.1] shows that $f_{i_1\dots i_k} = 0$; hence, $f = 0$. □

ACKNOWLEDGMENT. I am thankful to D. Leites for support and help.

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