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On the Zeros of Polynomials with Littlewood-Type Coefficient Constraints

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1. Introduction

There is a huge literature on the zeros of polynomials with restricted coefficients. See, for example, Amoroso [A], Bloch and Pólya [BP], Beaucoup, Borwein, Boyd, and Pinner [BBBP], Bombieri and Vaaler [BV], Hua [H], Erdős and Turán [ET], Borwein and Erdélyi [BE1; BE2], Borwein, Erdélyi, and Kós [BEK], Littlewood [L], Odlyzko and Poonen [OP], Schur [S], and Szegő [Sz].

In [BE2] we proved the following three essentially sharp theorems.

Theorem 1.1. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = 1, \ |a_j| \le 1, \ a_j \in \mathbb{C},$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle, where the constant $c > 0$ depends only on the polygon.

Theorem 1.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = |a_n| = 1, \ |a_j| \le 1, \ a_j \in \mathbb{C},$$

has at most $c(n\alpha + \sqrt{n})$ zeros in the strip

$$\{z \in \mathbb{C} : |\mathrm{Im}(z)| \le \alpha\}$$

and in the sector

$$\{z \in \mathbb{C} : |\arg(z)| \le \alpha\}.$$

Theorem 1.3. *Let $\alpha \in (0,1)$. Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = 1, \ |a_j| \le 1, \ a_j \in \mathbb{C},$$

has at most c/α zeros inside any polygon with vertices on the circle

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$$\{z \in \mathbb{C} : |z| = 1 - \alpha\},$$

where the constant $c > 0$ *depends only on the number of the vertices of the polygon*.

For $z_0 \in \mathbb{C}$ and $r > 0$, let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

In this paper we show that a polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = 1, \ |a_j| \le 1, \ a_j \in \mathbb{C}, \tag{1.1}$$

has at most $(c_1/\alpha)\log(1/\alpha)$ zeros in the disk $D(0, 1-\alpha)$ for every $\alpha \in (0,1)$, where $c_1 > 0$ is an absolute constant. This is a simple consequence of Jensen's formula. However it is not so simple to show that this estimate for the number of zeros in $D(0, 1-\alpha)$ is sharp. We will present two examples to show the existence of polynomials p_α $(\alpha \in (0,1))$ of the form (1.1) (with a suitable $n \in \mathbb{N}$ depending on α) with at least $\lfloor (c_2/\alpha)\log(1/\alpha) \rfloor$ zeros in $D(0, 1-\alpha)$ ($c_2 > 0$ is an absolute constant). In fact, we will show the existence of such polynomials from much smaller classes with more restrictions on the coefficients. Our first example has probabilistic background and shows the existence of polynomials p_α $(\alpha \in (0,1))$ with *complex* coefficients of modulus *exactly* 1 and with at least $\lfloor (c_2/\alpha)\log(1/\alpha) \rfloor$ zeros in $D(0, 1-\alpha)$ ($c_2 > 0$ is an absolute constant). Our second example is constructive and defines polynomials p_α $(\alpha \in (0,1))$ with *real* coefficients of modulus *at most* 1, with constant term 1, and with at least $\lfloor (c_2/\alpha)\log(1/\alpha) \rfloor$ zeros in $D(0, 1-\alpha)$ ($c_2 > 0$ is an absolute constant). So, in particular, the constant in Theorem 1.3 cannot be made independent of the number of vertices of the polygon.

Some other observations on polynomials with restricted coefficients are also formulated.

2. New Results

THEOREM 2.1. *Let* $\alpha \in (0,1)$. *Every polynomial of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = 1, \ |a_j| \le 1, \ a_j \in \mathbb{C}, \tag{2.1}$$

has at most $(2/\alpha)\log(1/\alpha)$ *zeros in the disk* $D(0, 1-\alpha)$.

THEOREM 2.2. *For every* $\alpha \in (0,1)$ *there is a polynomial* $Q := Q_\alpha$ *of the form*

$$Q_\alpha(x) = \sum_{j=0}^{n} a_{j,\alpha} x^j, \quad |a_{j,\alpha}| = 1, \ a_{j,\alpha} \in \mathbb{C}, \tag{2.2}$$

such that Q_α *has at least* $\lfloor (c_2/\alpha)\log(1/\alpha) \rfloor$ *zeros in the disk* $D(0, 1-\alpha)$, *where* $c_2 > 0$ *is an absolute constant*.

Theorem 2.2 is a consequence of the following.

THEOREM 2.3. *For every $n \in \mathbb{N}$ there is a polynomial p_n of the form*

$$p_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \quad |a_{j,n}| = 1, \ a_{j,n} \in \mathbb{C}, \tag{2.3}$$

such that p_n has no zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_3 \log n}{n} < |z| < 1 + \frac{c_3 \log n}{n} \right\},$$

where $c_3 > 0$ is an absolute constant.

In order to formulate some interesting corollaries of Theorems 2.1 and 2.3, we introduce some notation. Let $\mathcal{K}_n$ be the collection of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = |a_n| = 1, \ a_j \in [-1, 1].$$

Let $\mathcal{K}_n^c$ be the collection of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| = |a_n| = 1, \ a_j \in \mathbb{C}, \ |a_j| \le 1.$$

Let $\mathcal{L}_n$ be the collection of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad a_j \in \{-1, 1\}.$$

Finally, let $\mathcal{L}_n^c$ be the collection of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad a_j \in \mathbb{C}, \ |a_j| = 1.$$

For a polynomial p, let

$$d(p) := \min\{||1 - |z|| : z \in \mathbb{C}, \ p(z) = 0\}.$$

For a class of polynomials $\mathcal{A}$, we define

$$\gamma(\mathcal{A}) := \sup\{d(p) : p \in \mathcal{A}\}.$$

THEOREM 2.4. *There are absolute constants $c_4 > 0$ and $c_5 > 0$ such that*

$$\frac{c_4 \log n}{n} \le \gamma(\mathcal{L}_n^c) \le \gamma(\mathcal{K}_n^c) \le \frac{c_5 \log n}{n}.$$

THEOREM 2.5. *There is an absolute constant $c_6 > 0$ such that*

$$\gamma(\mathcal{L}_n) \le \gamma(\mathcal{K}_n) \le \frac{c_6 \log n}{n}.$$

There is an absolute constant $c_7 > 0$ *such that, for infinitely many values of* $n \in \mathbb{N}$, *we have*

$$\frac{c_7}{n} \le \gamma(\mathcal{L}_n) \le \gamma(\mathcal{K}_n).$$

THEOREM 2.6. *For every* $\alpha \in (0,1)$ *there is a polynomial* $P := P_\alpha$ *of the form*

$$P(x) = \sum_{j=0}^{n} a_{j,\alpha} x^j, \quad a_{0,\alpha} = 1, \; a_{j,\alpha} \in [-1,1], \tag{2.4}$$

that has at least $\lfloor (c_8/\alpha)\log(1/\alpha) \rfloor$ *zeros in the disk* $D(0, 1-\alpha)$, *where* $c_8 > 0$ *is an absolute constant.*

CONJECTURE 2.7. *Every polynomial* $p \in \mathcal{L}_n$ *has at least one zero in the annulus*

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_9}{n} < |z| < 1 + \frac{c_9}{n} \right\},$$

where $c_9 > 0$ *is an absolute constant.*

If a polynomial $p \in \mathcal{L}_n$ is self-reciprocal then we can prove more than the conclusion of Conjecture 2.7, as follows.

THEOREM 2.8. *Every self-reciprocal polynomial* $p \in \mathcal{L}_n$ *has at least one zero on the unit circle* $\{z \in \mathbb{C} : |z| = 1\}$.

We will also show that Conjecture 2.7 implies our next conjecture.

CONJECTURE 2.9. *There is no sequence* $(p_{n_m})_{m=1}^{\infty}$ *of "ultra-flat" polynomials* $p_{n_m} \in \mathcal{L}_{n_m}$ *satisfying*

$$(1-\varepsilon_m)(n_m+1)^{1/2} \le |p_{n_m}(z)| \le (1+\varepsilon_m)(n_m+1)^{1/2}$$

for all $z \in \mathbb{C}$ *with* $|z| = 1$ *and for all* $m \in \mathbb{N}$, *where* $(\varepsilon_m)_{m=1}^{\infty}$ *is a sequence of positive numbers converging to* 0.

THEOREM 2.10. *Conjecture 2.7 implies Conjecture 2.9.*

3. Auxiliary Results

The proof of Theorem 2.1 is based on the following result. For a proof, see for example [BE1, Sec. 4.2, E.10c].

THEOREM 3.1 (Jensen's Formula). *Suppose that* h *is a nonnegative integer and that*

$$f(z) = \sum_{k=h}^{\infty} c_k z^k, \quad c_h \neq 0,$$

is analytic on a disk of radius greater than R. *Suppose further that the zeros of* f *in* $D(0,R) \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < R\}$ *are* $a_1, a_2, \ldots, a_m$, *where each zero is listed as many times as its multiplicity. Then*

$$\log|c_h| + h \log R + \sum_{k=1}^{m} \log \frac{R}{|a_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta.$$

To prove Theorem 2.2, we will need the following deep result of Kahane [K].

THEOREM 3.2. *There is a sequence* $(p_n)_{n=1}^{\infty}$ *of polynomials* $p_n \in \mathcal{L}_n^c$ *of the form*

$$p_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \quad |a_{j,n}| = 1,\ a_{j,n} \in \mathbb{C}, \tag{3.1}$$

that satisfy

$$n^{1/2} - n^{0.31} < |p_n(z)| < n^{1/2} + n^{0.31}$$

for every $z \in \mathbb{C}$ *with* $|z| = 1$ *and for every sufficiently large* n.

In the proof of Theorem 2.2 we will also need the following simple polynomial inequality (see e.g. [BE1, Sec. 5.1, E17] for its proof). Let $D(0,1)$ and $\bar{D}(0,1)$ denote the open and closed complex unit disks, respectively.

THEOREM 3.3. *We have*

$$|p(z)| \le |z|^n \max_{u \in \bar{D}(0,1)} |p(u)|$$

for every polynomial p *of degree at most* n *with complex coefficients and for every* $z \in \mathbb{C}$ *with* $|z| > 1$.

The key step in proving Theorem 2.6 is the following lemma. We denote by $\mathcal{T}_n$ the class of all real trigonometric polynomials of degree at most n.

LEMMA 3.4. *For every* $r \in (0,1)$ *there is a real trigonometric polynomial* $P_n \in \mathcal{T}_n$ *of the form*

$$P_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}, \quad a_0 = 1,\ a_k \in [-r, r],\ k = \pm 1, \pm 2, \ldots, \pm n,$$

with $n \le c_1 r^{-13}$ $(c_1 > 0$ *is an absolute constant*$)$ *for which*

$$m(\{x \in [-\pi, \pi] : |P_n(x)| > r\} \le r.$$

We denote by $\mathcal{P}_n$ the collection of all polynomials of degree at most n with real coefficients. From Lemma 3.4 we will easily obtain Lemma 3.5.

LEMMA 3.5. *For every* $r \in (0,1)$ *we can find an integer* $n \in \mathbb{N}$, *a polynomial* $Q_{2n} \in \mathcal{P}_{2n}$ *of the form*

$$z^{-n} Q_{2n}(z) = \sum_{k=-n}^{n} a_k z^k, \quad a_0 = 1,\ a_k \in [-r, r],\ k = \pm 1, \pm 2, \ldots, \pm n,$$

with $n \le c_1 r^{-13}$ $(c_1 > 0$ *is an absolute constant*$)$, *and a set* $U_E \subset \mathbb{C}$ *such that*

$$|Q_{2n}(z)| \le 2r, \quad z \in U_E,$$

where U_E is of the form

$$U_E := \{z = \alpha e^{i\theta} : \alpha \in [1 - c_2 r^{26}, 1],\ \theta \in E\},$$

$E \subset [0, 2\pi]$ is the union of at most $2n+1$ intervals, and the Lebesgue measure $m(E)$ of E is at least $2\pi - r$ ($c_2 > 0$ is an absolute constant).

The following simple observation is due to Van der Corput; we will need it in the proof of Lemma 3.4 (see [Z, p. 197]).

LEMMA 3.6 (Van der Corput Lemma). *Let $A \neq 0$ and $B \in \mathbb{R}$. Let $I \subset \mathbb{R}$ be an interval. Then*

$$\left| \int_I \exp(i(Ax^2 + Bx))\, dx \right| \le C|A|^{-1/2},$$

where C is a constant independent of A, B, and I.

The Nikolskii-type inequality in Lemma 3.7 (see [DL, Thm. 2.6]) deals with the class $\mathcal{T}_n$ of all real trigonometric polynomials of degree at most n. This inequality will be needed in the proof of Theorem 2.8. In order to formulate this lemma, we need the following notation. Let $K := \mathbb{R} \pmod{2\pi}$. For $f \in C(K)$, let

$$\|f\|_\infty := \max_{\theta \in K} |f(\theta)|$$

and

$$\|f\|_p := \left(\int_0^{2\pi} |f(\theta)|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

LEMMA 3.7 (Nikolskii-Type Inequality for $\mathcal{T}_n$). *We have*

$$\|T_n\|_p \le \left(\frac{2rn+1}{2\pi} \right)^{1/q - 1/p} \|T_n\|_q$$

for all $T_n \in \mathcal{T}_n$ and $0 < q \le p \le \infty$, where $r := r(q)$ is the smallest integer not less than $q/2$.

Another two basic polynomial inequalities that we will need in the proof of Lemmas 3.4 and 3.5 are described in Lemmas 3.8 and 3.9. We denote by $\mathcal{T}_n^c$ the set of all trigonometric polynomials of degree at most n with complex coefficients; the set of all algebraic polynomials of degree at most n with complex coefficients will be denoted by $\mathcal{P}_n^c$.

LEMMA 3.8 (Bernstein's Inequality for Trigonometric Polynomials). *We have*

$$\max_{t \in [0, 2\pi]} |T_n'(t)| \le n \max_{t \in [0, 2\pi]} |T_n(t)|$$

for every $T_n \in \mathcal{T}_n^c$.

LEMMA 3.9 (Bernstein's Inequality for Algebraic Polynomials on the Unit Disk). *We have*

$$\max_{z \in \bar{D}(0,1)} |P_n'(z)| \le n \max_{z \in \bar{D}(0,1)} |P_n(z)|$$

for every $P_n \in \mathcal{P}_n^c$.

In the proof of Lemma 3.4 we will also need the following classical direct theorem of approximation (see e.g. [DL, Thm. 2.2, p. 204]).

LEMMA 3.10 (A Version of Jackson's Theorem). *Suppose that f is a continuously differentiable periodic function on $\mathbb{R}$. Then there is a $T_n \in \mathcal{T}_n$ such that*

$$\max_{t\in[0,2\pi]} |f(t) - T_n(t)| \le Cn^{-1} \max_{t\in[0,2\pi]} |f'(t)|,$$

where $C > 0$ is an absolute constant.

4. Proofs

Proof of Theorem 2.1. Let $\alpha \in (0,1)$, and let p be a polynomial of the form (2.1). It is easy to see that if $\alpha > 1/2$ then p does not have any zeros in $D(0, 1-\alpha)$, hence the conclusion of the theorem is true. So assume that $0 < \alpha \le 1/2$. Then

$$|p(z)| \le \frac{1}{1-|z|}, \quad z \in D(0,1).$$

Applying Jensen's formula with $R := 1 - \alpha/2$, we obtain

$$0 + \sum_{k=1}^{m} \log \frac{1-\alpha/2}{|a_k|} \le \frac{1}{2\pi} 2\pi \log \frac{2}{\alpha},$$

where the zeros of p in $D(0, 1-\alpha/2) \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1 - \alpha/2\}$ are $a_1, a_2, \dots, a_m$ and where each zero is listed as many times as its multiplicity. Therefore,

$$\sum_{\substack{k=1 \\ |a_k|<1-\alpha}}^{m} \log \frac{1-\alpha/2}{|a_k|} \le \log \frac{2}{\alpha}$$

and hence

$$\frac{M\alpha}{2} \le M \log \frac{1-\alpha/2}{1-\alpha} \le \log \frac{2}{\alpha},$$

where M is the number of zeros of p in $D(0, 1-\alpha)$. □

Proof of Theorem 2.3. Associated with a polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C},$$

we define

$$p^*(z) = z^n \overline{p(1/\bar{z})} = \sum_{j=0}^{n} \bar{a}_{n-j} z^j. \tag{4.1}$$

Let

$$p_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \quad |a_{j,n}| = 1, \; a_{j,n} \in \mathbb{C},$$

be the Kahane polynomials of Theorem 3.2 that satisfy

$$n^{1/2} - n^{0.31} < |p_n(z)| < n^{1/2} + n^{0.31}$$

for every $z \in \mathbb{C}$ with $|z| = 1$ and for every sufficiently large n. Then

$$(n^{1/2} - n^{0.31})^2 < z^{-n} p_n(z) p_n^*(z) = |p_n(z)|^2 < (n^{1/2} + n^{0.31})^2$$

for every $z \in \mathbb{C}$ with $|z| = 1$ and for every sufficiently large n. We define

$$q_n(z) = p_n(z) p_n^*(z) - nz^n. \tag{4.2}$$

Then q_n is a polynomial of degree $2n$ and

$$-3n^{0.81} < z^{-n} q_n(z) = |p_n(z)|^2 - n < 3n^{0.81}$$

for every $z \in \mathbb{C}$ with $|z| = 1$ and for every sufficiently large n. From this we conclude that

$$|q_n(z)| < 3n^{0.81} \tag{4.3}$$

for every $z \in \mathbb{C}$ with $|z| = 1$ and for every sufficiently large n. Using Theorem 3.3 and (4.3), we obtain that

$$|q_n(z)| \le |z|^n 3n^{0.81} < n \tag{4.4}$$

for every

$$\left\{ z \in \mathbb{C} : 1 \le |z| < 1 + \frac{c \log n}{n} \right\},$$

if $0 < c < 0.19$ and n is sufficiently large. Suppose that p_n has a zero in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c \log n}{2n} < |z| < 1 + \frac{c \log n}{2n} \right\},$$

where $0 < c < 1$. Then $p_n p_n^*$ has a zero z_0 in the annulus

$$\left\{ z \in \mathbb{C} : 1 \le |z| < 1 + \frac{c \log n}{n} \right\}.$$

Hence by (4.2) we have

$$|q_n(z_0)| = |p_n(z_0) p_n^*(z_0) - nz_0^n| = n|z_0|^n \ge n,$$

which is impossible by (4.4) if $0 < c < 0.19$ and n is sufficiently large. □

Proof of Theorem 2.2. By Theorem 2.3 there is a polynomial p_n of the form (2.3) such that p_n has no zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c \log n}{n} < |z| < \left(1 - \frac{c \log n}{n}\right)^{-1} \right\},$$

where $c > 0$ is an absolute constant. Since p_n is of the form (2.3), it follows that p_n^* (defined by (4.1)) is also of the form (2.3). Because p_n has exactly n complex zeros, either p_n or p_n^* has at least $n/2$ zeros in the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$. Let $q_n := p_n$ if p_n has at least $n/2$ zeros in the closed unit disk, and let $q_n := p_n^*$ otherwise. Then q_n has at least $n/2$ zeros in the disk

$$\left\{ z \in \mathbb{C} : |z| < 1 - \frac{c \log n}{n} \right\}$$

with an absolute constant $c > 0$. In proving the theorem, we may assume that $\alpha \in (0, 1/2]$. For $\alpha \in (0, 1/2]$, let n be the smallest integer such that

$$\frac{c \log n}{n} \geq \alpha.$$

Let $Q = Q_\alpha := q_n$. Then Q is of the form (2.2) and has at least

$$\lfloor (c_2/\alpha) \log(1/\alpha) \rfloor$$

zeros in the disk $D(0, 1-\alpha)$, where $c_2 > 0$ is an absolute constant. □

Proof of Theorem 2.4. First we show the upper bound. Let $p_n \in \mathcal{K}_n^c$. Observe that either $p_n \in \mathcal{K}_n^c$ or $p_n^* \in \mathcal{K}_n^c$ (defined by (4.1)) has at least $n/2$ zeros in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. By Theorem 2.1, both $p_n \in \mathcal{K}_n^c$ and $p_n^* \in \mathcal{K}_n^c$ have at most $n/4$ zeros in the disk $D(0, 1 - (c \log n)/n)$ with a sufficiently large absolute constant $c > 0$. Hence either p_n or p_n^* has at least $n/4$ zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c \log n}{n} \leq |z| \leq 1 \right\}$$

with a sufficiently large absolute constant $c > 0$. If p_n has at least $n/4$ zeros in the above annulus, then $d(p_n) \leq (c \log n)/n$. If p_n^* has at least $n/4$ zeros in the above annulus, then p_n has at least $n/4$ zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 \leq |z| \leq \left(1 - \frac{c \log n}{n}\right)^{-1} \right\},$$

and this yields $d(p_n) \leq (c_5 \log n)/n$ with a suitable absolute constant c_5 again. So the upper bound of the theorem is proved.

The lower bound of the theorem follows immediately from Theorem 2.3. □

Proof of Theorem 2.5. The upper bound is a special case of Theorem 2.4. To see the lower bound, we define

$$P_1(z) := z^2 - z - 1, \quad P_k(z) = P_{k-1}(z^3) P_1(z), \ k = 2, 3, \ldots .$$

Then it is easy to see that $P_k \in \mathcal{L}_{3^k - 1}$ and $d(P_k) \geq c 3^{-k}$ with an absolute constant $c > 0$. This proves the lower bound of the theorem. □

Proof of Lemma 3.4. In this proof $c_1, c_2, \ldots$ will denote suitable positive absolute constants. Let $h \in (0, 1)$. (The relation between r in the lemma and h will be specified later.) Take a nonnegative-valued function $g \in C^1(\mathbb{R})$ satisfying

$$g(x) = 0, \quad x \in \mathbb{R} \setminus (-1, 1),$$
$$0 \leq g(x) \leq 1, \quad x \in [-1, 1],$$

and

$$\int_{-\pi}^{\pi} g(x)\, dx = 1.$$

Let $g_h(x) := g((x - \pi)/h)$. Then

$$g_h(x) = 0, \quad x \in \mathbb{R} \setminus (\pi - h, \pi + h), \tag{4.5}$$

$$\int_0^{2\pi} g_h(x)\, dx = h, \tag{4.6}$$

and

$$\max_{x\in[0,2\pi]} |g_h'(x)| = h^{-1} \max_{x\in[0,2\pi]} |g'(x)| =: c_1 h^{-1} \tag{4.7}$$

(the function g is fixed in the proof so the constant c_1 is absolute). By Lemma 3.10 (Jackson's theorem), there is a $Q_m \in \mathcal{T}_m$ such that

$$\max_{x\in[0,2\pi]} |(Q_m - g_h)(x)| \le c_2 c_1 h^{-1} m^{-1} \le \frac{h^2}{4\pi},$$

assuming that

$$m = \lfloor 4\pi c_1 c_2 h^{-3} \rfloor + 1. \tag{4.8}$$

Hence, the 2π-periodic $Q_m \in \mathcal{T}_m$ satisfies

$$|Q_m(x)| \le \frac{h^2}{4\pi}, \quad x \in [0, \pi - h] \cup [\pi + h, 2\pi], \tag{4.9}$$

and

$$\begin{aligned} \int_0^{2\pi} Q_m(x)\, dx &= \int_0^{2\pi} g_h(x)\, dx + \int_0^{2\pi} (Q_m(x) - g_h(x))\, dx \\ &\ge h - \frac{2\pi h^2}{4\pi} \ge \frac{h}{2}. \end{aligned} \tag{4.10}$$

Denote the coefficients of Q_m by b_j, that is,

$$Q_m(x) = \sum_{j=-m}^{m} b_j e^{ijx}, \quad b_j \in \mathbb{R}.$$

Note that (4.9) implies

$$\begin{aligned} |b_j| &= \left| \frac{1}{2\pi} \int_0^{2\pi} Q_m(x) e^{-ijx}\, dx \right| \le \frac{1}{2\pi} \int_0^{2\pi} |Q_m(x)|\, dx \\ &\le \frac{1}{2\pi} \left(2h \max_{x\in[\pi-h,\pi+h]} |Q_m(x)| + 2\pi \frac{h^2}{4\pi} \right) \\ &\le \frac{1}{2\pi} \left(2h \left(\max_{x\in[\pi-h,\pi+h]} |g_h(x)| + \frac{h^2}{4\pi} \right) + \frac{h^2}{2} \right) \\ &\le \frac{1}{2\pi} \left(2h \left(\max_{x\in[0,2\pi]} |g(x)| \right) + h^2 \right) \le c_3 h \end{aligned} \tag{4.11}$$

(the function g is fixed in the proof so the constant $c_3 > 0$ is absolute). Also, by (4.10) we have

$$|b_0| = \left| \frac{1}{2\pi} \int_0^{2\pi} Q_m(x)\, dx \right| \ge \frac{h}{4\pi}. \tag{4.12}$$

Now let $S_n \in \mathcal{T}_n$ be the best uniform approximation from $\mathcal{T}_n$ to $f(x) := Q_m(Ax^2)$ on $[-\pi, \pi]$. Since f is even, so is S_n. Denote the coefficients of S_n by d_k, that is,

$$S_n(x) = \sum_{k=-n}^{n} d_k e^{ikx}, \quad d_k \in \mathbb{R},\ d_{-k} = d_k,\ k = \pm 1, \pm 2, \ldots, \pm n.$$

Combining Lemma 3.10 (Jackson's theorem) with Lemma 3.8 (Bernstein's inequality), we obtain

$$\begin{aligned} \max_{t\in[-\pi,\pi]} |f(t) - S_n(t)| &\le c_2 \Big(\max_{t\in[-\pi,\pi]} |f'(t)| \Big) n^{-1} \\ &\le c_2 2A\pi \Big(\max_{t\in[-\pi,\pi]} |Q_m'(t)| \Big) n^{-1} \\ &\le c_2 2A\pi m \Big(\max_{t\in[-\pi,\pi]} |Q_m(t)| \Big) n^{-1} \\ &\le 2A\pi(\lfloor 4\pi c_1 c_2 h^{-3} \rfloor + 1) 2n^{-1} \le h^2 \end{aligned} \tag{4.13}$$

for $n := \lfloor c_4 A h^{-5} \rfloor + 1$ with an absolute constant $c_4 > 0$. We write the coefficients d_k of S_n as follows:

$$\begin{aligned} d_k &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_n(x) e^{-ikx}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} (S_n(x) - f(x)) e^{-ikx}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Q_m(Ax^2) e^{-ikx}\, dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} (S_n(x) - f(x)) e^{-ikx}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{j=-m}^{m} b_j \exp(i(Ajx^2 - kx)) \Bigg) dx \\ &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} (S_n(x) - f(x)) e^{-ikx}\, dx. \end{aligned} \tag{4.14}$$

Now we choose $A := c_5^2 h^{-8}$, where the absolute constant $c_5 > 0$ will be chosen later. Applying Lemma 3.6 (Van der Corput's lemma) in (4.14) and using (4.13), (4.11), and (4.8), we obtain

$$\begin{aligned} |d_k| &\le c_6 A^{-1/2} \Bigg(\sum_{j=-m}^{m} |b_j| \Bigg) + h^2 \le c_6 A^{-1/2} c_3 h (2m+1) + \frac{h^2}{2\pi} \\ &\le c_7 A^{-1/2} h^{-2} + h^2 \le c_8 h^2, \quad k = \pm 1, \pm 2, \ldots, \pm n, \end{aligned} \tag{4.15}$$

where $c_6 > 0$, $c_7 > 0$, and $c_8 > 0$ are suitable absolute constants. Also, applying Lemma 3.6 in (4.14) and using (4.8), (4.13), and (4.12), we obtain

$$\begin{aligned} |d_0| &\ge |b_0| - c_6 A^{-1/2} \Bigg(\sum_{j=-m}^{-1} |b_j| + \sum_{j=1}^{m} |b_j| \Bigg) - h^2 \\ &\ge |b_0| - c_6 A^{-1/2} c_3 h (2m+1) - h^2 \ge |b_0| - c_9 A^{-1/2} h^{-2} - h^2 \\ &\ge \frac{h}{4\pi} - \frac{c_9 h^2}{c_5} - h^2 \ge \frac{h}{8\pi} \end{aligned} \tag{4.16}$$

with some absolute constant $c_9 > 0$, where $c_5 > 0$ is chosen so that the last inequality in (4.16) is satisfied. Observe that (4.9) and $f(x) = Q_m(Ax^2)$ for $x \in [-\pi, \pi]$ imply that

$$\left\{x \in [-\pi, \pi] : |f(x)| > \frac{h^2}{4\pi}\right\} \subset \bigcup_{k=0}^{N}([a_k, b_k] \cup [-b_k, -a_k]), \tag{4.17}$$

where

$$[a_k, b_k] := \left[\left(\frac{(2k+1)\pi - h}{A}\right)^{1/2}, \left(\frac{(2k+1)\pi + h}{A}\right)^{1/2}\right]$$

and

$$N := \lfloor A/2 \rfloor + 1.$$

A straightforward calculation gives that, for $h \in (-1, 1)$,

$$\begin{aligned} 2\sum_{k=0}^{N}(b_k - a_k) &= 2\sum_{k=0}^{N}\left(\frac{(2k+1)\pi + h}{A}\right)^{1/2} - \left(\frac{(2k+1)\pi - h}{A}\right)^{1/2} \\ &\le 2\sum_{k=0}^{N}\frac{2h}{2\left(A\left(2k+\frac{1}{2}\right)\pi\right)^{1/2}} \le 2c_{10}hA^{-1/2}N^{1/2} \le c_{11}h \end{aligned} \tag{4.18}$$

with some absolute constants $c_{10} > 0$ and $c_{11} > 0$. Combining (4.17), (4.18), and (4.13) gives for $h \in (0, 1)$ that

$$m\left(\left\{x \in [-\pi, \pi] : |S_n(x)| > 2h^2\right\}\right) \le c_{11}h. \tag{4.19}$$

Now let $R_n := d_0^{-1}S_n \in \mathcal{T}_n$, where (as before)

$$n := \lfloor c_4 Ah^{-5} \rfloor + 1 \le \lfloor c_{12}h^{-13} \rfloor \tag{4.20}$$

with an absolute constant $c_{12} > 0$. Since S_n is even, so is R_n. Hence, by (4.15) and (4.16) we have

$$\begin{gathered} R_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}, \\ a_0 = 1, \quad -8\pi c_8 h \le a_k \le 8\pi c_8 h, \quad k = \pm 1, \pm 2, \ldots, \pm n. \end{gathered} \tag{4.21}$$

Finally we conclude from (4.19) that

$$m\left(\left\{x \in [0, 2\pi] : |R_n(x)| > 16\pi h\right\}\right) \le c_{11}h. \tag{4.22}$$

Now (4.20), (4.21), and (4.22) give the lemma. □

Proof of Lemma 3.5. For $r \in (0, 1)$, let $P_n \in \mathcal{T}_n$ be the same as in Lemma 3.4 ($n \le c_1 r^{-13}$). Let

$$Q_{2n}(e^{it}) := e^{int}P_n(t).$$

Then $Q_{2n} \in \mathcal{P}_{2n}$ is of the required form. Also, there exists a set $E \subset [0, 2\pi]$ with $m(E) \ge 2\pi - r$ such that

$$|Q_{2n}(z)| \le r, \quad z = e^{i\theta}, \; \theta \in E. \tag{4.23}$$

Since the set

$$\{z \in \mathbb{C} : |z| = 1, \ |Q_{2n}(z)| < r\}$$

is the union of at most $2n$ subarcs, we may assume that $E \subset [0, 2\pi]$ is the union of at most $2n + 1$ intervals. Now let $z_\alpha := \alpha e^{i\theta}$ with $\alpha \in [1 - c_2 r^{26}, 1]$. Using Lemma 3.9 (Bernstein's inequality) and (4.23), we obtain

$$\begin{aligned}|Q_{2n}(z_\alpha)| &\le |Q_{2n}(z_1)| + |Q_{2n}(z_\alpha) - Q_{2n}(z_1)| \\ &\le r + |z_\alpha - z_1| \max_{|w|\le 1} |Q'_{2n}(w)| \\ &\le r + 2nc_2 r^{26} \max_{|w|\le 1} |Q_{2n}(w)| \le r + 2nc_2 r^{26}(1 + 2nr) \\ &\le r + 2c_1 r^{-13} c_2 r^{26}(1 + 2c_1 r^{-12}) \le 2r\end{aligned}$$

for a sufficiently small absolute constant $c_2 > 0$. □

Proof of Theorem 2.6. Without loss of generality we may assume that α^{-1} is an integer. Let M be defined by

$$M := \lfloor c_3 \log(1/\alpha) \rfloor \tag{4.24}$$

with a sufficiently small absolute constant $c_3 > 0$ that will be specified later. We define

$$R(z) := 2^M z^{M/\alpha} - 1. \tag{4.25}$$

Then R has M/α zeros on a circle centered at the origin with radius $2^{-\alpha}$. These are given explicitly by the formulas

$$z_k := 2^{-\alpha} \exp\left(\frac{2\pi k i}{M/\alpha}\right), \quad k = 0, 1, \ldots, M/\alpha - 1.$$

Let B_k, $k = 0, 1, \ldots, (M/\alpha) - 1$, be the regions described as the union of the points $z = \beta e^{i\theta}$ for which

$$\beta \in [2^{-2\alpha}, 2^{-\alpha/2}] \quad \text{and} \quad \theta \in \left[\frac{(2k-1)\pi}{M/\alpha}, \frac{(2k+1)\pi}{M/\alpha}\right].$$

Then $z_k \in B_k$ and an easy calculation shows that

$$|R(z)| \ge c_4, \quad z \in \partial B_k, \tag{4.26}$$

where ∂B_k denotes the boundary of B_k and $c_4 > 0$ is an absolute constant. Associated with $r := \alpha^{1/52}$, let n, Q_{2n}, E, U_E be as in Lemma 3.5. Then the radial width of U_E is $c_2 \alpha^{1/2}$. Also, $m(E) \ge 2\pi - \alpha^{1/52}$, E is the union of at most $2c_1\alpha^{-1/4} + 1$ intervals, and $|Q_{2n}(z)| \le 2\alpha^{1/52}$ on U_E. From these we conclude that

$$|2^M Q_{2n}(z)| < 2^M 2\alpha^{1/52} < e^{(\log 2)c_3 \log(1/\alpha)} < c_4, \quad z \in U_E, \tag{4.27}$$

assuming that the absolute constant $c_3 > 0$ in (4.24) is sufficiently small. Note that by Lemma 3.5 we have $n \le c_1 r^{-13}$, so if $\alpha < c_6$ with a sufficiently small absolute constant $c_6 > 0$ then

$$\frac{M}{\alpha} - n \geq \frac{M}{\alpha} - c_1 r^{-13} = \frac{M}{\alpha} - c_1 \alpha^{-1/4}$$
$$\geq \frac{\lfloor c_3 \log(1/\alpha) \rfloor}{\alpha} - c_1 \alpha^{-1/4} > 0. \tag{4.28}$$

Also, if $c_3 > 0$ in (4.24) is sufficiently small, then

$$2^M r = e^{(\log 2)\lfloor c_3 \log(1/\alpha)\rfloor} \alpha^{1/52} \leq 1. \tag{4.29}$$

Now let

$$P(z) := R(z) - 2^M z^{M/\alpha - n} Q_{2n}(z).$$

By (4.28) and (4.29), if $\alpha < c_6$ with a sufficiently small absolute constant $c_6 > 0$ and if the absolute constant $c_3 > 0$ in (4.24) is sufficiently small, then the polynomial P is of the form

$$P(z) = \sum_{k=0}^{N} a_k z^k, \quad a_0 = -1, \ a_k \in [-1, 1], \ k = 1, 2, \ldots, N.$$

It is also routine to observe that, for $\alpha < c_7$ (with a sufficiently small absolute constant $c_7 > 0$), the number of the indices $k = 0, 1, \ldots, (M/\alpha) - 1$ for which $B_k \subset U_E$ is at least $M/(2\alpha)$. Using (4.26), (4.27), and Rouche's theorem, we conclude that if $\alpha < c_7$ and the absolute constant $c_3 > 0$ in (4.24) is sufficiently small then P has at least

$$M/(2\alpha) = \lfloor c_3 \log(1/\alpha) \rfloor / (2\alpha)$$

zeros in the disk centered at 0 with radius $2^{-\alpha/2} \leq 1 - \alpha/4$. The proof is now finished. □

Proof of Theorem 2.8. Suppose $p \in \mathcal{L}_n$ is self-reciprocal and suppose p does not have a zero on the unit circle. If n is odd, then $p(-1) = 0$ and the theorem is proved. If n is even, then $T_n(t) := e^{-nt/2} p(e^{it})$ is a real trigonometric polynomial of degree at most $n/2$; that is, $T_n \in \mathcal{T}_{n/2}$, and T_n does not have any real zeros. Without loss of generality we may assume that T_n is positive on the real line (this implies that the constant term in T_n is 1). We fix an $\varepsilon \in (0, 1)$ so that $T_n - \varepsilon$ does not have a real zero. Then we have

$$\|T_n - \varepsilon\|_1 = \int_0^{2\pi} |T_n(\theta) - \varepsilon| \, d\theta = \int_0^{2\pi} (T_n(\theta) - \varepsilon) \, d\theta = 2\pi(1 - \varepsilon).$$

Using the Parseval formula, we also have

$$\|T_n - \varepsilon\|_2 = (2\pi(n + 1 - 2\varepsilon + \varepsilon^2))^{1/2}.$$

But then, by Lemma 3.7 (the Nikolskii-type inequality for $\mathcal{T}_n$), we have

$$(2\pi(n + 1 - 2\varepsilon + \varepsilon^2))^{1/2} = \|T_n - \varepsilon\|_2 \leq \left(\frac{n+1}{2\pi}\right)^{1/2} \|T_n - \varepsilon\|_1$$
$$= \left(\frac{n+1}{2\pi}\right)^{1/2} 2\pi(1 - \varepsilon).$$

Hence, for $\varepsilon \in (0, 1)$ we have

$$n(2\varepsilon - \varepsilon^2) \leq 0,$$

a contradiction. □

Proof of Theorem 2.10. The proof is similar to that of Theorem 2.2. We omit the details. □

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